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Chromatic Uniqueness of Certain Bipartite Graphs with Six Edges Deleted

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Abstract : For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of 2−connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. F.M.Dong et al. (Discrete Math. vol.224 (2000) 107–124) proved that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq \min\{4, q-1\}$, then G is chromatically unique. In this paper, we study the chromaticity of any graph $G \in \mathcal{K}_2^{-s}(p,q)$ when $p \geq 6$, $q = 4$ and $s = 6$.

Keywords : Chromatic polynomial; Chromatic uniqueness; Bipartite graph.
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1 Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$equivalent), symbolically $G \sim H$, if $P(G,l) = P(H,l)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply $\chi-$*disjoint*.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2−connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.

In [4, 5], Dong et al. proved the following results.

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Lemma 1.1. *If $p \geq q \geq 3$ and $s \leq p+q-4$, then for any $G \in K^{-s}(p,q)$ with $\delta(G) \geq 2$, then G is 2-connected.*

Theorem 1.2. *For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q-1$, $K_2^{-s}(p,q)$ is χ-closed.*

Teo and Koh [13] showed that every graph in $\mathcal{K}(p,q) \cup \mathcal{K}^{-1}(p,q)$ is $\chi-$unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p-q$ is *small enough*, then each graph in $\mathcal{K}^{-s}(p,q)$ is $\chi-$unique;

(ii) If $G \in \mathcal{K}^{-s}(p,q)$, where $0 \leq p-q \leq 1$, such that the set of s edges deleted forms a matching, then G is $\chi-$unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p,q)$, where $3 \leq s \leq p-q$ and

$$q \geq \max \left\{ \frac{1}{2}(p-q)(s-1)+\frac{3}{2},\ \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6 \right\},$$

and the set of s edges deleted forms a matching or a star, then G is $\chi-$unique. In [5], Dong et al. proved that any $2-$connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q-1$ or that induces a star is chromatically unique.

Very recently, Dong et al. [4] showed that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q-1\}$. In [9], we proved that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 6$ and $s = 5$; or $p \geq q \geq 7$ and $s = 6$. In [10, 11], we extended this study for the case $p > q = 5$ and $s = 5$; or $p > q = 4$ and $s = 5$. In this paper, we shall study the chromaticity of any graph in $K_2^{-s}(p,q)$ when $p \geq 6$, $q = 4$ and $s = 6$.

2 Preliminary Results and Notation

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E,\ x \in A, y \in B\ \}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [8]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 2.1. *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{2.1}$$

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{ \, Q \mid Q \text{ is an independent set in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \, \}.$$

Lemma 2.2. *(Dong et al. [5]) For $G \in \mathcal{K}^{-s}(p,q)$,*

$$\alpha'(G,3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4−independent partitions $\{ A_1, A_2, A_3, A_4 \}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$\begin{aligned}(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3)\\ = (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.\end{aligned}$$

Define

$$\alpha'(G,4) = \alpha(G,4) - \{ \, (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2 \, \}.$$

Observe that for G, $H \in \mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4) = \alpha(H,4) \quad \text{if and only if} \quad \alpha'(G,4) = \alpha'(H,4).$$

The following results will be used to prove our main theorem.

Lemma 2.3. *(Dong et al. [3]) For $G = (A, B; E) \in \mathcal{K}^{-s}(p,q)$ with $|A| = p$ and $|B| = q$,*

$$\begin{aligned}\alpha'(G,4) \quad &= \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) + \\ &\quad \left| \{ \, \{ \, Q_1, Q_2 \, \} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \, \} \right|.\end{aligned}$$

Lemma 2.4. *(Dong et al. [5]) For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,*

$$\alpha(G,k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1).$$

Theorem 2.5. *(Dong et al. [5]) For integers p,q,s with $p \geq q \geq 3$ and $0 \leq s \leq 2q - 3$, and $G \in \mathcal{K}_2^{-s}(p,q)$,*

$$[G] \ \subseteq \mathcal{K}_2^{-s}(p,q),$$

if one of the following conditions is satisfied:

(i) $s \leq q-1$;

(ii) $s = q \geq 6$ *and* $p \geq 2$;

(iii) $p \geq q+4$;

(iv) $p \in \{q+3, q+1\}$ *and* $0 \leq s \leq 2q-4$;

(v) $p = q+2$ *and* $\triangle(G') \geq s+3-q$;

(vi) $p = q$ *and* $\alpha'(G_i, 3) < 2^{p-2}$.

3 Main Result

In [9], we proved that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 6$ and $s = 5$ or $s = 6$. In [10], we showed that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p > q = 5$ and $s = 5$. In [11], we proved that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p > q = 4$ and $s = 5$. In this section, we shall prove that every graph in $\mathcal{K}_2^{-s}(p,q)$ is χ-unique if $p \geq 6$, $q = 4$ and $s = 6$.

Let G be any graph in $\mathcal{K}_2^{-6}(p,q)$, and $G' = K(p,q) - G$. By construction method and Lemma 1.1, one can easily verify that there are 44 structures of G' ($q = 4$ and G is $2-$connected), which are named as G'_1, G'_2, …, G'_{44} (see Table 1). We group the graphs G_1, G_2, …, G_{44} according to their values of $\alpha'(G_i, 3)$, which can be calculated by using Lemma 2.2 and these values are in column three of Table 1. Thus we have the following observations.

(i) $\alpha'(G_i, 3) = 8$, for i=1;

(ii) $\alpha'(G_i, 3) = 9$, for i=2,3,4,5;

(iii) $\alpha'(G_i, 3) = 10$, for i=6,7,…,11;

(iv) $\alpha'(G_i, 3) = 11$, for i=12,13,…,17;

(v) $\alpha'(G_i, 3) = 12$, for i=18,19,…,25;

(vi) $\alpha'(G_i, 3) = 13$, for i=26,27,28;

(vii) $\alpha'(G_i, 3) = 14$, for i=29,30;

(viii) $\alpha'(G_i, 3) = 15$, for i=31,32;

(ix) $\alpha'(G_i, 3) = 17$, for i=33,34;

(x) $\alpha'(G_i, 3) = 18$, for i=35,36,37;

(xi) $\alpha'(G_i, 3) = 19$, for i=38,39;

(xii) $\alpha'(G_i, 3) = 20$, for i=40;

(xiii) $\alpha'(G_i, 3) = 21$, for i=41;

(xiv) $\alpha'(G_i, 3) = 32$, for i=42;

(xv) $\alpha'(G_i, 3) = 33$, for i=43;

(xvi $\alpha'(G_i, 3) = 63$, for i=44.

We then group these graphs according to their $\alpha'(G_i, 3)$. Hence we have the following classification of the graphs.

$$\begin{aligned}
\mathcal{T}_1 &= \{ G_1 \} \\
\mathcal{T}_2 &= \{ G_2, G_3, G_4, G_5 \} \\
\mathcal{T}_3 &= \{ G_6, G_7, \ldots, G_{11} \} \\
\mathcal{T}_4 &= \{ G_{12}, G_{13}, \ldots, G_{17} \} \\
\mathcal{T}_5 &= \{ G_{18}, G_{19}, \ldots, G_{25} \} \\
\mathcal{T}_6 &= \{ G_{26}, G_{27}, G_{28} \} \\
\mathcal{T}_7 &= \{ G_{29}, G_{30} \} \\
\mathcal{T}_8 &= \{ G_{31}, G_{32} \} \\
\mathcal{T}_9 &= \{ G_{33}, G_{34} \} \\
\mathcal{T}_{10} &= \{ G_{35}, G_{36}, G_{37} \} \\
\mathcal{T}_{11} &= \{ G_{38}, G_{39} \} \\
\mathcal{T}_{12} &= \{ G_{40} \} \\
\mathcal{T}_{13} &= \{ G_{41} \} \\
\mathcal{T}_{14} &= \{ G_{42} \} \\
\mathcal{T}_{15} &= \{ G_{43} \} \\
\mathcal{T}_{16} &= \{ G_{44} \}
\end{aligned}$$

We also calculate the values of $\alpha'(G_i, 4)$ by using Lemma 2.3 and we list them in column four of Table 1. We now present our main result in the following theorem.

Theorem 3.1. *Every graph in $\mathcal{K}_2^{-6}(p, q)$ with $p > q = 4$ is χ-unique if one of the following conditions is satisfied:*

(i) $p \geq 7$,

(ii) $p = 6$ *and* $\triangle(G') \geq 5$.

Proof Observe that for any i, j with $1 \leq i < j \leq 16$, $\alpha'(G, 3) < \alpha'(H, 3)$ if $G \in \mathcal{T}_i$ and $H \in \mathcal{T}_j$. Thus by Lemma 2.1 and Equation (2.1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 16)$ are $\chi-$disjoint and since $\mathcal{K}_2^{-6}(p, 4)$ is $\chi-$closed under the conditions (iii) or (iv) of Theorem 2.5, then each $\mathcal{T}_i$ $(1 \leq i \leq 16)$ is $\chi-$closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are $\chi-$unique, it suffices to show that for any two graphs, G, $H \in \mathcal{T}_i$, if $G \not\cong H$, then either $\alpha'(G, 4) \neq \alpha'(H, 4)$ or $\alpha(G, 5) \neq \alpha(H, 5)$. Note that $\mathcal{T}_1$, $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_{14}$, $\mathcal{T}_{15}$ and $\mathcal{T}_{16}$ contains only one graph G_1, G_{41}, G_{42}, G_{43} and G_{44}, respectively and hence G_1, G_{40}, G_{41}, G_{42}, G_{43} and G_{44} are $\chi-$unique. The remaining work is to compare every two graphs in $\mathcal{T}_i$

for $2 \le i \le 11$. Note that all graphs in $\mathcal{T}_i$ $(2 \le i \le 11)$ are not considerable for the case $p = 6$ since $\triangle(G') < 5$. Thus, for all $\mathcal{T}_i$ $(2 \le i \le 11)$, we only consider the case $p \ge 7$.

[1] $\mathcal{T}_2$

$$\begin{aligned}
&\alpha'(G_4,4) - \alpha'(G_2,4)\\
&\quad = \left[3 \cdot 2^{p-3} + 3 \cdot 2^{q-2} + 21 \right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 14 \right]\\
&\quad = -2^{p-3} + 2^{q-3} + 7 < 0,\\
&\alpha'(G_2,4) - \alpha'(G_3,4)\\
&\quad = \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 14 \right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 18 \right]\\
&\quad = -4 < 0,\\
&\alpha'(G_3,4) - \alpha'(G_5,4)\\
&\quad = \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 18 \right] - \left[4 \cdot 2^{p-3} + 5 \cdot 2^{q-3} + 21 \right]\\
&\quad = -3 < 0.
\end{aligned}$$

Thus, we can conclude that $\alpha'(G_i,4) \ne \alpha'(G_j,4)$ for $2 \le i < j \le 5$.

[2] $\mathcal{T}_3$

$$\begin{aligned}
&\alpha'(G_{11},4) - \alpha'(G_7,4)\\
&\quad = \left[7 \cdot 2^{p-4} + 4 \cdot 2^{q-2} + 16 \right] - \left[5 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 18 \right]\\
&\quad = -3 \cdot 2^{p-4} + 2^{q-3} - 2 < 0,\\
&\alpha'(G_7,4) - \alpha'(G_6,4)\\
&\quad = \left[5 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 18 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 14 \right]\\
&\quad = -2^{p-3} + 2^{q-3} + 4 < 0,\\
&\alpha'(G_6,4) - \alpha'(G_8,4)\\
&\quad = \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 14 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] = -4 < 0,\\
&\alpha'(G_8,4) - \alpha'(G_9,4)\\
&\quad = \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] = 0,\\
&\alpha'(G_8,4) - \alpha'(G_{10},4)\\
&\quad = \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18 \right] - \left[6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 19 \right] = -1 < 0.
\end{aligned}$$

Thus, we can conclude that $\alpha'(G_i,4) \ne \alpha'(G_j,4)$ for $6 \le i < j \le 12$ except for the graphs G_8 and G_9. Since $\alpha'(G_8,4)=\alpha'(G_9,4)$, we need to compare $\alpha(G_8,5)$ and $\alpha(G_9,5)$. By using Lemma 2.4, we can show that $\alpha(G_8,5)\neq\alpha(G_9,5)$ (see [12]).

[3] $\mathcal{T}_4$

$$\begin{aligned}
&\alpha'(G_{15},4)-\alpha'(G_{17},4)\\
&=\left[\,9\cdot 2^{p-4}+5\cdot 2^{q-2}+21\,\right]-\left[\,11\cdot 2^{p-4}+9\cdot 2^{q-3}+12\,\right]\\
&=-2^{p-3}+2^{q-3}+9<0,\\
&\alpha'(G_{17},4)-\alpha'(G_{16},4)\\
&=\left[\,11\cdot 2^{p-4}+9\cdot 2^{q-3}+12\,\right]-\left[\,11\cdot 2^{p-4}+9\cdot 2^{q-3}+21\,\right]=-9<0,\\
&\alpha'(G_{16},4)-\alpha'(G_{14},4)\\
&=\left[\,11\cdot 2^{p-4}+9\cdot 2^{q-3}+21\,\right]-\left[\,7\cdot 2^{p-3}+7\cdot 2^{q-3}+11\,\right]\\
&=-3\cdot 2^{p-4}+2\cdot 2^{q-3}+10<0,\\
&\alpha'(G_{14},4)-\alpha'(G_{12},4)\\
&=\left[\,7\cdot 2^{p-3}+7\cdot 2^{q-3}+11\,\right]-\left[\,7\cdot 2^{p-3}+8\cdot 2^{q-3}+15\,\right]\\
&=-2^{q-3}-4<0,\\
&\alpha'(G_{12},4)-\alpha'(G_{13},4)\\
&=\left[\,7\cdot 2^{p-3}+8\cdot 2^{q-3}+15\,\right]-\left[\,7\cdot 2^{q-3}+8\cdot 2^{p-3}+15\,\right]\\
&=-2^{p-3}+2^{q-3}<0.
\end{aligned}$$

Thus, we can conclude that $\alpha'(G_i,4)\neq\alpha'(G_j,4)$ for $13\leq i<j\leq 18$.

[4] $\mathcal{T}_5$**:** We consider two cases $p=7$ and $p\geq 8$.

(4.1) **Case 1:** When $p=7$.

$$\alpha'(G_{25},4) - \alpha'(G_{23},4)$$
$$= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 12 \right] - \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 18 \right] - 6 < 0,$$
$$\alpha'(G_{23},4) - \alpha'(G_{22},4)$$
$$= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 18 \right] - \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 8 \right]$$
$$= -2 \cdot 2^{p-4} + 2^{q-3} + 10 < 0,$$
$$\alpha'(G_{22},4) - \alpha'(G_{21},4)$$
$$= \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 8 \right] - \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 24 \right]$$
$$= -2 \cdot 2^{p-4} - 2^{q-3} - 16 < 0,$$
$$\alpha'(G_{21},4) - \alpha'(G_{24},4)$$
$$= \left[13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 24 \right] - \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 18 \right]$$
$$= -2 \cdot 2^{p-4} + 2^{q-3} + 6 < 0,$$
$$\alpha'(G_{24},4) - \alpha'(G_{18},4)$$
$$= \left[15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 18 \right] - \left[8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17 \right]$$
$$= -2^{p-4} + 2^{q-3} + 1 < 0,$$
$$\alpha'(G_{18},4) - \alpha'(G_{20},4)$$
$$= \left[8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17 \right] - \left[9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 12 \right]$$
$$= -2^{p-3} + 5 < 0,$$
$$\alpha'(G_{20},4) - \alpha'(G_{19},4)$$
$$= \left[9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 12 \right] - \left[8 \cdot 2^{q-3} + 9 \cdot 2^{p-3} + 17 \right]$$
$$= 2^{q-3} - 5 < 0.$$

Thus, we have $\alpha'(G_{25},4) < \alpha'(G_{23},4) < \alpha'(G_{22},4) < \alpha'(G_{21},4) < \alpha'(G_{24},4) < \alpha'(G_{18},4) < \alpha'(G_{20},4) < \alpha'(G_{19},4)$.

(4.2) Case 2: When $p \geq 8$, we can easily show that $\alpha'(G_{25},4) < \alpha'(G_{23},4) < \alpha'(G_{21},4) < \alpha'(G_{22},4) < \alpha'(G_{24},4) < \alpha'(G_{18},4) < \alpha'(G_{20},4) < \alpha'(G_{19},4)$.

Thus, we conclude that $\alpha'(G_i,4) \neq \alpha'(G_j,4)$ for $18 \leq i < j \leq 25$.

Similarly, we can show that for any two graphs, $G, H \in \mathcal{T}_i$ $(6 \leq i \leq 11)$, then $\alpha'(G,4) \neq \alpha'(H,4)$. For details, see [12]. Hence, the proof of the theorem is now completed. □

In view of Theorem 3.1 and results in [9], we posed the following problem:

Problem. Study the chromaticity of any graph in $\mathcal{K}_2^{-6}(p,q)$ with $p > q$ and $q = 5, 6$.

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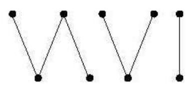
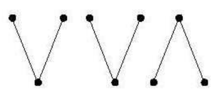
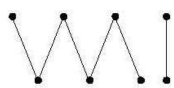
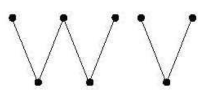
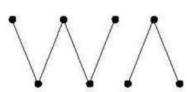
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p, \lvert B'\rvert = 4$	$\alpha'(G_i,3)$	$\alpha'(G_i,4) - 6(2^{p-2}+2^{q-2}-2)$
G_1	 A' B'	8	$2\cdot 2^{p-3}+2\cdot 2^{q-3}+18$
G_2	 A' B'	9	$4\cdot 2^{p-3}+5\cdot 2^{q-3}+14$
G_3	 A' B'	9	$4\cdot 2^{p-3}+5\cdot 2^{q-3}+18$
G_4	 A' B'	9	$3\cdot 2^{p-3}+3\cdot 2^{q-2}+21$
G_5	 A' B'	9	$4\cdot 7^{p-3}+5\cdot 2^{q-3}+21$
G_6	 A' B'	10	$6\cdot 2^{p-3}+6\cdot 2^{q-3}+14$
G_7	 A' B'	10	$5\cdot 2^{p-3}+7\cdot 2^{q-3}+18$
G_8	 A' B'	10	$6\cdot 2^{p-3}+6\cdot 2^{q-3}+18$

TABLE 1 (1 of 6): Graphs in $\mathcal{K}_2^{-6}(p,4)$, $p \geq 6$

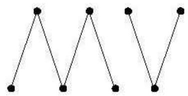
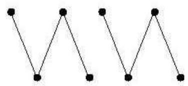
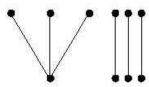
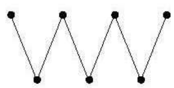
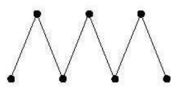
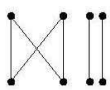
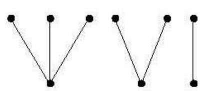
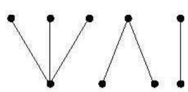
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p, \lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_9	 A' B'	10	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 18$
G_{10}	 A' B'	10	$6 \cdot 2^{p-3} + 6 \cdot 2^{q-3} + 19$
G_{11}	 A' B'	10	$7 \cdot 2^{p-4} + 4 \cdot 2^{q-2} + 16$
G_{12}	 A' B'	11	$7 \cdot 2^{p-3} + 8 \cdot 2^{q-3} + 15$
G_{13}	 A' B'	11	$2 \cdot 7^{p-3} + 8 \cdot 2^{q-3} + 15$
G_{14}	 A' B'	11	$7 \cdot 2^{p-3} + 7 \cdot 2^{q-3} + 11$
G_{15}	 A' B'	11	$9 \cdot 2^{p-4} + 5 \cdot 2^{q-2} + 21$
G_{16}	 A' B'	11	$11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 21$

TABLE 1 (2 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

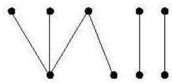
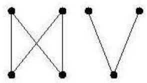
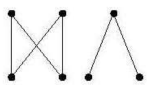
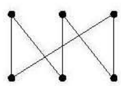
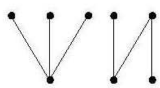
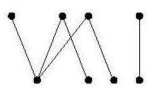
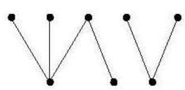
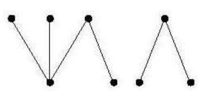
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,4} - G_i$) $\lvert A'\rvert = p, \lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{17}	 A' B'	11	$11 \cdot 2^{p-4} + 9 \cdot 2^{q-3} + 12$
G_{18}	 A' B'	12	$8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17$
G_{19}	 A' B'	12	$8 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 17$
G_{20}	 A' B'	12	$9 \cdot 2^{p-3} + 9 \cdot 2^{q-3} + 12$
G_{21}	 A' B'	12	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 24$
G_{22}	 A' B'	12	$15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 8$
G_{23}	 A' B'	12	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 18$
G_{24}	 A' B'	12	$15 \cdot 2^{p-4} + 10 \cdot 2^{q-3} + 18$

TABLE 1 (3 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

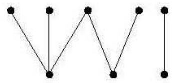
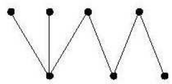
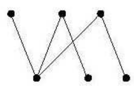
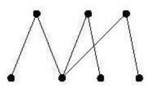
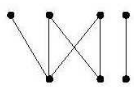
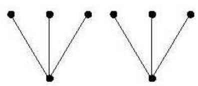
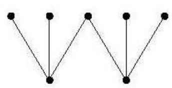
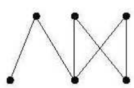
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p, \lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{17}	 A' B'	12	$13 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 12$
G_{18}	 A' B'	13	$17 \cdot 2^{p-4} + 12 \cdot 2^{q-3} + 15$
G_{27}	 A' B'	13	$17 \cdot 2^{p-4} + 12 \cdot 2^{q-3} + 9$
G_{28}	 A' B'	13	$19 \cdot 2^{p-4} + 11 \cdot 2^{q-3} + 4$
G_{29}	 A' B'	14	$9 \cdot 2^{p-3} + 11 \cdot 2^{q-3} + 6$
G_{30}	 A' B'	14	$14 \cdot 2^{p-4} + 8 \cdot 2^{q-2} + 33$
G_{31}	 A' B'	15	$18 \cdot 2^{p-4} + 17 \cdot 2^{q-3} + 15$
G_{32}	 A' B'	15	$23 \cdot 2^{p-4} + 14 \cdot 2^{q-3} + 1$

TABLE 1 (4 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$

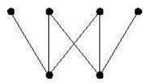
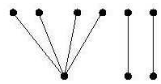
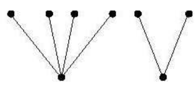
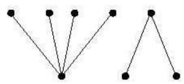
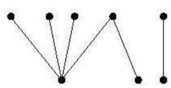
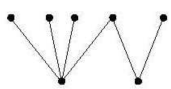
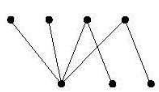
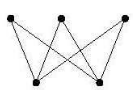
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p, \lvert B'\rvert = 4$	$\alpha'(G_i,3)$	$\alpha'(G_i,4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{33}	 A' B'	17	$24 \cdot 2^{p-4} + 19 \cdot 2^{q-3} - 1$
G_{34}	 A' B'	17	$33 \cdot 2^{p-5} + 11 \cdot 2^{q-3} + 9$
G_{35}	 A' B'	18	$37 \cdot 2^{p-5} + 12 \cdot 2^{q-2} + 21$
G_{36}	 A' B'	18	$12 \cdot 2^{p-2} + 37 \cdot 2^{q-5} + 21$
G_{37}	 A' B'	18	$41 \cdot 2^{p-5} + 23 \cdot 2^{q-3}$
G_{38}	 A' B'	19	$45 \cdot 2^{p-5} + 25 \cdot 2^{q-3} + 3$
G_{39}	 A' B'	19	$49 \cdot 2^{p-5} + 25 \cdot 2^{q-3} + 3$
G_{40}	 A' B'	20	$49 \cdot 2^{p-4} + 17 \cdot 2^{q-2} - 22$

TABLE 1 (5 of 6): Graphs in $\mathcal{K}_2^{-6}(p,4)$, $p \geq 6$

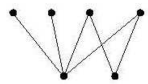
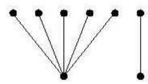
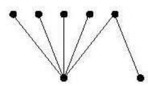
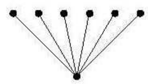
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,4} - G_i$) $\lvert A'\rvert = p, \lvert B'\rvert = 4$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4) - 6(2^{p-2} + 2^{q-2} - 2)$
G_{41}	 A' B'	21	$?? \cdot 2^{p-4} + ? \cdot 2^{q-3} + ???$
G_{42}	 A' B'	32	$?? \cdot 2^{p-5} + ?? \cdot 2^{q-3} + ???$
G_{43}	 A' B'	33	$?? \cdot 2^{p-5} + ?? \cdot 2^{q-2} + ???$
G_{44}	 A' B'	63	$?? \cdot 2^{p-2} + ?? \cdot 2^{q-5} + ???$

TABLE 1 (6 of 6): Graphs in $\mathcal{K}_2^{-6}(p, 4)$, $p \geq 6$