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On the Spaces of λ-Convergent and Bounded Sequences

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Abstract : In the present paper, we introduce the notion of λ-convergent and bounded sequences. Further, we define some related BK spaces and construct their bases. Moreover, we establish some inclusion relations concerning with those spaces and determine their α-, β- and γ-duals. Finally, we characterize some related matrix classes.

Keywords : Sequence spaces; BK spaces; Schauder basis; α-, β- and γ-duals; Matrix mappings.
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1 Introduction

By w, we denote the space of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. Also, we shall use the conventions that $e = (1, 1, \ldots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the n^{th} place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Any vector subspace of w is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Further, by ℓ_p $(1 \le p < \infty)$, we denote the sequence space of all p-absolutely convergent series, that is $\ell_p = \{x = (x_k) \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \le p < \infty$. Moreover, we write bs, cs and cs_0 for the sequence spaces of all bounded, convergent and null series, respectively.

A sequence space X is called an FK space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A normed FK space is called a BK space, that is, a BK space is a Banach sequence space with continuous coordinates.

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The sequence spaces ℓ_∞, c and c_0 are BK spaces with the usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k \in \mathbb{N}$. Also, the space ℓ_p is a BK space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, where $1 \leq p < \infty$.

A sequence $(b_n)_{n=0}^{\infty}$ in a normed space X is called a *Schauder basis* for X if for every $x \in X$ there is a unique sequence $(\alpha_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \alpha_n b_n$, i.e., $\lim_{m\to\infty} \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$.

The α-, β- and γ-*duals* of a sequence space X are respectively defined by

$$X^\alpha = \big\{a = (a_k) \in w : \ ax = (a_k x_k) \in \ell_1 \ \textit{for all} \ x = (x_k) \in X\big\},$$

$$X^\beta = \big\{a = (a_k) \in w : \ ax = (a_k x_k) \in cs \ \textit{for all} \ x = (x_k) \in X\big\}$$

and

$$X^\gamma = \big\{a = (a_k) \in w : \ ax = (a_k x_k) \in bs \ \textit{for all} \ x = (x_k) \in X\big\}.$$

If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the n^{th} row of A, that is $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. Further, if $x = (x_k) \in w$ then we define the A-*transform* of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \qquad (n \in \mathbb{N}) \tag{1.1}$$

provided the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$. Furthermore, the sequence x is said to be A-*summable* to $a \in \mathbb{C}$ if Ax converges to a which is called the A-*limit* of x.

In addition, let X and Y be sequence spaces. Then, we say that A defines a *matrix mapping* from X into Y if for every sequence $x \in X$ the A-transform of x exists and is in Y. Moreover, we write (X, Y) for the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

For an arbitrary sequence space X, the *matrix domain* of an infinite matrix A in X is defined by

$$X_A = \{x \in w : \ Ax \in X\} \tag{1.2}$$

which is a sequence space.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors, see for instance [1, 2, 3, 4, 8, 10, 12, 13] and [14].

In this paper, we introduce the notion of λ-convergent and bounded sequences. Further, we define some related BK spaces and construct their bases. Moreover, we establish some inclusion relations concerning with those spaces and determine their α-, β- and γ-duals. Finally, we characterize some related matrix classes.

2 Notion of λ-convergent and bounded sequences

Throughout this paper, let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \cdots \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty. \tag{2.1}$$

We say that a sequence $x = (x_k) \in w$ is λ-*convergent* to the number $l \in \mathbb{C}$, called as the λ-*limit* of x, if $\Lambda_n(x) \to l$ as $n \to \infty$, where

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k; \qquad (n \in \mathbb{N}). \tag{2.2}$$

In particular, we say that x is a λ-*null sequence* if $\Lambda_n(x) \to 0$ as $n \to \infty$. Further, we say that x is λ-*bounded* if $\sup_n |\Lambda_n(x)| < \infty$.

Here and in the sequel, we shall use the convention that any term with a negative subscript is equal to naught, e.g. $\lambda_{-1} = 0$ and $x_{-1} = 0$.

Now, it is well known [11] that if $\lim_{n\to\infty} x_n = a$ in the ordinary sense of convergence, then

$$\lim_{n\to\infty}\left(\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k - a|\right) = 0.$$

This implies that

$$\lim_{n\to\infty}|\Lambda_n(x) - a| = \lim_{n\to\infty}\left|\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(x_k - a)\right| = 0$$

which yields that $\lim_{n\to\infty}\Lambda_n(x) = a$ and hence x is λ-convergent to a. We therefore deduce that the ordinary convergence implies the λ-convergence to the same limit. This leads us to the following basic result:

Lemma 2.1. *Every convergent sequence is λ-convergent to the same ordinary limit.*

We shall later show that the converse implication need not be true. Before that, the following result is immediate by Lemma 2.1.

Lemma 2.2. *If a λ-convergent sequence converges in the ordinary sense, then it must converge to the same λ-limit.*

Now, let $x = (x_k) \in w$ and $n \geq 1$. Then, by using (2.2), we derive that

$$\begin{aligned} x_n - \Lambda_n(x) &= \frac{1}{\lambda_n}\sum_{i=0}^{n}(\lambda_i - \lambda_{i-1})(x_n - x_i) \\ &= \frac{1}{\lambda_n}\sum_{i=0}^{n-1}(\lambda_i - \lambda_{i-1})(x_n - x_i) \\ &= \frac{1}{\lambda_n}\sum_{i=0}^{n-1}(\lambda_i - \lambda_{i-1})\sum_{k=i+1}^{n}(x_k - x_{k-1}) \\ &= \frac{1}{\lambda_n}\sum_{k=1}^{n}(x_k - x_{k-1})\sum_{i=0}^{k-1}(\lambda_i - \lambda_{i-1}) \\ &= \frac{1}{\lambda_n}\sum_{k=1}^{n}\lambda_{k-1}(x_k - x_{k-1}). \end{aligned}$$

Therefore, we have for every $x = (x_k) \in w$ that

$$x_n - \Lambda_n(x) = S_n(x); \quad (n \in \mathbb{N}), \tag{2.3}$$

where the sequence $S(x) = (S_n(x))_{n=0}^{\infty}$ is defined by

$$S_0(x) = 0 \text{ and } S_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n}\lambda_{k-1}(x_k - x_{k-1}); \quad (n \geq 1). \tag{2.4}$$

Now, the following result is obtained from Lemma 2.2 by using (2.3).

Lemma 2.3. *A λ-convergent sequence x converges in the ordinary sense if and only if $S(x) \in c_0$.*

Similarly, the following results are obvious.

Lemma 2.4. *Every bounded sequence is λ-bounded.*

Lemma 2.5. *A λ-bounded sequence x is bounded in the ordinary sense if and only if $S(x) \in \ell_\infty$.*

Now, we define the infinite matrix $\Lambda = (\lambda_{nk})_{n,k=0}^{\infty}$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n} & ; \ (0 \leq k \leq n), \\ 0 & ; \ (k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. Then, the Λ-transform of a sequence $x \in w$ is the sequence $\Lambda(x) = (\Lambda_n(x))_{n=0}^{\infty}$, where $\Lambda_n(x)$ is given by (2.2) for every $n \in \mathbb{N}$. Thus, the sequence x is λ-convergent if and only if x is Λ-summable. Further, if x is λ-convergent then the λ-limit of x is nothing but the Λ-limit of x.

Finally, it is obvious that the matrix Λ is a triangle, that is $\lambda_{nn} \neq 0$ and $\lambda_{nk} = 0$ for $k > n$ $(n = 0, 1, 2 \ldots)$. Also, it follows by Lemma 2.1 that the method Λ is regular.

Remark 2.6. We may note that if we put $q_k = \lambda_k - \lambda_{k-1}$ for all k, then the matrix Λ is the special case $Q_n \to \infty$ $(n \to \infty)$ of the matrix $\bar{N}_q$ of weighted means [8], where $Q_n = \sum_{k=0}^{n} q_k = \lambda_n$ for all n. On the other hand, the matrix Λ is reduced, in the special case $\lambda_k = k+1$ $(k \in \mathbb{N})$, to the matrix C_1 of Cesàro means [13, 14].

3 The spaces of λ-convergent and bounded sequences

In the present section, we introduce the sequence spaces ℓ_∞^λ, c^λ and c_0^λ as the sets of all λ-bounded, λ-convergent and λ-null sequences, respectively, that is

$$\ell_\infty^\lambda = \Big\{ x \in w : \ \sup_n |\Lambda_n(x)| < \infty \Big\},$$

$$c^\lambda = \Big\{ x \in w : \ \lim_{n\to\infty} \Lambda_n(x) \ \text{exists} \Big\}$$

and

$$c_0^\lambda = \Big\{ x \in w : \ \lim_{n\to\infty} \Lambda_n(x) = 0 \Big\}.$$

With the notation of (1.2), we can redefine the spaces ℓ_∞^λ, c^λ and c_0^λ as the matrix domains of the triangle Λ in the spaces ℓ_∞, c and c_0, respectively, that is

$$\ell_\infty^\lambda = (\ell_\infty)_\Lambda, \ c^\lambda = c_\Lambda \ \text{ and } \ c_0^\lambda = (c_0)_\Lambda. \tag{3.1}$$

Now, we may begin with the following result which is essential in the text.

Theorem 3.1. *The sequence spaces ℓ_∞^λ, c^λ and c_0^λ are BK spaces with the same norm given by*

$$\|x\|_{\ell_\infty^\lambda} = \|\Lambda(x)\|_{\ell_\infty} = \sup_n |\Lambda_n(x)|. \tag{3.2}$$

Proof. This result follows from [5, Lemma 2.1] by using (3.1). □

Remark 3.2. It can easily be seen that the absolute property does not hold on the spaces ℓ_∞^λ, c^λ and c_0^λ, that is $\|x\|_{\ell_\infty^\lambda} \neq \||x|\|_{\ell_\infty^\lambda}$ for at least one sequence x in each of these spaces, where $|x| = (|x_k|)$. Thus, the spaces ℓ_∞^λ, c^λ and c_0^λ are BK spaces of non-absolute type.

Theorem 3.3. *The sequence spaces ℓ_∞^λ, c^λ and c_0^λ are norm isomorphic to the spaces ℓ_∞, c and c_0, respectively, that is $\ell_\infty^\lambda \cong \ell_\infty$, $c^\lambda \cong c$ and $c_0^\lambda \cong c_0$.*

Proof. Let X denote any of the spaces ℓ_∞, c or c_0 and X^λ be the respective one of the spaces ℓ_∞^λ, c^λ or c_0^λ. Since the matrix Λ is a triangle, it has a unique inverse which is also a triangle [9, Proposition 1.1]. Therefore, the linear operator $L_\Lambda : X^\lambda \to X$, defined by $L_\Lambda(x) = \Lambda(x)$ for all $x \in X^\lambda$, is bijective and is norm preserving by (3.2) of Theorem 3.1. Hence $X^\lambda \cong X$. □

Finally, we conclude this section with the following consequence of Theorems 3.1 and 3.3.

Corollary 3.4. *Define the sequence $e_\lambda^{(n)} \in c_0^\lambda$ for every fixed $n \in \mathbb{N}$ by*

$$\left(e_\lambda^{(n)}\right)_k = \begin{cases} (-1)^{k-n} \dfrac{\lambda_n}{\lambda_k - \lambda_{k-1}} & ;\ (n \leq k \leq n+1), \\ \\ 0 & ;\ (\text{otherwise}) \end{cases} \qquad (k \in \mathbb{N}).$$

Then, we have

(a) *The sequence $(e_\lambda^{(0)}, e_\lambda^{(1)}, \ldots)$ is a Schauder basis for the space c_0^λ and every $x \in c_0^\lambda$ has a unique representation $x = \sum_{n=0}^{\infty} \Lambda_n(x)\, e_\lambda^{(n)}$.*

(b) *The sequence $(e, e_\lambda^{(0)}, e_\lambda^{(1)}, \ldots)$ is a Schauder basis for the space c^λ and every $x \in c^\lambda$ has a unique representation $x = le + \sum_{n=0}^{\infty}(\Lambda_n(x) - l)e_\lambda^{(n)}$, where $l = \lim_{n\to\infty} \Lambda_n(x)$.*

Proof. This result is immediate by [9, Corollary 2.3], since $\Lambda(e) = e$ and $\Lambda\big(e_\lambda^{(n)}\big) = e^{(n)}$ for all n. □

Remark 3.5. It is obvious by Remark 2.6 that the spaces ℓ_∞^λ, c^λ and c_0^λ are the special case $q = \Delta\lambda$ of the spaces $(\bar{N}, q)_\infty$, $(\bar{N}, q)$ and $(\bar{N}, q)_0$ of weighted means [8], that is $\ell_\infty^\lambda = (\bar{N}, \Delta\lambda)_\infty$, $c^\lambda = (\bar{N}, \Delta\lambda)$ and $c_0^\lambda = (\bar{N}, \Delta\lambda)_0$. On the other hand, the spaces ℓ_∞^λ, c^λ and c_0^λ are reduced in the special case $\lambda_k = k+1$ $(k \in \mathbb{N})$ to the Cesàro sequence spaces X_∞, $\tilde{c}$ and $\tilde{c}_0$ of non-absolute type [13, 14], respectively.

4 Some inclusion relations

In this section, we establish some inclusion relations concerning with the spaces c_0^λ, c^λ and ℓ_∞^λ, and we may begin with the following basic result:

Theorem 4.1. *The inclusions $c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ strictly hold.*

Proof. It is clear that the inclusions $c_0^\lambda \subset c^\lambda \subset \ell_\infty^\lambda$ hold. Further, since the inclusion $c_0 \subset c$ is strict, it follows by Lemma 2.1 that the inclusion $c_0^\lambda \subset c^\lambda$ is also strict. Moreover, consider the sequence $x = (x_k)$ defined by $x_k = (-1)^k(\lambda_k + \lambda_{k-1})/(\lambda_k - \lambda_{k-1})$ for all $k \in \mathbb{N}$. Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(-1)^k(\lambda_k + \lambda_{k-1}) = (-1)^n.$$

This shows that $\Lambda(x) \in \ell_\infty \backslash c$. Thus, the sequence x is in ℓ_∞^λ but not in c^λ. Hence, the inclusion $c^\lambda \subset \ell_\infty^\lambda$ strictly holds. This completes the proof. □

Now, the following result is immediate by the regularity of the matrix Λ and by Lemma 2.3.

Lemma 4.2. *The inclusions $c_0 \subset c_0^\lambda$ and $c \subset c^\lambda$ hold. Furthermore, the equalities hold if and only if $S(x) \in c_0$ for every sequence x in the spaces c_0^λ and c^λ, respectively.*

Proof. The first part is obvious by Lemma 2.1. Thus, we turn to the second part. For this, suppose firstly that the equality $c_0^\lambda = c_0$ holds. Then, we have for every $x \in c_0^\lambda$ that $x \in c_0$ and hence $S(x) \in c_0$ by Lemma 2.3.

Conversely, let $x \in c_0^\lambda$. Then, we have by the hypothesis that $S(x) \in c_0$. Thus, it follows, by Lemma 2.3 and then Lemma 2.2, that $x \in c_0$. This shows that the inclusion $c_0^\lambda \subset c_0$ holds. Hence, by combining the inclusions $c_0^\lambda \subset c_0$ and $c_0 \subset c_0^\lambda$, we get the equality $c_0^\lambda = c_0$.

Similarly, one can show that the equality $c^\lambda = c$ holds if and only if $S(x) \in c_0$ for every $x \in c^\lambda$. This concludes the proof. □

Moreover, the following result can be proved similarly by means of Lemmas 2.4 and 2.5.

Lemma 4.3. *The inclusion $\ell_\infty \subset \ell_\infty^\lambda$ holds. Furthermore, the equality $\ell_\infty^\lambda = \ell_\infty$ holds if and only if $S(x) \in \ell_\infty$ for every $x \in \ell_\infty^\lambda$.*

Now, it is obvious by Lemma 4.2 that $c_0 \subset c_0^\lambda \cap c$. Conversely, it follows by Lemma 2.2 that $c_0^\lambda \cap c \subset c_0$. This yields the following result:

Theorem 4.4. *The equality $c_0^\lambda \cap c = c_0$ holds.*

It is worth mentioning that the equality $c^\lambda \cap \ell_\infty = c$ need not be held. For example, let $\lambda_k = k+1$ and $x_k = (-1)^k$ for all k. Then $x \in c^\lambda \cap \ell_\infty$ while $x \notin c$.

Now, let $x = (x_k) \in w$ and $n \geq 1$. Then, by bearing in mind the relations (2.3) and (2.4), we derive that

$$\begin{aligned}
S_n(x) &= \frac{1}{\lambda_n}\sum_{k=1}^{n}\lambda_{k-1}(x_k - x_{k-1}) \\
&= \frac{1}{\lambda_n}\Big[\sum_{k=1}^{n}\lambda_{k-1}x_k - \sum_{k=1}^{n}\lambda_{k-1}x_{k-1}\Big] \\
&= \frac{1}{\lambda_n}\Big[\sum_{k=0}^{n}\lambda_{k-1}x_k - \sum_{k=0}^{n-1}\lambda_k x_k\Big] \\
&= \frac{1}{\lambda_n}\Big[\lambda_{n-1}x_n - \sum_{k=0}^{n-1}(\lambda_k - \lambda_{k-1})x_k\Big] \\
&= \frac{\lambda_{n-1}}{\lambda_n}\big[x_n - \Lambda_{n-1}(x)\big] \\
&= \frac{\lambda_{n-1}}{\lambda_n}\big[S_n(x) + \Lambda_n(x) - \Lambda_{n-1}(x)\big].
\end{aligned}$$

Hence, we have for every $x \in w$ that

$$S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\big[\Lambda_n(x) - \Lambda_{n-1}(x)\big]; \qquad (n \in \mathbb{N}). \tag{4.1}$$

On the other hand, by taking into account the definition of the sequence λ given by (2.1), we have $\lambda_{k+1}/\lambda_k > 1$ for all $k \in \mathbb{N}$. Thus, there are only two distinct cases of the sequence λ, either $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 1$ or $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k = 1$. Obviously, the first case holds if and only if $\liminf_{k\to\infty}(\lambda_{k+1}-\lambda_k)/\lambda_{k+1} > 0$ which is equivalent to say that the sequence $(\lambda_k/(\lambda_k - \lambda_{k-1}))_{k=0}^{\infty}$ is a bounded sequence. Similarly, the second case holds if and only if the above sequence is unbounded. Therefore, we have the following lemma:

Lemma 4.5. *For any sequence $\lambda = (\lambda_k)_{k=0}^{\infty}$ satisfying* (2.1), *we have*

(a) $\left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=0}^{\infty} \notin \ell_\infty$ *if and only if* $\liminf\limits_{k\to\infty} \dfrac{\lambda_{k+1}}{\lambda_k} = 1$.

(b) $\left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)_{k=0}^{\infty} \in \ell_\infty$ *if and only if* $\liminf\limits_{k\to\infty} \dfrac{\lambda_{k+1}}{\lambda_k} > 1$.

It is clear that Lemma 4.5 still holds if the sequence $(\lambda_k/(\lambda_k - \lambda_{k-1}))_{k=0}^{\infty}$ is replaced by $(\lambda_k/(\lambda_{k+1} - \lambda_k))_{k=0}^{\infty}$.

Now, we are going to prove the following result which gives the necessary and sufficient condition for the matrix Λ to be stronger than convergence and boundedness both, i.e., for the inclusions $c_0 \subset c_0^\lambda$, $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ to be strict.

Theorem 4.6. *The inclusions $c_0 \subset c_0^\lambda$, $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ strictly hold if and only if* $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$.

Proof. Suppose that the inclusion $\ell_\infty \subset \ell_\infty^\lambda$ is strict. Then, Lemma 4.3 implies the existence of a sequence $x \in \ell_\infty^\lambda$ such that $S(x) = (S_n(x))_{n=0}^{\infty} \notin \ell_\infty$. Since $x \in \ell_\infty^\lambda$, we have $\Lambda(x) = (\Lambda_n(x))_{n=0}^{\infty} \in \ell_\infty$ and hence $(\Lambda_n(x) - \Lambda_{n-1}(x))_{n=0}^{\infty} \in \ell_\infty$. Therefore, we deduce from (4.1) that $(\lambda_{n-1}/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \notin \ell_\infty$ and hence $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \notin \ell_\infty$. This leads us with Lemma 4.5 (a) to the consequence that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. Similarly, by using Lemma 4.2 instead of Lemma 4.3, it can be shown that if the inclusions $c_0 \subset c_0^\lambda$ and $c \subset c^\lambda$ are strict, then $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. This proves the necessity of the condition.

To prove the sufficiency, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. Then, we have by Lemma 4.5 (a) that $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \notin \ell_\infty$. Let us now define the sequence $x = (x_k)$ by $x_k = (-1)^k \lambda_k/(\lambda_k - \lambda_{k-1})$ for all k. Then, we have for every $n \in \mathbb{N}$ that

$$|\Lambda_n(x)| = \frac{1}{\lambda_n}\left|\sum_{k=0}^{n}(-1)^k \lambda_k\right| \le \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1}) = 1$$

which shows that $\Lambda(x) \in \ell_\infty$. Thus, the sequence x is in ℓ_∞^λ but not in ℓ_∞. Therefore, by combining this with the fact that the inclusion $\ell_\infty \subset \ell_\infty^\lambda$ always holds by Lemma 4.3, we conclude that this inclusion is strict.

Similarly, if $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k = 1$ then we deduce from Lemma 4.5 (a) that $\liminf_{k\to\infty}(\lambda_k - \lambda_{k-1})/\lambda_k = 0$. Thus, there is a subsequence $(\lambda_{k_r})_{r=0}^{\infty}$ of the sequence $\lambda = (\lambda_k)_{k=0}^{\infty}$ such that

$$\lim_{r\to\infty}\left(\frac{\lambda_{k_r} - \lambda_{k_r-1}}{\lambda_{k_r}}\right) = 0. \tag{4.2}$$

Obviously, our subsequence can be chosen such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$.

Now, let us define the sequence $y = (y_k)_{k=0}^{\infty}$ by

$$y_k = \begin{cases} 1 & ;\ (k = k_r), \\ -\left(\dfrac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}}\right) & ;\ (k = k_r + 1), \qquad (r \in \mathbb{N}) \\ 0 & ;\ (\text{otherwise}) \end{cases} \tag{4.3}$$

for all $k \in \mathbb{N}$. Then $y \notin c$. On the other hand, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n} & ;\ (n = k_r), \\ & \qquad\qquad (r \in \mathbb{N}). \\ 0 & ;\ (n \neq k_r) \end{cases}$$

This and (4.2) together imply that $\Lambda(y) \in c_0$ and hence $y \in c_0^{\lambda}$. Therefore, the sequence y is in the both spaces c_0^{λ} and c^{λ} but not in any one of the spaces c_0 or c. Hence, by combining this with Lemma 4.2, we deduce that the inclusions $c_0 \subset c_0^{\lambda}$ and $c \subset c^{\lambda}$ are strict. This concludes the proof. □

Now, as a consequence of Theorem 4.6, we have the following result which gives the necessary and sufficient condition for the matrix Λ to be equivalent to convergence and boundedness both.

Corollary 4.7. *The equalities $c_0^{\lambda} = c_0$, $c^{\lambda} = c$ and $\ell_{\infty}^{\lambda} = \ell_{\infty}$ hold if and only if $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.*

Proof. The necessity is immediate by Theorem 4.6. For, if the equalities hold then the inclusions in Theorem 4.6 cannot be strict and hence $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n \neq 1$ which implies that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$.

Conversely, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$. Then, it follows by part (b) of Lemma 4.5 that $(\lambda_n/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \in \ell_{\infty}$ and hence $(\lambda_{n-1}/(\lambda_n - \lambda_{n-1}))_{n=0}^{\infty} \in \ell_{\infty}$.

Now, let $x \in c^{\lambda}$ be given. Then, we have $\Lambda(x) = (\Lambda_n(x))_{n=0}^{\infty} \in c$ and hence $(\Lambda_n(x) - \Lambda_{n-1}(x))_{n=0}^{\infty} \in c_0$. Thus, we obtain by (4.1) that $(S_n(x))_{n=0}^{\infty} \in c_0$. This shows that $S(x) \in c_0$ for every $x \in c^{\lambda}$ and hence for every $x \in c_0^{\lambda}$. Consequently, we deduce by Lemma 4.2 that the equalities $c_0^{\lambda} = c_0$ and $c^{\lambda} = c$ hold. Similarly, by using Lemma 4.3 instead of Lemma 4.2, one can shows that if

$\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$, then the equality $\ell_\infty^\lambda = \ell_\infty$ holds. This completes the proof. □

Finally, we conclude this section with the following results concerning with the spaces c_0^λ and c^λ.

Lemma 4.8. *The following statements are true:*

(a) *Although the spaces c_0^λ and c overlap, the space c_0^λ does not include the space c.*

(b) *Although the spaces c^λ and ℓ_∞ overlap, the space c^λ does not include the space ℓ_∞.*

Proof. Part (a) is immediate by Theorem 4.4. To prove (b), it is obvious by Lemma 4.2 that $c \subset c^\lambda \cap \ell_\infty$, that is, the spaces c^λ and ℓ_∞ overlap. Furthermore, due to the Steinhaus Theorem [6] (essentially saying that any regular matrix cannot sum all bounded sequences), the regularity of the matrix Λ implies the existence of a sequence $x \in \ell_\infty$ which is not Λ-summable, i.e. $\Lambda(x) \notin c$. Thus, such a sequence x is in ℓ_∞ but not in c^λ. Hence, the inclusion $\ell_\infty \subset c^\lambda$ does not hold. This concludes the proof. □

Theorem 4.9. *If $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$, then the following hold:*

(a) *Neither of the spaces c_0^λ and c includes the other.*

(b) *Neither of the spaces c_0^λ and ℓ_∞ includes the other.*

(c) *Neither of the spaces c^λ and ℓ_∞ includes the other.*

Proof. For (a), it has been shown in Lemma 4.8 (a) that the inclusion $c \subset c_0^\lambda$ does not hold. Further, if $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$ then the converse inclusion is also not held. For example, the sequence y defined by (4.3), in the proof of Theorem 4.6, belongs to the set $c_0^\lambda \backslash c$. Hence, part (a) follows.

To prove (b), we deduce from Lemma 4.8 that the inclusion $\ell_\infty \subset c_0^\lambda$ does not hold. Moreover, we are going to show that the converse inclusion does not hold if $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. For this, suppose that $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n = 1$. Then, as we have seen in the proof of Theorem 4.6, there is a subsequence $(\lambda_{k_r})_{r=0}^\infty$ of the sequence $\lambda = (\lambda_k)_{k=0}^\infty$ such that (4.2) holds and $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$.

Now, let $0 < \alpha < 1$ and define the sequence $x = (x_k)_{k=0}^\infty$ by

$$x_k = \begin{cases} \left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right)^\alpha & ; \ (k = k_r), \\ -\left(\dfrac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}}\right) x_{k-1} & ; \ (k = k_r + 1), \qquad (r \in \mathbb{N}) \\ 0 & ; \ (\text{otherwise}) \end{cases}$$

for all $k \in \mathbb{N}$. Then, it follows by (4.2) that $x \notin \ell_\infty$. On the other hand, the straightforward computations yield that

$$\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})x_k=\begin{cases}(\lambda_n-\lambda_{n-1})\Big(\dfrac{\lambda_n}{\lambda_n-\lambda_{n-1}}\Big)^{\alpha} & ;\ (n=k_r),\\ \\ 0 & ;\ (n\neq k_r)\end{cases}\qquad (r\in\mathbb{N})$$

holds for every $n\in\mathbb{N}$, and hence

$$\Lambda_n(x)=\begin{cases}\Big(\dfrac{\lambda_n-\lambda_{n-1}}{\lambda_n}\Big)^{1-\alpha} & ;\ (n=k_r),\\ \\ 0 & ;\ (n\neq k_r)\end{cases}\qquad (r\in\mathbb{N}).$$

This, together with (4.2), implies that $\Lambda(x)\in c_0$. Thus, the sequence x is in c_0^λ but not in ℓ_∞. Consequently, the inclusion $c_0^\lambda\subset\ell_\infty$ fails.

Finally, part (c) is immediate by combining part (b) and Lemma 4.8 (b). □

Remark 4.10. The results of this section may extend to the spaces $Z(u,v;c_0)$, $Z(u,v;c)$ and $Z(u,v;\ell_\infty)$ of generalized weighted means [10] with some conditions on the sequences u and v.

5 The α-, β- and γ-duals of the spaces c_0^λ, c^λ and ℓ_∞^λ

In the present section, we determine the α-, β- and γ-duals of the sequence spaces c_0^λ, c^λ and ℓ_∞^λ.

Throughout, let $\mathcal{F}$ denote the collection of all nonempty and finite subsets of $\mathbb{N}=\{0,1,2,\ldots\}$. Then, the following known results [15] are fundamental for our investigation.

Lemma 5.1. *We have* $(c_0,\ell_1)=(c,\ell_1)=(\ell_\infty,\ell_1)$. *Further* $A\in(c_0,\ell_1)$ *if and only if*

$$\sup_{K\in\mathcal{F}}\left(\sum_{n=0}^{\infty}\left|\sum_{k\in K}a_{nk}\right|\right)<\infty.$$

Lemma 5.2. *We have* $(c_0,\ell_\infty)=(c,\ell_\infty)=(\ell_\infty,\ell_\infty)$. *Furthermore* $A\in(\ell_\infty,\ell_\infty)$ *if and only if*

$$\sup_{n}\left(\sum_{k=0}^{\infty}|a_{nk}|\right)<\infty.$$

Moreover, we shall assume throughout that the sequences $x=(x_k)$ and $y=(y_k)$ are connected by the relation $y=\Lambda(x)$, that is y is the Λ-transform of x.

Then, the sequence x is in any of the spaces c_0^λ, c^λ or ℓ_∞^λ if and only if y is in the respective one of the spaces c_0, c or ℓ_∞. In addition, one can easily derive that

$$x_k = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} y_j; \qquad (k \in \mathbb{N}). \tag{5.1}$$

Now, we may begin the following result which determines the α-dual of the spaces c_0^λ, c^λ and ℓ_∞^λ.

Theorem 5.3. *The α-dual of the spaces c_0^λ, c^λ and ℓ_∞^λ is the set*

$$a_1^\lambda = \left\{ a = (a_n) \in w : \sum_{n=0}^{\infty} \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} |a_n| < \infty \right\}.$$

Proof. For any fixed sequence $a = (a_n) \in w$, we define the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ by

$$b_{nk} = \begin{cases} (-1)^{n-k} \dfrac{\lambda_k}{\lambda_n - \lambda_{n-1}} a_n & ; \ (n-1 \le k \le n), \\ 0 & ; \ (k < n-1 \ \text{or} \ k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. Also, for every $x \in w$ we put $y = \Lambda(x)$. Then, it follows by (5.1) that

$$a_n x_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{\lambda_k}{\lambda_n - \lambda_{n-1}} a_n y_k = B_n(y); \qquad (n \in \mathbb{N}). \tag{5.2}$$

Thus, we observe by (5.2) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in c_0^\lambda$ if and only if $By \in \ell_1$ whenever $y \in c_0$, that is $a \in (c_0^\lambda)^\alpha$ if and only if $B \in (c_0, \ell_1)$. Therefore, it follows by Lemma 5.1, with B instead of A, that $a \in (c_0^\lambda)^\alpha$ if and only if

$$\sup_{K \in \mathcal{F}} \left(\sum_{n=0}^{\infty} \left| \sum_{k \in K} b_{nk} \right| \right) < \infty. \tag{5.3}$$

On the other hand, let $n \in \mathbb{N}$ be given. Then, we have for any $K \in \mathcal{F}$ that

$$\left| \sum_{k \in K} b_{nk} \right| = \begin{cases} 0 & ; \ (n-1 \notin K \ \text{and} \ n \notin K), \\ \dfrac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}} |a_n| & ; \ (n-1 \in K \ \text{and} \ n \notin K), \\ \dfrac{\lambda_n}{\lambda_n - \lambda_{n-1}} |a_n| & ; \ (n-1 \notin K \ \text{and} \ n \in K), \\ |a_n| & ; \ (n-1 \in K \ \text{and} \ n \in K). \end{cases}$$

Hence, we deduce that (5.3) holds if and only if

$$\sum_{n=0}^{\infty} \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} |a_n| < \infty$$

which shows that $(c_0^\lambda)^\alpha = a_1^\lambda$. Finally, we have by Lemma 5.1 that $(c_0,\ell_1) = (c,\ell_1) = (\ell_\infty,\ell_1)$. Thus, it can similarly be shown that $(c^\lambda)^\alpha = (\ell_\infty^\lambda)^\alpha = a_1^\lambda$. This completes the proof. □

Remark 5.4. Let $\mu = (\mu_n)_{n=0}^\infty$ be defined by $\mu_n = (\lambda_n - \lambda_{n-1})/\lambda_n$ for all n. Then, we have by Theorem 5.3 that $(c_0^\lambda)^\alpha = (c^\lambda)^\alpha = (\ell_\infty^\lambda)^\alpha = \ell_\mu^1$, where ℓ_μ^1 denotes the space of de Malafosse [7] which is defined as the set of all sequences $x = (x_n) \in w$ such that $x/\mu = (x_n/\mu_n) \in \ell_1$. On the other hand, we may note by Lemma 4.5 (b) that if $\liminf_{n\to\infty} \lambda_{n+1}/\lambda_n > 1$, then there is $M > 1$ such that $1 \le \lambda_n/(\lambda_n - \lambda_{n-1}) \le M$ for all n. In this special case, we obtain by Theorem 5.3 that $(c_0^\lambda)^\alpha = (c^\lambda)^\alpha = (\ell_\infty^\lambda)^\alpha = \ell_1$ which is compatible with the fact that $c_0^\lambda = c_0$, $c^\lambda = c$ and $\ell_\infty^\lambda = \ell_\infty$ by Corollary 4.7.

Now, let $x, y \in w$ be connected by the relation $y = \Lambda(x)$. Then, by using (5.1), we can easily derive that

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n-1} \bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k y_k + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n y_n; \qquad (n \in \mathbb{N}), \tag{5.4}$$

where

$$\bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big) = \frac{a_k}{\lambda_k - \lambda_{k-1}} - \frac{a_{k+1}}{\lambda_{k+1} - \lambda_k}; \qquad (k \in \mathbb{N}).$$

This leads us to the following result:

Theorem 5.5. *Define the sets a_2^λ, a_3^λ, a_4^λ and a_5^λ as follows:*

$$a_2^\lambda = \Big\{a = (a_k) \in w : \sum_{k=0}^{\infty}\Big|\bar{\Delta}\Big(\frac{a_k}{\lambda_k - \lambda_{k-1}}\Big)\lambda_k\Big| < \infty\Big\},$$

$$a_3^\lambda = \Big\{a = (a_k) \in w : \sup_k \Big|\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\Big| < \infty\Big\},$$

$$a_4^\lambda = \Big\{a = (a_k) \in w : \lim_{k\to\infty}\Big(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\Big) \text{ exists}\Big\}$$

and

$$a_5^\lambda = \Big\{a = (a_k) \in w : \lim_{k\to\infty}\Big(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k\Big) = 0\Big\}.$$

Then, we have $(c_0^\lambda)^\beta = a_2^\lambda \cap a_3^\lambda$, $(c^\lambda)^\beta = a_2^\lambda \cap a_4^\lambda$ and $(\ell_\infty^\lambda)^\beta = a_2^\lambda \cap a_5^\lambda$.

Proof. This result is an immediate consequence of [10, Theorem 2]. □

Remark 5.6. Let us consider the special case $x = y = e$ of the equality (5.4). Then, it follows by Theorem 5.5 that the inclusions $(c_0^\lambda)^\beta \subset bs$, $(c^\lambda)^\beta \subset cs$ and $(\ell_\infty^\lambda)^\beta \subset cs$ hold.

Finally, we conclude this section with the following result concerning with the γ-dual of the spaces c_0^λ, c^λ and ℓ_∞^λ.

Theorem 5.7. *The γ-dual of the spaces c_0^λ, c^λ and ℓ_∞^λ is the set $a_2^\lambda \cap a_3^\lambda$.*

Proof. This result can be obtained from Lemma 5.2 by using (5.4). □

6 Certain matrix mappings on the spaces c_0^λ, c^λ and ℓ_∞^λ

In this final section, we state some results which characterize various matrix mappings on the spaces c_0^λ, c^λ and ℓ_∞^λ and between them. The most of these results are immediate by those of Malkowsky and Rakočević [8] and some of them are the impoved versions.

For an infinite matrix $A = (a_{nk})$, we shall write for brevity that

$$\tilde{a}_{nk} = \Big(\frac{a_{nk}}{\lambda_k - \lambda_{k-1}} - \frac{a_{n,k+1}}{\lambda_{k+1} - \lambda_k}\Big)\lambda_k$$

for all $n, k \in \mathbb{N}$. Further, let $x, y \in w$ be connected by the relation $y = \Lambda(x)$. Then, we have by (5.4) that

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \tilde{a}_{nk} y_k + \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} y_m; \qquad (n, m \in \mathbb{N}). \tag{6.1}$$

In particular, let $x \in c^\lambda$ and $A_n = (a_{nk})_{k=0}^{\infty} \in (c^\lambda)^\beta$ for all $n \in \mathbb{N}$. Then, we obtain, by passing to the limits in (6.1) as $m \to \infty$ and using Theorem 5.5, that

$$\begin{aligned} \sum_{k=0}^{\infty} a_{nk} x_k &= \sum_{k=0}^{\infty} \tilde{a}_{nk} y_k + l a_n \\ &= \sum_{k=0}^{\infty} \tilde{a}_{nk} (y_k - l) + l\Big(\sum_{k=0}^{\infty} \tilde{a}_{nk} + a_n\Big) \end{aligned}$$

which can be written as follows

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk} (y_k - l) + l\Big(\sum_{k=0}^{\infty} a_{nk}\Big); \qquad (n \in \mathbb{N}), \tag{6.2}$$

where $l = \lim_{k\to\infty} y_k$ and $a_n = \lim_{k\to\infty}(\lambda_k a_{nk}/(\lambda_k - \lambda_{k-1}))$ for all $n \in \mathbb{N}$.

Now, let us consider the following conditions:

$$\sup_n \Big(\sum_{k=0}^{\infty} |\tilde{a}_{nk}|\Big) < \infty, \tag{6.3}$$

$$\Big(\frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk}\Big)_{k=0}^{\infty} \in c_0 \ \text{ for every } n \in \mathbb{N}, \tag{6.4}$$

$$\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_{nk}\right)_{k=0}^{\infty}\in c \text{ for every } n\in\mathbb{N}, \tag{6.5}$$

$$\left(\frac{\lambda_k}{\lambda_k-\lambda_{k-1}}a_{nk}\right)_{k=0}^{\infty}\in \ell_\infty \text{ for every } n\in\mathbb{N}, \tag{6.6}$$

$$\sup_n\Big|\sum_{k=0}^{\infty}a_{nk}\Big|<\infty, \tag{6.7}$$

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty}a_{nk}\Big)=a, \tag{6.8}$$

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty}a_{nk}\Big)=0, \tag{6.9}$$

$$\sum_{n=0}^{\infty}\Big|\sum_{k=0}^{\infty}a_{nk}\Big|<\infty, \tag{6.10}$$

$$\sum_{n=0}^{\infty}\Big|\sum_{k=0}^{\infty}a_{nk}\Big|^p<\infty;\qquad(1<p<\infty), \tag{6.11}$$

$$\lim_{n\to\infty}\tilde{a}_{nk}=\tilde{a}_k \text{ for every } k\in\mathbb{N}, \tag{6.12}$$

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{a}_k|\Big)=0, \tag{6.13}$$

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty}|\tilde{a}_{nk}|\Big)=0, \tag{6.14}$$

$$\lim_{n\to\infty}\tilde{a}_{nk}=0 \text{ for every } k\in\mathbb{N}, \tag{6.15}$$

$$\sup_{N\in\mathcal{F}}\Big(\sum_{k=0}^{\infty}\Big|\sum_{n\in N}\tilde{a}_{nk}\Big|\Big)<\infty, \tag{6.16}$$

$$\sup_{K\in\mathcal{F}}\Big(\sum_{n=0}^{\infty}\Big|\sum_{k\in K}\tilde{a}_{nk}\Big|^p\Big)<\infty;\qquad(1<p<\infty), \tag{6.17}$$

$$\sum_{k=0}^{\infty}|\tilde{a}_{nk}| \text{ converges for every } n\in\mathbb{N}. \tag{6.18}$$

Then, by combining Theorem 5.5 with the results of Stieglitz and Tietz [15], we immediately deduce the following results by using (6.1) and (6.2).

Theorem 6.1. *We have*

(a) $A \in (\ell_\infty^\lambda, \ell_\infty)$ *if and only if* (6.3) *and* (6.4) *hold.*

(b) $A \in (c^\lambda, \ell_\infty)$ *if and only if* (6.3), (6.5) *and* (6.7) *hold.*

(c) $A \in (c_0^\lambda, \ell_\infty)$ *if and only if* (6.3) *and* (6.6) *hold.*

Theorem 6.2. *We have*

(a) $A \in (\ell_\infty^\lambda, c)$ *if and only if* (6.3), (6.4), (6.12) *and* (6.13) *hold. Further, if* $A \in (\ell_\infty^\lambda, c)$ *then we have for every* $x \in \ell_\infty^\lambda$ *that*

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \tilde{a}_k \Lambda_k(x). \tag{6.19}$$

(b) $A \in (c^\lambda, c)$ *if and only if* (6.3), (6.5), (6.8) *and* (6.12) *hold. Further, if* $A \in (c^\lambda, c)$ *then we have for every* $x \in c^\lambda$ *that*

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \tilde{a}_k (\Lambda_k(x) - l) + la,$$

where $l = \lim_{k\to\infty} \Lambda_k(x)$.

(c) $A \in (c_0^\lambda, c)$ *if and only if* (6.3), (6.6) *and* (6.12) *hold. Furthermore, if* $A \in (c_0^\lambda, c)$ *then* (6.19) *holds for every* $x \in c_0^\lambda$.

Theorem 6.3. *We have*

(a) $A \in (\ell_\infty^\lambda, c_0)$ *if and only if* (6.4) *and* (6.14) *hold.*

(b) $A \in (c^\lambda, c_0)$ *if and only if* (6.3), (6.5), (6.9) *and* (6.15) *hold.*

(c) $A \in (c_0^\lambda, c_0)$ *if and only if* (6.3), (6.6) *and* (6.15) *hold.*

Theorem 6.4. *We have*

(a) $A \in (\ell_\infty^\lambda, \ell_1)$ *if and only if* (6.4) *and* (6.16) *hold.*

(b) $A \in (c^\lambda, \ell_1)$ *if and only if* (6.5), (6.10) *and* (6.16) *hold.*

(c) $A \in (c_0^\lambda, \ell_1)$ *if and only if* (6.6) *and* (6.16) *hold.*

Theorem 6.5. *Let* $1 < p < \infty$. *Then, we have*

(a) $A \in (\ell_\infty^\lambda, \ell_p)$ *if and only if* (6.4), (6.17) *and* (6.18) *hold.*

(b) $A \in (c^\lambda, \ell_p)$ *if and only if* (6.5), (6.11), (6.17) *and* (6.18) *hold.*

(c) $A \in (c_0^\lambda, \ell_p)$ *if and only if* (6.6), (6.17) *and* (6.18) *hold.*

Finally, we conclude our work with the following corollaries which are immediate by [8, Proposition 3.3].

Corollary 6.6. *Let $\lambda' = (\lambda'_k)$ be a strictly increasing sequence of positive reals tending to infinity, $A = (a_{nk})$ an infinite matrix and define the matrix $B = (b_{nk})$ by*

$$b_{nk} = \frac{1}{\lambda'_n}\sum_{j=0}^{n}(\lambda'_j - \lambda'_{j-1})a_{jk}; \qquad (n, k \in \mathbb{N}).$$

Then, the necessary and sufficient conditions for the matrix A to belong to any of the classes $(\ell_\infty^\lambda, \ell_\infty^{\lambda'})$, $(c^\lambda, \ell_\infty^{\lambda'})$, $(c_0^\lambda, \ell_\infty^{\lambda'})$, $(\ell_\infty^\lambda, c^{\lambda'})$, $(c^\lambda, c^{\lambda'})$, $(c_0^\lambda, c^{\lambda'})$, $(\ell_\infty^\lambda, c_0^{\lambda'})$, $(c^\lambda, c_0^{\lambda'})$ or $(c_0^\lambda, c_0^{\lambda'})$ are obtained from the respective one in Theorems 6.1, 6.2 *or* 6.3 *by replacing the entries of the matrix A by those of the matrix B.*

Corollary 6.7. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $B = (b_{nk})$ by*

$$b_{nk} = \sum_{j=0}^{n} a_{jk}; \qquad (n, k \in \mathbb{N}).$$

Then, the necessary and sufficient conditions for the matrix A to belong to any of the classes $(\ell_\infty^\lambda, bs)$, (c^λ, bs), (c_0^λ, bs), $(\ell_\infty^\lambda, cs)$, (c^λ, cs), (c_0^λ, cs), $(\ell_\infty^\lambda, cs_0)$, (c^λ, cs_0) or (c_0^λ, cs_0) are obtained from the respective one in Theorems 6.1, 6.2 *or* 6.3 *by replacing the entries of the matrix A by those of the matrix B.*

Corollary 6.8. *Let $0 < r < 1$, $A = (a_{nk})$ an infinite matrix and define the matrix $B = (b_{nk})$ by*

$$b_{nk} = \sum_{j=0}^{n} \binom{n}{j}(1 - r)^{n-j} r^j a_{jk}; \qquad (n, k \in \mathbb{N}).$$

Then, the necessary and sufficient conditions for the matrix A to belong to any of the classes $(\ell_\infty^\lambda, e_p^r)$, (c^λ, e_p^r), (c_0^λ, e_p^r), $(\ell_\infty^\lambda, e_c^r)$, (c^λ, e_c^r), (c_0^λ, e_c^r), $(\ell_\infty^\lambda, e_0^r)$, (c^λ, e_0^r) or (c_0^λ, e_0^r) are obtained from the respective ones in Theorems 6.1–6.5 *by replacing the entries of the matrix A by those of the matrix B, where $1 \le p \le \infty$ and e_0^r, e_c^r and e_p^r denote the Euler sequence spaces which have been studied by Altay and Başar* [1] *and by Altay, Başar and Mursaleen* [2, 12].

Remark 6.9. By following the same technique used in Corollaries 6.6, 6.7 and 6.8, we can deduce the characterization of matrix operators that map any of the spaces c_0^λ, c^λ and ℓ_∞^λ into the sequence spaces defined in [3, 4, 13] and [14].

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