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Quasi Almost Convergence in a Normed Space for Double Sequences

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Abstract : Let $x = (x_{i,j})$ be a double sequence. We prove that a sequence $(x_{i,j}) \in T$, (where T is the real vector space of all bounded sequences in a real normed space X) is quasi almost convergent to $s \in X$ if and only if

$$\left\|\frac{1}{pq}\sum_{i=np}^{(n+1)p-1}\sum_{j=mq}^{(m+1)q-1} x_{i,j} - s\right\|_X \to 0 \;\; as \;\; p,q \to \infty,$$

uniformly in $n, m(= 0, 1, 2, \cdots)$. In this paper we extend the results of Dimitrije Hajduković [Mathematica Moravica, 6(2002), 65-70] for double sequence spaces.

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1 Introduction

The idea of almost convergence for single sequence space was introduced by Lorentz [4] and for double sequences by Moricz and Rhoades [5]. A double sequence $x = (x_{i,j})$ of real numbers is said to be almost convergent to a limit s if

$$\lim_{p,q\to\infty}\sup_{n,m\geq 0}|\frac{1}{pq}\sum_{i=n}^{n+p-1}\sum_{j=m}^{n+q-1} x_{i,j} - s| = 0.$$

In [1] was shown the existence of the functionals of the kind of Banach limits defined on the real vector space T of all bounded sequences in a real normed space X. In [2] by these functionals was defined the almost convergence of a sequence

$(x_i) \in T$ and shown that (x_i) almost converges to $s \in X$ if

$$\left\|\frac{1}{p}\sum_{i=0}^{p-1} x_{k+1} - s\right\|_X \to 0 \quad as \quad p \to \infty, \tag{1.1}$$

uniformly in $k(= 0, 1, 2, \cdots)$.

Let us define a family of functionals q (of the kind of Banach limits) defined on the space T by

$$\begin{aligned} q(x) &= q(x_{i,j}) \\ &= \overline{\lim_{p,q\to\infty}} \left\{ \sup_{m,n} \frac{1}{pq} \left\| \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} x_{i,j} \right\|_X \right\} \quad (x_{i,j}) = x \in T, \end{aligned} \tag{1.2}$$

where the supremum is taken over all $n, m(= 0, 1, 2, \cdots)$.
The functional q clearly is real valued and it satisfies the conditions

$$q(x) \geq 0, q(ax) = |a|q(x), q(x + y) \leq q(x) + q(y) \quad (a \in R; x, y \in T);$$

that is q is a symmetric convex functional on the space T.

Lemma . Let X be a real linear space and $q : X \to \mathbb{R}$ a functional such that the following assertions are valid

$$|L(x_{ij})| \leq q(x_{i,j}) \quad ((x_{i,j}) \in T). \tag{1.3}$$

$$q(x) \geq 0, q(ax) = |a|q(x), q(x + y) \leq q(x) + q(y) \quad (a \in R; x, y \in T).$$

Then, for each $x_0 \in X$, there exists a linear functional L on T such that

$$|L(x)| \leq q(x), \quad L(x_0) = q(x_0) \ \text{ for all } \ x \in T.$$

Denoting now by Π the family of functionals satisfying the above conditions, then for each $s \in T$ we have

$$L(x_{i,j} - s) = 0 \ \text{ if } \ q(x_{i,j} - s) = 0 \ \ (\forall \ \ L \in \Pi \ \ , (x_{i,j}) \in T). \tag{1.4}$$

We define the following definition

Definition. A sequence $(x_{i,j}) \in T$ is quasi almost convergent to $s \in X$ or quasi F-summable to s is its quasi almost limit we will write

$$(Q - F) - \lim_{ij\to\infty} x_{i,j} = s$$

if

$$L(x_{i,j} - s) = 0 \ \text{ for all } \ L \in \Pi. \tag{1.5}$$

In this paper, we extend the results of Dimitrije Hajduković [3] for double sequence spaces.

2 Main Results

Theorem 1. A sequence $(x_{ij}) \in T$ quasi almost converges to $s \in X$ if and only if

$$\left\| \frac{1}{pq} \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} x_{i,j} - s \right\|_X \to 0 \quad as \quad p, q \to \infty, \tag{2.1}$$

uniformly in $n, m (= 0, 1, 2, \cdots)$.

Proof. Let $(x_{ij}) \in T$. Then $(Q - F) - \lim_{ij \to \infty} x_{ij} = s$. By (1.2), (1.4) and (1.5) we have

$$\bar{\lim_{p,q \to \infty}} \left\{ \sup_{m,n>0} \frac{1}{pq} \left\| \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} (x_{i,j} - s) \right\|_X \right\} = 0.$$

Then for any $\epsilon > 0$ there exists an integer $p_0, q_0 > 0$ such that for all $p, q > 0$ and n $(p > p_0,\ q > q_0,\ \ n = 0, 1, 2, \cdots)$, we have

$$\frac{1}{pq} \left\| \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} (x_{i,j} - s) \right\|_X < \epsilon.$$

Since $\epsilon > 0$ is arbitrary we have

$$\left\| \frac{1}{pq} \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} x_{i,j} - s \right\|_X \to 0 \quad as \quad p, q \to \infty,$$

uniformly in n, m so the condition (2.1) is necessary.

Conversely , suppose that (2.1) be true. This implies that

$$\left\{ \sup_{m,n>0} \frac{1}{pq} \left\| \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} x_{i,j} - s \right\|_X \right\} \to 0 \quad as \quad p, q \to \infty$$

or

$$q(x_{ij} - s) = \lim_{p,q \to \infty} \left\{ \sup_{m,n>0} \frac{1}{pq} \left\| \sum_{i=np}^{(n+1)p-1} \sum_{j=mq}^{(m+1)q-1} x_{i,j} - s \right\|_X \right\} = 0.$$

Hence , by (1.4), we have

$$L(x_{i,j} - s) = 0, \quad for\ all \quad L \in \Pi$$

which implies that

$$(Q - F) - \lim_{ij \to \infty} x_{i,j} = s.$$

The proof is now completed.

Remark. By definition , that a sequence (x_{ij}), $x_{ij} \in X$ is C-summable to $s \in X$ if and only if

$$\left\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{i,j} - s\right\|_X \to 0 \quad as \quad p,q \to \infty \tag{2.2}$$

Theorem 2. If a sequence $(x_{ij}) \in T$ almost converges to $s \in X$, then it quasi almost converges to s.

Proof. Suppose a sequence $(x_{ij}) \in T$ almost converges to $s \in X$. Then by the definition of almost convergence for double sequences, for any $\epsilon > 0$ there exists an integer $p_0 > 0$ and $q_0 > 0$ such that

$$\left\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{k+i,l+j} - s\right\|_X < \epsilon \qquad (k,l = 0,1,2,3,\cdots).$$

Hence for $k = np$ and $l = mq$ we have

$$\left\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}(x_{np+i,mq+j} - s)\right\|_X$$

$$= \left\|\frac{1}{pq}\sum_{i=np}^{((n+1)p-1}\sum_{j=mq}^{(m+1)q-1} x_{i,j} - s\right\|_X < \epsilon.$$

Since $\epsilon > 0$ is arbitrary we have

$$\left\|\frac{1}{pq}\sum_{i=np}^{((n+1)p-1}\sum_{j=mq}^{(m+1)q-1} x_{i,j} - s\right\|_X \to 0 \quad as \quad p,q \to \infty$$

uniformly in m, n which implies that $(x_{i,j}) \in T$ quasi almost converges to s.

Theorem 3. If a sequence $(x_{i,j}) \in T$ quasi almost converges to $s \in X$, then it is C- summable to s.

Proof. Suppose that $(x_{i,j}) \in T$ quasi almost converges to $s \in X$. Then (2.1) is true which for $n = 0$ and $m = 0$ implies (2.2), so $(x_{i,j})$ is C- summable to s.

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