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Idempotent and Regular Elements in $Hyp_G(3)$

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Abstract : The concepts of an idempotent element and a regular element are important role in semigroup theory. In this paper we characterize idempotent and regular elememts of the set of all generalized hypersubstitutions of type $\tau = (3)$.

Keywords : Generalized hypersubstitution, Idempotent element, Regular element.
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1 Introduction

All idempotent elements and all regular elements of the set of all generalized hypersubstitutions of type $\tau = (2)$ were studied by W. Puninagool and S. Leeratanavalee [3], [4]. In this paper we characterize idempotent and regular elememts of the set of all generalized hypersubstitutions of type $\tau = (3)$.

A generalized hypersubstitution of type $\tau = (n_i)_{i \in I}$ is a mapping σ which maps each n_i-ary operation symbol of type τ to the set $W_\tau(X)$ of all terms of type τ built up by operation symbols from $\{f_i \mid i \in I\}$ where f_i is n_i-ary and variables from a countably infinite alphabet $X := \{x_1, x_2, x_3, \ldots\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we define at first the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

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We extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and σ_{id} is the identity element.

Proposition 1.1. ([2]) *For arbitrary terms $t, t_1, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], ..., \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, ..., t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2. ([2]) *$\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$.*

For more details on generalized hypersubstitutions see [2].

2 Idempotent elements in $Hyp_G(3)$

In this section we characterize idempotent generalized hypersubstitutions of type $\tau = (3)$. We have only one ternary operation symbol, say f. The generalized hypersubstitution σ which maps f to the term t is denoted by σ_t. For any term $t \in W_{(3)}(X)$, the set of all variables occurring in t is denoted by $var(t)$. First, we will recall the definition of an idempotent element.

Definition 2.1. *([1]) For any semigroup S, an element $e \in S$ is called* idempotent *if $ee = e$. In general, by $E(S)$ we denote the set of all idempotent elements of S.*

Proposition 2.2. *An element $\sigma_t \in Hyp_G(3)$ is idempotent if and only if $\hat{\sigma}_t[t] = t$.*

Proof. Assume that σ_t is idempotent, i.e. $\sigma_t^2 = \sigma_t$. Then

$$\hat{\sigma}_t[t] = \hat{\sigma}_t[\sigma_t(f)] = \sigma_t^2(f) = \sigma_t(f) = t.$$

Conversely, let $\hat{\sigma}_t[t] = t$. We have $(\sigma_t \circ_G \sigma_t)(f) = \hat{\sigma}_t[\sigma_t(f)] = \hat{\sigma}_t[t] = t = \sigma_t(f)$. Thus $\sigma_t^2 = \sigma_t$, i.e. σ_t is idempotent. ■

Proposition 2.3. *For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.*

Proof. Since for every $x_i \in X$, $\hat{\sigma}_{x_i}[x_i] = x_i$. By Proposition 2.2 we have σ_{x_i} is idempotent. σ_{id} is idempotent because it is a neutral element. ■

Note that for any $t \in W_{(3)}(X)\setminus X$ and $x_1, x_2, x_3 \notin var(t), \sigma_t$ is idempotent. Because there has nothing to substitute in the term $\hat{\sigma}_t[t]$. Thus $\hat{\sigma}_t[t] = t$.

Theorem 2.4. *Let $\tau = (3)$ be a type with a ternary operation symbol f. Let $t = f(t_1, t_2, t_3) \in W_{(3)}(X)$ and $var(t) \cap X_3 \neq \emptyset$. Then σ_t is idempotent if and only $t_i = x_i$ for all $x_i \in var(t) \cap X_3$.*

Proof. Assume that σ_t is idempotent. Then

$$\begin{aligned} S^3(f(t_1,t_2,t_3), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3]) &= \sigma^2_{f(t_1,t_2,t_3)}(f) \\ &= \sigma_{f(t_1,t_2,t_3)}(f) \\ &= f(t_1,t_2,t_3). \end{aligned}$$

Suppose that there exists $x_i \in var(t) \cap X_3$ such that $t_i \neq x_i$. If $t_i \in X$, then $\hat{\sigma}_{f(t_1,t_2,t_3)}[t_i] = t_i \neq x_i$. So

$$S^3(f(t_1,t_2,t_3), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3]) \neq f(t_1,t_2,t_3)$$

and it is a contradiction. If $t_i \notin X$, then $\hat{\sigma}_{f(t_1,t_2,t_3)}[t_i] \notin X$. We obtain

$$op(t) = op(S^3(f(t_1,t_2,t_3), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3])) > op(t)$$

where $op(t)$ denotes the number of all operation symbols occurring in t. This is a contradiction. For the converse direction, consider

$$\begin{aligned} \hat{\sigma}_t[t] &= \hat{\sigma}_{f(t_1,t_2,t_3)}[f(t_1,t_2,t_3)] \\ &= S^3(\sigma_{f(t_1,t_2,t_3)}(f), \hat{\sigma}_{f(t_1,t_2,t_3)}[t_1], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_2], \hat{\sigma}_{f(t_1,t_2,t_3)}[t_3]). \end{aligned}$$

Since $var(t) \cap X_3 \neq \emptyset$ and $t_i = x_i$ for all $x_i \in var(t) \cap X_3$. Then after substitution in the term t we get the term t again. Thus σ_t is idempotent. ■

Let $i, j, k \in \mathbb{N}$. For convenience, we denote:

$E_0 := \{\sigma_t \mid t \in X\} \cup \{\sigma_t \mid t \in W_{(3)}(X) \setminus X \text{ and } x_1, x_2, x_3 \notin var(t)\}$,

$E_1 := \{\sigma_{f(x_1,x_2,x_2)}, \sigma_{f(x_i,x_3,x_3)}, \sigma_{f(x_3,x_j,x_3)}, \sigma_{f(x_2,x_2,x_k)} \mid i \neq 2, j, k \neq 1\}$,

$E_2 := \{\sigma_{f(x_1,x_j,x_k)} \mid j \neq 3, k \neq 2\}$,

$E_3 := \{\sigma_{f(x_i,x_2,x_k)} \mid i > 3, k \neq 1\}$,

$E_4 := \{\sigma_{f(x_i,x_j,x_k)} \mid i, j > 3, k \geq 3\}$,

$E_5 := \{\sigma_{f(x_1,x_j,t)} \mid j \notin \{2,3\}, t \notin X \text{ and } x_2, x_3 \notin var(t)\} \cup \{\sigma_{f(x_1,x_2,t)} \mid t \notin X \text{ and } x_3 \notin var(t)\} \cup \{\sigma_{f(x_i,x_2,t)} \mid i \notin \{1,3\}, t \notin X \text{ and } x_1, x_3 \notin var(t)\}$,

$E_6 := \{\sigma_{f(x_1,t,x_k)} \mid t \notin X, x_2, x_3 \notin var(t) \text{ and } k \notin \{2,3\}\} \cup \{\sigma_{f(x_1,t,x_3)} \mid t \notin X \text{ and } x_2 \notin var(t)\} \cup \{\sigma_{f(x_i,t,x_3)} \mid i \notin \{1,2\}, t \notin X \text{ and } x_1, x_2 \notin var(t)\}$,

$E_7 := \{\sigma_{f(t,x_2,x_k)} \mid t \notin X, x_1, x_3 \notin var(t) \text{ and } k \notin \{1,3\}\} \cup \{\sigma_{f(t,x_2,x_3)} \mid t \notin X \text{ and } x_1 \notin var(t)\} \cup \{\sigma_{f(t,x_j,x_3)} \mid t \notin X, x_1, x_2 \notin var(t) \text{ and } j \notin \{1,2\}\}$,

$E_8 := \{\sigma_{f(x_1,t_1,t_2)} \mid t_1, t_2 \notin X \text{ and } x_2, x_3 \notin var(t_1) \cup var(t_2)\} \cup \{\sigma_{f(t_1,x_2,t_2)} \mid t_1, t_2 \notin X \text{ and } x_1, x_3 \notin var(t_1) \cup var(t_2)\} \cup \{\sigma_{f(t_1,t_2,x_3)} \mid t_1, t_2 \notin X \text{ and } x_1, x_2 \notin var(t_1) \cup var(t_2)\}$.

By Theorem 2.4, we have

Corollary 2.5. $E(Hyp_G(3)) = E_0 \cup E_1 \cup E_2 \cup \ldots \cup E_8$.

3 The Regular Elements in $Hyp_G(3)$

In this section we will determine all regular elements of $Hyp_G(3)$. At first we want to recall the definition of a regular element.

Definition 3.1. *An element a of a semigroup S is called* regular *if there exists $x \in S$ such that $axa = a$. The semigroup S is called* regular *if all its elements are regular.*

It is clear that for all σ_{x_i} where $i \in \mathbb{N}$ and $x_i \in X$ is regular and σ_{id} is also regular. If $var(t) \cap X_3 = \emptyset$ where $X_3 = \{x_1, x_2, x_3\}$, then $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ where $\sigma_s \in W_{(3)}(X)$ and thus σ_t is regular. Then we consider only the case $var(t) \cap X_3 \neq \emptyset$.

Proposition 3.2. *Let $t = f(t_1, t_2, t_3)$, $s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_1\}$.*
If $t_j = x_1$ and $s_1 = x_j$ where $j \in \{1, 2, 3\}$, then σ_t is regular. Otherwise σ_t is not regular.

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$\begin{aligned}
\hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(x_j, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_1 = x_j \\
&= f(\hat{\sigma}_s[t_j], t_4, t_5) \quad \text{where } t_4, t_5 \in W_{(3)}(X) \\
&= f(x_1, t_4, t_5) \quad \text{since } t_j = x_1.
\end{aligned}$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(x_1, t_4, t_5)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1\}$, so $x_1 \in var(t)$ is substituted by the term x_1 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_j \neq x_1$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1\}$. So $u_1 = x_1$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_1 = x_j$. Hence $s_1 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_1] = x_1$ which contradicts to $u_1 = x_1$. Therefore σ_t is not regular. ■

Proposition 3.3. *Let $t = f(t_1, t_2, t_3)$, $s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_2\}$.*
If $t_j = x_2$ and $s_2 = x_j$ where $j \in \{1, 2, 3\}$, then σ_t is regular. Otherwise σ_t is not regular.

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$\begin{aligned}
\hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, x_j, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_2 = x_j \\
&= f(t_4, \hat{\sigma}_s[t_j], t_5) \quad \text{where } t_4, t_5 \in W_{(3)}(X) \\
&= f(t_4, x_2, t_5) \quad \text{since } t_j = x_2.
\end{aligned}$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(t_4, x_2, t_5)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_2\}$, so $x_2 \in var(t)$ is substituted by the term x_2 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_j \neq x_2$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_2\}$. So $u_2 = x_2$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_2 = x_j$. Hence $s_2 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_2] = x_2$ which contradicts to $u_2 = x_2$. Therefore σ_t is not regular.

■

Proposition 3.4. *Let $t = f(t_1, t_2, t_3), s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_3\}$.*
If $t_j = x_3$ and $s_3 = x_j$ where $j \in \{1, 2, 3\}$, then σ_t is regular. Otherwise σ_t is not regular.

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$
\begin{aligned}
\hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, s_2, x_j), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_3 = x_j \\
&= f(t_4, t_5, \hat{\sigma}_s[t_j]) \quad \text{where } t_4, t_5 \in W_{(3)}(X) \\
&= f(t_4, t_5, x_3) \quad \text{since } t_j = x_3.
\end{aligned}
$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(t_4, t_5, x_3)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_3\}$, so $x_3 \in var(t)$ is substituted by the term x_3 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_j \neq x_3$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_3\}$. So $u_3 = x_3$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_3 = x_j$. Hence $s_3 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_3] = x_3$ which contradicts to $u_3 = x_3$. Therefore σ_t is not regular.

■

Proposition 3.5. *Let $t = f(t_1, t_2, t_3)$, $s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_1, x_2\}$.*
If $t_j = x_1$, $t_k = x_2$ and $s_1 = x_j$, $s_2 = x_k$ where $j \neq k$ and $j, k \in \{1, 2, 3\}$, then σ_t is regular. Otherwise σ_t is not regular.

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$
\begin{aligned}
\hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(x_j, x_k, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_1 = x_j, s_2 = x_k \\
&= f(\hat{\sigma}_s[t_j], \hat{\sigma}_s[t_k], t_4) \quad \text{where } t_4 \in W_{(3)}(X) \\
&= f(x_1, x_2, t_4) \quad \text{since } t_j = x_1, t_k = x_2.
\end{aligned}
$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(x_1, x_2, t_4)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1, x_2\}$, so $x_1 \in var(t)$ is substituted by the term x_1, $x_2 \in var(t)$ is substituted by the term x_2 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_j \neq x_1$ or $t_k \neq x_2$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1, x_2\}$. So $u_1 = x_1$ and $u_2 = x_2$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_1 = x_j$, $s_2 = x_k$. Hence $s_1 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_1] = x_1$ which contradicts to $u_1 = x_1$ or $s_2 \in var(s)$ is substituted by the term $\hat{\sigma}[t_k] \neq \hat{\sigma}[x_2] = x_2$ which contradicts to $u_2 = x_2$. Therefore σ_t is not regular.

■

Proposition 3.6. *Let $t = f(t_1, t_2, t_3)$, $s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_1, x_3\}$.*
If $t_j = x_1$, $t_k = x_3$ and $s_1 = x_j$, $s_3 = x_k$ where $j \neq k$ and $j, k \in \{1, 2, 3\}$, then σ_t is regular. Otherwise σ_t is not regular.

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$\begin{aligned}
\hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\
&= S^3(f(x_j, s_2, x_k), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_1 = x_j, s_3 = x_k \\
&= f(\hat{\sigma}_s[t_j], t_4, \hat{\sigma}_s[t_k]) \quad \text{where } t_4 \in W_{(3)}(X) \\
&= f(x_1, t_4, x_3) \quad \text{since } t_j = x_1, t_k = x_3.
\end{aligned}$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(x_1, t_4, x_3)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1, x_3\}$, so $x_1 \in var(t)$ is substituted by the term x_1, $x_3 \in var(t)$ is substituted by the term x_3 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_j \neq x_1$ or $t_k \neq x_3$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1, x_3\}$. So $u_1 = x_1$ and $u_3 = x_3$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_1 = x_j$, $s_3 = x_k$. Hence $s_1 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_1] = x_1$ which contradicts to $u_1 = x_1$ or $s_3 \in var(s)$ is substituted by the term $\hat{\sigma}[t_k] \neq \hat{\sigma}[x_3] = x_3$ which contradicts to $u_3 = x_3$. Therefore σ_t is not regular.

■

Proposition 3.7. *Let $t = f(t_1, t_2, t_3)$, $s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_2, x_3\}$.*
If $t_j = x_2$, $t_k = x_3$ and $s_2 = x_j$, $s_3 = x_k$ where $j \neq k$ and $j, k \in \{1, 2, 3\}$, then σ_t is regular. Otherwise σ_t is not regular.

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$\begin{aligned} \hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\ &= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\ &= S^3(f(s_1, x_j, x_k), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_2 = x_j, s_3 = x_k \\ &= f(t_4, \hat{\sigma}_s[t_j], \hat{\sigma}_s[t_k]) \quad \text{where } t_4 \in W_{(3)}(X) \\ &= f(t_4, x_2, x_3) \quad \text{since } t_j = x_2, t_k = x_3. \end{aligned}$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(t_4, x_2, x_3)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_2, x_3\}$, so $x_2 \in var(t)$ is substituted by the term x_2, $x_3 \in var(t)$ is substituted by the term x_3 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_j \neq x_2$ or $t_k \neq x_3$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_2, x_3\}$. So $u_2 = x_2$ and $u_3 = x_3$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_2 = x_j$, $s_3 = x_k$. Hence $s_2 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_2] = x_2$ which contradicts to $u_2 = x_2$ or $s_3 \in var(s)$ is substituted by the term $\hat{\sigma}[t_k] \neq \hat{\sigma}[x_3] = x_3$ which contradicts to $u_3 = x_3$. Therefore σ_t is not regular.

■

Proposition 3.8. *Let $t = f(t_1, t_2, t_3)$, $s = f(s_1, s_2, s_3)$ and $\emptyset \neq var(t) \cap X_3 = \{x_1, x_2, x_3\}$. If $t_i = x_1$, $t_j = x_2$, $t_k = x_3$ and $s_1 = x_i$, $s_2 = x_j$, $s_3 = x_k$ where $i, j, k \in \{1, 2, 3\}$ and all are distinct, then σ_t is regular. Otherwise σ_t is not regular.*

Proof. Consider $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]]$. Since

$$\begin{aligned} \hat{\sigma}_s[t] &= S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\ &= S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \\ &= S^3(f(x_i, x_j, x_k), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) \quad \text{since } s_1 = x_i, s_2 = x_j, s_3 = x_k \\ &= f(\hat{\sigma}_s[t_i], \hat{\sigma}_s[t_j], \hat{\sigma}_s[t_k]) \\ &= f(x_1, x_2, x_3) \quad \text{since } t_i = x_1, t_j = x_2, t_k = x_3. \end{aligned}$$

Next, we consider $\hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[f(x_1, x_2, x_3)]$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1, x_2, x_3\}$, so $x_1 \in var(t)$ is substituted by the term x_1, $x_2 \in var(t)$ is substituted by the term x_2, $x_3 \in var(t)$ is substituted by the term x_3 and $x_m \in var(t)$ is untouched. Hence $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. Therefore σ_t is regular.

Now, let $t_i \neq x_1$ or $t_j \neq x_2$ or $t_k \neq x_3$. Suppose that σ_t is regular, thus there exists $\sigma_s \in Hyp_G(3)$ such that $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. We let $\hat{\sigma}_s[t] = f(u_1, u_2, u_3)$. Since $\emptyset \neq var(t) \cap X_3 = \{x_1, x_2, x_3\}$. So $u_1 = x_1$, $u_2 = x_2$ and $u_3 = x_3$. But since $f(u_1, u_2, u_3) = S^3(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3]) = S^3(f(s_1, s_2, s_3), \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], \hat{\sigma}_s[t_3])$ and $s_1 = x_i$, $s_2 = x_j$, $s_3 = x_k$. Hence $s_1 \in var(s)$ is substituted by the term $\hat{\sigma}[t_i] \neq \hat{\sigma}[x_1] = x_1$ which contradicts to $u_1 = x_1$ or $s_2 \in var(s)$ is substituted by the term $\hat{\sigma}[t_j] \neq \hat{\sigma}[x_2] = x_2$ which contradicts to $u_2 = x_2$ or $s_3 \in var(s)$ is substituted by the term $\hat{\sigma}[t_k] \neq \hat{\sigma}[x_3] = x_3$ which contradicts to $u_3 = x_3$. Therefore σ_t is not regular.

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