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On the Spectrum and the Fine Spectrum of the Zweier Matrix as an Operator on Some Sequence Spaces

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Abstract : The main purpose of this paper is to determine the fine spectrum of the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv.

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1 Preliminaries, Background and Notation

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \Big\{y \in Y : y = Tx,\ x \in X\Big\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\emptyset\}$ be a non trivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on a subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $D(T)$ is dense in X, or that T has a closed graph

$$\Big\{(x, Tx) : x \in D(T)\Big\} \subseteq X \times X.$$

We mean by the expression T is *invertible* that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number α is the perturbed operator

$$T_\alpha = T - \alpha I,$$

defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T,X)$ consists of those $\alpha \in \mathbb{C}$ for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T,X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \mapsto T_\alpha^{-1}$.

The name resolvent is appropriate, since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y$ provided T_α^{-1} exists. More important, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. A *regular value* α of T is a complex number such that T_α^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_α and T_α^{-1}, we need some basic concepts in spectral theory which are given as follows (see [6, pages 370-371]):

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values α of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into the following three disjoint sets:

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_α^{-1} does not exist. A $\alpha \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_α^{-1} exists and unbounded and the domain of T_α^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X.

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

From Goldberg [4, pages 58-71], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1}:

(i) $R(T) = X$,

(ii) $R(T) \neq \overline{R(T)} = X$,

(iii) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

Applying Golberg's classification to T_α, we have three possibilities for T_α and T_α^{-1},

(i) T_α is surjective,

(ii) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,

(iii) $\overline{R(T_\alpha)} \neq X$

and

(i) T_α is injective and T_α^{-1} is continuous,

(ii) T_α is injective and T_α^{-1} is discontinuous,

(iii) T_α is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2 and III_3. If α is a complex number such that $T_\alpha \in I_1$ or $T_\alpha \in II_1$, then α is in the resolvent set $\rho(T, X)$ of T. The further classification gives rise to the *fine spectrum* of T. If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\alpha \in II_2\sigma(T, X)$.

By a *sequence space*, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_p, we denote the space of all p-absolutely summable sequences, where $1 \leq p < \infty$.

Let $n, k \in \mathbb{N}$ and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, and write

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad , (n \in \mathbb{N}, x \in D_{00}(A)), \tag{1.1}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum on the right side of (1.1) exists as a finite sum. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (1.1) converges in the norm of μ. We shall write

$$(\lambda : \mu) = \Big\{ A : \lambda \subseteq D_\mu(A) \Big\}$$

for the space of those matrices which send the whole of the sequence space λ into the sequence space μ in this sense.

Let us consider the Zweier matrix Z^s represented by the following band matrix,

$$Z^s = \begin{bmatrix} s & 0 & 0 & \ldots \\ 1-s & s & 0 & \ldots \\ 0 & 1-s & s & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where s is a real number such that $s \neq 0, 1$.

Now, we may give:

Lemma 1.1 *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Corollary 1.2 *$Z^s : \ell_1 \to \ell_1$ is a bounded linear operator with the norm $\|Z^s\|_{(\ell_1,\ell_1)} = |s| + |1-s|$.*

Lemma 1.3 *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(bv)$ from bv to itself if and only if*

$$\sup_l \sum_{n=0}^{\infty} \left| \sum_{k=l}^{\infty} (a_{nk} - a_{n-1,k}) \right| < \infty$$

Corollary 1.4 *$Z^s : bv \to bv$ is a bounded linear operator with the norm $\|Z^s\|_{(\ell_1,\ell_1)} = \|Z^s\|_{(bv,bv)}$.*

In [12] Wenger examined the fine spectrum of the integer power of the Cesàro operator in c and Rhoades [11] generalized this result to the weighted mean methods. The fine spectrum of the Cesàro operator on the sequence space ℓ_p has been studied by Gonzàlez [5], where $1 < p < \infty$. The spectrum of the Cesàro operator on the sequence spaces c_0, bv and $bv_0 = bv \cap c_0$ have also been investigated by Reade [10], Akhmedov and Başar [1], and Okutoyi [9, 8], respectively. The spectrum and the fine spectrum of the Rhally operators on the sequence spaces c_0 and c, under assumption that $\lim_{n\to\infty}(n+1)a_n = L \neq 0$, have been examined by Yıldırım [13]. Furthermore, Coşkun [3] has studied the spectrum and fine spectrum for p-Cesàro operator acting on the space c_0. More recently, de Malafosse [7] and Altay and Başar [2] have respectively studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k}\ , \ (r > 0).$$

In this work, our purpose is to determine the spectrum and the fine spectrum of the Zweier matrix as an operator on the sequence spaces ℓ_1 and bv.

2 The Spectrum of The Operator Z^s on the sequence spaces ℓ_1 and bv

In this section, the spectrum, the point spectrum, the continuous spectrum, the residual spectrum and the fine spectrum in according to the Goldberg's classification of the operator Z^s on the sequence spaces ℓ_1 and bv have been examined. Let us define the set D_s by

$$D_s = \left\{ \alpha \in \mathbb{C} : |\alpha - s| \leq |1 - s| \right\}$$

.

Theorem 2.1 $\sigma(Z^s, \ell_1) = D_s$.

Proof. First, we prove that $(Z^s - \alpha I)^{-1}$ exists and is in $(\ell_1 : \ell_1)$ for $\alpha \notin D_s$ and nextly show that the operator $Z^s - \alpha I$ is not invertible for $\alpha \in D_s$.

Let $\alpha \notin D_s$. Since $Z^s - \alpha I$ is triangle, $(Z^s - \alpha I)^{-1}$ exists. Solving the equation $(Z^s - \alpha I)x = y$ for x in terms of y gives the matrix of $(Z^s - \alpha I)^{-1}$. The n^{th} row turns out to be

$$\frac{(s-1)^{n-k}}{(s-\alpha)^{n-k+1}}$$

in the k^{th} place for $k \leq n$ and zero otherwise. Thus, we observe that

$$\begin{aligned} ||Z^s - \alpha I)^{-1}||_{(\ell_1,\ell_1)} &= \sup_k \sum_{n=k} \left|\frac{s-1}{s-\alpha}\right|^{n-k} \left|\frac{1}{s-\alpha}\right| \\ &= \left|\frac{1}{s-\alpha}\right| \sum_{n=0}^{\infty} \left|\frac{s-1}{s-\alpha}\right|^{n} < \infty, \end{aligned} \tag{2.2}$$

i.e., $(Z^s - \alpha I)^{-1} \in (\ell_1 : \ell_1)$.

Let $\alpha \in D_s$ and $\alpha \neq s$. Since $Z^s - \alpha I$ is triangle, $(Z^s - \alpha I)^{-1}$ exists and one can see by (2.2) that

$$||(Z^s - \alpha I)^{-1}||_{(\ell_1:\ell_1)} = \infty$$

whenever $\alpha \in D_s$, i.e., $(Z^s - \alpha I)^{-1}$ is not in $B(\ell_1)$. If $\alpha = s$, then the operator $Z^s - sI$ is represented by the matrix

$$Z^s = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1-s & 0 & 0 & \dots \\ 0 & 1-s & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $(Z^s - sI)x = \theta$ implies $x = \theta$, $Z^s - sI : \ell_1 \to \ell_1$ is injective but has not a dense range. Hence, $Z^s - sI$ is not invertible. This completes the proof. $\square$

Theorem 2.2 $\sigma_p(Z^s, \ell_1) = \emptyset$.

Proof. Suppose $Z^s x = \alpha x$ for $x \neq \theta = (0, 0, 0, \dots)$ in ℓ_1. Then by solving the system of linear equations

$$\left.\begin{array}{rcl} sx_0 &=& \alpha x_0 \\ (1-s)x_0 + sx_1 &=& \alpha x_1 \\ (1-s)x_1 + sx_2 &=& \alpha x_2 \\ &\vdots& \\ (1-s)x_k + sx_{k+1} &=& \alpha x_k \\ &\vdots& \end{array}\right\}$$

we find that if x_t is the first non–zero entry of the sequence $x = (x_n)$, then $\alpha = s$ and from the equation

$$(1-s)x_t + sx_{t+1} = \alpha x_{t+1}$$

we get $(1-s)x_t = 0$. Since $s \neq 1$ we must have $x_t = 0$, contradicting the fact that $x_t \neq 0$. This completes the proof. $\square$

If $T : \ell_1 \to \ell_1$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_1^* \to \ell_1^*$ is defined by the transpose of the matrix A. The dual space of ℓ_1 is isomorphic to ℓ_∞, the space of bounded sequences with the norm $\|x\| = \sup_k |x_k|$.

Theorem 2.3 $\sigma_p((Z^s)^*, \ell_1^*) = D_s$.

Proof. Suppose $(Z^s)^* x = \alpha x$ for $x \neq \theta \in \ell_1^* \cong \ell_\infty$. Then by solving the system of linear equations

$$\left.\begin{array}{rcl} sx_0 + (1-s)x_1 & = & \alpha x_0 \\ sx_1 + (1-s)x_2 & = & \alpha x_1 \\ sx_2 + (1-s)x_3 & = & \alpha x_2 \\ & \vdots & \\ sx_k + (1-s)x_{k+1} & = & \alpha x_k \\ & \vdots & \end{array}\right\}$$

we obtain that

$$x_n = \left(\frac{\alpha - s}{1-s}\right)^n x_0.$$

This shows that $x = (x_k) \in \ell_1^*$ if and only if $|\alpha - s| \leq |1-s|$. This completes the proof. $\square$

Now, we give the following lemmas required in the proof of next theorems.

Lemma 2.4 *[4, p. 59] T has a dense range if and only if T^* is one to one.*

Lemma 2.5 *[4, p. 60] The adjoint operator T^* of T is onto if and only if T has a bounded inverse.*

Theorem 2.6 $\sigma_r(Z^s, \ell_1) = D_s$.

Proof. We show that the operator $Z^s - \alpha I$ has an inverse and $\overline{R(Z^s - \alpha I)} \neq \ell_1$ for α satisfying $|\alpha - s| \leq |1-s|$. For $\alpha \neq s$, the operator $Z^s - \alpha I$ is triangle and has an inverse. For $\alpha = s$, the operator $Z^s - \alpha I$ is one to one hence has an inverse. But $(Z^s)^* - \alpha I$ is not one to one by Theorem 2.3. Now, Lemma 2.4 yields the fact that the range of the operator $Z^s - \alpha I$ is not dense in ℓ_1 and this step completes the proof. $\square$

Theorem 2.7 $s \in III_1\sigma(Z^s, \ell_1)$.

Proof. By Theorem 2.3 and Lemma 2.4, $Z^s - sI \in III$. On the other hand, since $\sigma_p(Z^s, \ell_1) = \emptyset$ by Theorem 2.2, $Z^s - sI$ has an inverse. Then, $Z^s - sI \in 1 \cup 2$.

To show that $Z^s - sI \in 1$, it is enough to establish by Lemma 2.5 that $(Z^s)^* - sI$ is onto. Given $y = (y_n) \in \ell_\infty$, then we must find $x = (x_n) \in \ell_\infty$ such that $((Z^s)^* - sI)x = y$. Direct calculation gives that

$$x_n = \frac{1}{1-s} y_{n-1},$$

for all $n \in \mathbb{N}$. This shows that $(Z^s)^* - sI$ is onto. $\square$

Theorem 2.8 *If $\alpha \neq s$ and $\alpha \in D_s$, then $\alpha \in III_2\sigma(Z^s, \ell_1)$.*

Proof. Since $\alpha \neq s$, the operator $Z^s - \alpha I$ is triangle, hence it has an inverse. By (2.2), the inverse of the operator $Z^s - \alpha I$ is discontinuous. Therefore, $Z^s - \alpha I \in 2$.

By Theorem 2.3, $(Z^s)^* - \alpha I$ is not one to one. By Lemma 2.4, $Z^s - \alpha I$ does not have a dense range. Therefore $Z^s - \alpha I \in III$. This completes the proof. $\square$

Theorem 2.9 $\sigma_c(Z^s, \ell_1) = \emptyset$.

Proof. Since $\sigma_p(Z^s, \ell_1) = \emptyset$ and $\sigma(Z^s, \ell_1)$ is the disjoint union of the parts $\sigma_p(Z^s, \ell_1)$, $\sigma_r(Z^s, \ell_1)$ and $\sigma_c(Z^s, \ell_1)$ we must have $\sigma_c(Z^s, \ell_1) = \emptyset$. $\square$

Now, we may investigate the spectrum of Z^s over the sequence space bv. If $T : bv \to bv$ is a bounded linear operator with matrix $A = (a_{nk})$, then $T^* : bv^* \to bv^*$ acting on $\mathbb{C} \oplus bs$ has the matrix representation of the form

$$\begin{bmatrix} \overline{\chi} & v_0 - \overline{\chi} & v_1 - \overline{\chi} & v_2 - \overline{\chi} & \dots \\ a_0 & a_{00} - a_0 & a_{10} - a_0 & a_{20} - a_0 & \dots \\ a_1 & a_{01} - a_1 & a_{11} - a_1 & a_{21} - a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix};$$

where

$$\overline{\chi} = \lim_{n\to\infty} \sum_{v=0}^{\infty} a_{nv}, \qquad a_n = \lim_{k\to\infty} a_{kn}, \qquad v_k = P_k(T(\delta)),$$

where $\delta = (1, 1, 1, \dots)$ and P_k is the k^{th} coordinate function for each $k \in \mathbb{N}$, [9]. For $Z^s : bv \to bv$, the matrix $(Z^s)^* \in B(\mathbb{C} \oplus bs)$ is the following:

$$(Z^s)^* = \begin{bmatrix} 1 & s-1 & 0 & 0 & 0 & \dots \\ 0 & r & 1-s & 0 & 0 & \dots \\ 0 & 0 & s & 1-s & 0 & \dots \\ 0 & 0 & 0 & s & 1-s & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

Theorem 2.10 $\sigma_p((Z^s)^*, bv^*) = D_s$.

Proof. Suppose $(Z^s)^*x = \alpha x$ for $x \neq \theta \in bv^* = \mathbb{C} \oplus bs$. Then by solving the system of linear equations

$$\left.\begin{array}{rcl} x_0 + (s-1)x_1 & = & \alpha x_0 \\ sx_1 + (1-s)x_2 & = & \alpha x_1 \\ sx_2 + (1-s)x_3 & = & \alpha x_2 \\ & \vdots & \\ sx_k + (1-s)x_{k+1} & = & \alpha x_k \\ & \vdots & \end{array}\right\} \tag{2.3}$$

we obtain that

$$x_n = \left(\frac{\alpha - s}{1-s}\right)^{n-1}\left(\frac{1-\alpha}{1-s}\right)x_0; \ (n \geq 1). \tag{2.4}$$

If $(\alpha - s)/1 - s = 1$, i.e. $\alpha = 1$, then since $s \neq 1$ $x_1 = 0, x_2 = 0, ..., x_n = 0, ...$; $x = (x_0, 0, 0, ...)$ is an eigenvector corresponding to $\alpha = 1$. If $(\alpha - s)/(1-s) \neq 1$, then $x \in \mathbb{C} \oplus bs$ if and only if

$$\sup_n \left|\sum_{k=0}^{n-1}\left(\frac{\alpha - s}{1-s}\right)^k \left(\frac{1-\alpha}{1-s}\right)\right| < \infty$$

which leads us to the consequence that

$$\left|\frac{1-\alpha}{1-s}\right| \sup_n \frac{1 - \left(\dfrac{\alpha - s}{1-s}\right)^n}{1 - \left(\dfrac{\alpha - s}{1-s}\right)} < \infty$$

if and only if $|\alpha - s| \leq |1 - s|$, as asserted. □

Since the proof of the results related to the spectrum and fine spectrum of Zweier matrix as an oprator on bv are similiar to that of ℓ_1, we give the results without proof by the following theorem:

Theorem 2.11 *(a)* $\sigma(Z^s, bv) = D_s$.

(b) $\sigma_p(Z^s, bv) = \emptyset$.

(c) $\sigma_r(Z^s, bv) = D_s$.

(d) $\sigma_c(Z^s, bv) = \emptyset$.

(e) $s \in III_1\sigma(Z^s, bv)$.

(f) If $\alpha \neq s$ *and* $\alpha \in \sigma_r(Z^s, bv)$*, then* $\alpha \in III_2\sigma(Z^s, bv)$.

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