Thai Journal of Mathematics
Volume 8 (2010) Number 2 : 275–292

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

The Hybrid Steepest Descent Method for Addressing Fixed Point Problems and System of Equilibrium Problems

C. Jaiboon

Abstract : In this paper, we suggest and analyze an iterative scheme based on the hybrid steepest descent method for finding a common element of the set of solutions of a system of equilibrium problems, the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problems for inverse-strongly monotone mappings in Hilbert spaces. We obtain a strong convergence theorem for the sequence generate by these processes in Hilbert spaces. The results in this paper improve and extend the corresponding results given by many others.

Keywords : Nonexpansive mapping; Variational inequality; Fixed points; System of equilibrium problems; hybrid steepest descent method
2000 Mathematics Subject Classification: 47H09, 47H10, 47J20

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. A mapping S of C into itself is called *nonexpansive* (see [11]) if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. We denote $F(S) = \{x \in C : x = Sx\}$ be the set of fixed points of S. Recall also that a self-mapping $f : C \to C$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$, $\forall x, y \in C$. In addition, let $B : C \to H$ be a nonlinear mapping. The variational inequality problem is to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(C, B)$.

This research was partially supported by Rajamangala University of Technology Rattanakosin Research and Development Institute.

Let $\{F_i,\ i = 1, 2, \ldots, N\}$ be a finite family of bifunctions from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The system of equilibrium problems for $\{F_1, F_2, \ldots, F_N\}$ is to find a common element $x \in C$ such that

$$\begin{cases} F_1(x,y) \geq 0, \ \ \forall y \in C, \\ F_2(x,y) \geq 0, \ \ \forall y \in C, \\ \ \vdots \\ F_N(x,y) \geq 0, \ \ \forall y \in C. \end{cases} \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\cap_{i=1}^{N} SEP(F_i)$, where $SEP(F_i)$ is the set of solutions of the equilibrium problem, that is,

$$F_i(x,y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

If $N = 1$, then the problem (1.2) is reduced to the equilibrium problem (EP).
If $N = 1$ and $F(x,y) = \langle Bx, y - x\rangle$, then the problem (1.2) is reduced to the variational inequality problem.

The system of equilibrium problems includes fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems and the equilibrium problem as special cases (see, for instance, [1, 2, 3]). In 1997, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

Definition 1.1. *Let $B : C \to H$ be a nonlinear mapping. Then B is called*

(1) monotone *if*

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C,$$

(2) β-strongly monotone *if there exists a constant $\beta > 0$ such that*

$$\langle Bx - By, x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C,$$

(3) η-Lipschitz continuous *if there exists a positive real number η such that*

$$\|Bx - By\| \leq \eta\|x - y\|, \quad \forall x, y \in C,$$

(4) ξ-inverse-strongly monotone *if there exists a constant $\xi > 0$ such that*

$$\langle Bx - By, x - y\rangle \geq \xi\|Bx - By\|^2, \ \ \forall x, y \in C.$$

Remark 1.2. *It is obvious that any ξ-inverse-strongly monotone mappings B are monotone and $\frac{1}{\xi}$-Lipschitz continuous.*

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is called *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the *normal cone* to C at $v \in C$, that is, $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0, \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then T is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [9].

For finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequalities for a ξ-inverse-strongly monotone, Takahashi and Toyoda [12] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n B x_n), \ \ \forall n \geq 0, \end{cases} \tag{1.4}$$

where B is a ξ-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in $(0, 2\xi)$. They showed that if $F(S) \cap VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to some $z \in F(S) \cap VI(C, B)$.

For finding an element of $VI(C, B)$, Iiduka et al. [5] introduced the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrary}, \\ x_{n+1} = P_C\big(\alpha_n x_n + (1 - \alpha_n) P_C(x_n - \lambda_n B x_n)\big), \ \ \forall n \geq 0, \end{cases} \tag{1.5}$$

where B is a ξ-inverse-strongly monotone mapping, $\{\alpha_n\}$ is a sequence in (-1, 1) and $\{\lambda_n\}$ is a sequence in $(0, 2\xi)$. They proved that if $VI(C, B)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.5) converges weakly to some $z \in VI(C, B)$.

For finding a common element of $F(S) \cap VI(C, B)$, let $S : H \to H$ be a nonexpansive mapping, Yamada [14] introduced the following iterative scheme called the hybrid steepest descent method:

$$x_{n+1} = Sx_n - \lambda_n \mu B S x_n, \ \ \forall n \geq 1, \tag{1.6}$$

where $x_1 = x \in H$, $\{\lambda_n\} \subset (0, 1)$, $B : H \to H$ be a strongly monotone and Lipschitz continuous mapping and μ is a positive real number. He proved that the sequence $\{x_n\}$ generated by (1.6) converging strongly to the unique solution of the $F(S) \cap VI(C, B)$.

In 2009, Peng and Yao [8] introduced an iterative scheme for finding a common element of the set of solutions for a system equilibrium problems, the set of solutions to the variational inequality for a monotone and Lipschitz continuous mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space and proved a strong convergence theorem.

Motivated and inspired by the work in the literature, in this paper, we introduce an iterative scheme for finding a common element of the set of solutions of the system equilibrium problems, the set of fixed points of a nonexpansive mapping and the set of solutions of variational inequality problems for inverse-strongly monotone mappings in Hilbert spaces. Furthermore, we prove that the proposed iterative scheme converges strongly to a common element of the above three sets by using the hybrid steepest descent method. Our results extend and improve the corresponding results of Iiduka et al. [5], Yamada [14], Peng and Yao [8] and many others.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For a sequence $\{x_n\}$, the notation of $x_n \rightharpoonup x$ and $x_n \to x$ means that the sequence $\{x_n\}$ converges weakly and strongly to x, respectively. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For any $x \in H$, there exists a *unique nearest point* in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is well known that P_C is a firmly nonexpansive mapping of H onto C, that is,

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall x \in H, \ y \in C \tag{2.2}$$

Lemma 2.1. [11] *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let B be a mapping of C into H. Let $x \in C$. Then for $\lambda > 0$,*

$$x \in VI(C,B) \Longleftrightarrow x = P_C(x - \lambda Bx),$$

where P_C is the metric projection of H onto C.

Lemma 2.1. [7] *Let $(C, \langle ., .\rangle)$ be an inner product space. Then for all $x, y, z \in C$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.2. [6] *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

hold for each $y \in H$ with $y \neq x$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.3. [1] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4. [3] *Assume that $F : C \times C \to R$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $J_r^F : H \to C$ as follows:*

$$J_r^F(x) = \left\{ \, z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \, \right\}$$

for all $z \in H$. Then, the following hold:

(1) J_r^F is single- valued;

(2) J_r^F is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\left\| J_r^F x - J_r^F y \right\|^2 \leq \left\langle J_r^F x - J_r^F y, x - y \right\rangle;$$

(3) $F(J_r^F) = EP(F)$; and

(4) $EP(F)$ is closed and convex.

Lemma 2.5. [10] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.6. [13] *Let $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - l_n)a_n + \varrho_n, \ \forall n \geq 1,$$

where $\{l_n\}$ is a sequence in $(0, 1)$ and $\{\varrho_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} l_n = \infty$,

(2) $\limsup_{n\to\infty}(\varrho_n \, / \, l_n) \leq 0$ or $\sum_{n=1}^{\infty} |\varrho_n| < \infty$; then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. *In a real Hilbert space H, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \text{ for all } x, y \in H.$$

3 Main Results

In this following, we establish a strong convergence theorem which solved the problem of finding a common element of the set of solutions of a system of equilibrium problems, the set of solutions of variational inequality problems and the set of fixed points of a nonexpansive mapping in Hilbert spaces.

We first prove the following lemma.

Lemma 3.1. *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H, $S : C \to C$ be a nonexpansive mapping and let $B : C \to H$ be ξ-inverse-strongly monotone. It $0 < \lambda_n \leq 2\xi$, then $S - \lambda_n BS$ is a nonexpansive mapping in H.*

Proof. For all $x, y \in C$ and $0 < \lambda_n \leq 2\xi$, we have

$$\begin{aligned} & \|(S-\lambda_n BS)x - (S-\lambda_n BS)y\|^2 \\ = & \|(Sx - Sy) - \lambda_n(BSx - BSy)\|^2 \\ = & \|Sx - Sy\|^2 - 2\lambda_n\langle Sx - Sy, BSx - BSy\rangle + \lambda_n^2\|BSx - BSy\|^2 \\ \leq & \|x-y\|^2 - 2\lambda_n\xi\|BSx - BSy\| + \lambda_n^2\|BSx - BSy\|^2 \\ = & \|x-y\|^2 + \lambda_n(\lambda_n - 2\xi)\|BSx - BSy\|^2 \\ \leq & \|x-y\|^2. \end{aligned}$$

So, $S - \alpha_n BS$ is a nonexpansive mapping of C into H.

Now we can prove that a strong convergence theorem is a real Hilbert space.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $F_i, i \in \{1, 2, 3, \ldots, N\}$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let S be a nonexpansive mapping of C into itself and let B be a ξ-inverse-strongly monotone mapping of C into H such that*

$$\Theta := F(S) \cap \left(\cap_{k=1}^N SEP(F_k)\right) \cap VI(C, B) \neq \emptyset.$$

Let f be a contraction of C into itself. Let $\{x_n\}$ and $\left\{u_n^{(k)}\right\}_{k=1}^N$ be sequences generated by

$$\begin{cases} x_1 = x \in C \text{ chosen arbitrary}, \\ F_1(u_n^{(1)}, y) + \frac{1}{r_1}\left\langle y - u_n^{(1)}, u_n^{(1)} - x_n\right\rangle \geq 0, \ \forall y \in C, \\ F_2(u_n^{(2)}, y) + \frac{1}{r_2}\left\langle y - u_n^{(2)}, u_n^{(2)} - u_n^{(1)}\right\rangle \geq 0, \ \forall y \in C, \\ \vdots \\ F_N(u_n^{(N)}, y) + \frac{1}{r_N}\left\langle y - u_n^{(N)}, u_n^{(N)} - u_n^{(N-1)}\right\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n P_C(Su_n^{(N)} - \lambda_n BSu_n^{(N)}) + \beta_n x_n + \gamma_n f(Sx_n), \end{cases} \tag{3.1}$$

for all $n = 1, 2, 3, \ldots,$ *where*

$$\begin{cases} u_n^{(1)} = J_{r_{1,n}}^{F_1} x_n, \\ u_n^{(2)} = J_{r_{2,n}}^{F_2} u_n^{(1)} \\ u_n^{(i)} = J_{r_{i,n}}^{F_i} u_n^{(i-1)} = J_{r_{i,n}}^{F_i} J_{r_{i-1,n}}^{F_{i-1}} u_n^{(i-2)} = J_{r_{i,n}}^{F_i} J_{r_{i-1,n}}^{F_{i-1}} J_{r_{i-2,n}}^{F_{i-2}} \ldots J_{r_{2,n}}^{F_2} u_n^{(1)} \\ \quad\;\; = J_{r_{i,n}}^{F_i} J_{r_{i-1,n}}^{F_{i-1}} J_{r_{i-2,n}}^{F_{i-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1} x_n, \;\; i = 2, 3, \ldots N \end{cases} \tag{3.2}$$

and $J_{r_{i,n}}^{F_i} : C \to C,\ i = 1, 2, 3, \ldots N,$ *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequence in* $(0, 1)$, $\{\lambda_n\} \subset [a, b] \subset (0, 2\xi)$ *and* $\{r_{i,n}\}, i \in \{1, 2, 3, \ldots, N\}$ *are a real sequence in* $(0, \infty)$ *satisfy the following conditions:*

(C1) $\alpha_n + \beta_n + \gamma_n = 1,$

(C2) $\liminf_{n\to\infty} \alpha_n > 0$ *and* $\lim_{n\to\infty} \alpha_n = 0,$

(C3) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty,$

(C4) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(C5) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0.$

Then, $\{x_n\}$ *and* $\left\{u_n^{(k)}\right\}_{k=1}^{N}$ *converge strongly to* $z = P_\Theta z$ *provided* $J_{r_{k,n}}^{F_k}$, $k = 1, 2, 3, \ldots N$ *is firmly nonexpansive, that is,*

$$\left\| J_{r_{k,n}}^{F_k} x - J_{r_{k,n}}^{F_k} y \right\|^2 \leq \left\langle J_{r_{k,n}}^{F_k} x - J_{r_{k,n}}^{F_k} y, x - y \right\rangle.$$

Proof. We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded.
Let $\tilde{x} \in \Theta$. Denote by $\mathfrak{B}_n^k = J_{r_{k,n}}^{F_k} J_{r_{k-1,n}}^{F_{k-1}} J_{r_{k-2,n}}^{F_{k-2}} \ldots J_{r_{2,n}}^{F_2} J_{r_{1,n}}^{F_1}$ for $k \in \{1, 2, 3, \ldots, N\}$ and $\mathfrak{B}_n^0 = I$ for all n. In fact, by the definition that for each $k \in \{1, 2, 3, \ldots, N\}$, $J_{r_{k,n}}^{F_k}$ is nonexpansive and $\tilde{x} = \mathfrak{B}_n^k \tilde{x}$ and we note that $u_n^{(k)} = \mathfrak{B}_n^k x_n$. If follows that

$$\|u_n^{(k)} - \tilde{x}\| = \|\mathfrak{B}_n^k y_n - \mathfrak{B}_n^k \tilde{x}\| \leq \|y_n - \tilde{x}\|.$$

Put $\psi_n = P_C(Su_n^{(k)} - \lambda_n BSu_n^{(k)})$ and form Lemma 3.1 $S - \alpha_n BS$ is a nonexpansive mapping and P_C is a nonexpansive mapping. For $\tilde{x} \in VI(C, B)$, we have $\tilde{x} = P_C(\tilde{x} - \lambda_n B\tilde{x})$ from Lemma 2.1, we have

$$\begin{aligned} \|\psi_n - \tilde{x}\| &= \|P_C(Su_n^{(k)} - \lambda_n BSu_n^{(k))} - P_C(\tilde{x} - \lambda_n B\tilde{x})\| \\ &\leq \|(Su_n^{(k)} - \lambda_n BSu_n^{(k))} - (\tilde{x} - \lambda_n B\tilde{x})\| \\ &= \|(Su_n^{(k)} - \lambda_n BSu_n^{(k)}) - (S\tilde{x} - \lambda_n BS\tilde{x})\| \\ &= \|(S - \lambda_n BS)u_n^{(k)} - (S - \lambda_n BS)\tilde{x}\| \\ &\leq \|u_n^{(k)} - \tilde{x}\| \leq \|x_n - \tilde{x}\|. \end{aligned}$$

From (3.1), we obtain

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\| &= \|\alpha_n\psi_n+\beta_n x_n+\gamma_n f(Sx_n)-\tilde{x}\| \\
&\le \alpha_n\|\psi_n-\tilde{x}\|+\beta_n\|x_n-\tilde{x}\|+\gamma_n\|f(Sx_n)-\tilde{x}\| \\
&\le \alpha_n\|x_n-\tilde{x}\|+\beta_n\|x_n-\tilde{x}\|+\gamma_n\|f(Sx_n)-f(\tilde{x})\|+\gamma_n\|f(\tilde{x})-\tilde{x}\| \\
&\le \alpha_n\|x_n-\tilde{x}\|+\beta_n\|x_n-\tilde{x}\|+\gamma_n\alpha\|x_n-\tilde{x}\|+\gamma_n\|f(\tilde{x})-\tilde{x}\| \\
&= (1-\gamma_n)\|x_n-\tilde{x}\|+\gamma_n\alpha\|x_n-\tilde{x}\|+\gamma_n\|f(\tilde{x})-\tilde{x}\| \\
&= (1-\gamma_n+\gamma_n\alpha)\|x_n-\tilde{x}\|+\gamma_n\|f(\tilde{x})-\tilde{x}\| \\
&= [1-\gamma_n(1-\alpha)]\|x_n-\tilde{x}\|+\gamma_n(1-\alpha)\frac{\|f(\tilde{x})-\tilde{x}\|}{(1-\alpha)} \\
&\le \max\left\{\|x_n-\tilde{x}\|,\frac{\|f(\tilde{x})-\tilde{x}\|}{(1-\alpha)}\right\} \\
&\le \dots \\
&\le \max\left\{\|x_1-\tilde{x}\|,\frac{\|f(\tilde{x})-\tilde{x}\|}{(1-\alpha)}\right\}, \quad \forall n\ge 1.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded. We also obtain that $\{u_n^{(k)}\}$, $\{\psi_n\}$, $\{BSu_n^{(k)}\}$, $\{Sx_n\}$, $\big\{f(Sx_n)\big\}$ are all bounded.

Step 2. We claim that if ω_n be a bounded sequence in C then

$$\lim_{n\to\infty}\|\mathfrak{B}_n^k\omega_n-\mathfrak{B}_{n+1}^k\omega_n\|=0, \tag{3.3}$$

for every $k\in\{1,2,3,\dots,N\}$. From Step 2 of the proof Theorem 3.1 in [4], we have that for $k\in\{1,2,3,\dots,N\}$,

$$\lim_{n\to\infty}\|J_{r_{k,n+1}}^{F_k}\omega_n-J_{r_{k,n}}^{F_k}\omega_n\|=0. \tag{3.4}$$

Note that for every $k\in\{1,2,3,\dots,N\}$, we obtain

$$\mathfrak{B}_n^k=J_{r_{k,n}}^{F_k}J_{r_{k-1,n}}^{F_{k-1}}J_{r_{k-2,n}}^{F_{k-2}}\dots J_{r_{2,n}}^{F_2}J_{r_{1,n}}^{F_1}=J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}.$$

So, we have

$$\begin{aligned}
&\|\mathfrak{B}_n^k\omega_n-\mathfrak{B}_{n+1}^k\omega_n\| \qquad (3.5)\\
=\ &\|J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n-J_{r_{k,n+1}}^{F_k}\mathfrak{B}_{n+1}^{k-1}\omega_n\| \\
\le\ &\|J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n-J_{r_{k,n+1}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n\|+\|J_{r_{k,n+1}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n-J_{r_{k,n+1}}^{F_k}\mathfrak{B}_{n+1}^{k-1}\omega_n\| \\
\le\ &\|J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n-J_{r_{k,n+1}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n\|+\|\mathfrak{B}_n^{k-1}\omega_n-\mathfrak{B}_{n+1}^{k-1}\omega_n\| \\
\le\ &\|J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n-J_{r_{k,n+1}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n\|+\|J_{r_{k-1,n}}^{F_{k-1}}\mathfrak{B}_n^{k-2}\omega_n-J_{r_{k-1,n+1}}^{F_{k-1}}\mathfrak{B}_n^{k-2}\omega_n\| \\
&+\|\mathfrak{B}_n^{k-2}\omega_n-\mathfrak{B}_{n+1}^{k-2}\omega_n\| \\
\le\ &\|J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n-J_{r_{k,n+1}}^{F_k}\mathfrak{B}_n^{k-1}\omega_n\|+\|J_{r_{k-1,n}}^{F_{k-1}}\mathfrak{B}_n^{k-2}\omega_n-J_{r_{k-1,n+1}}^{F_{k-1}}\mathfrak{B}_n^{k-2}\omega_n\| \\
&+\dots+\|J_{r_{2,n}}^{F_2}\mathfrak{B}_n^1\omega_n-J_{r_{2,n+1}}^{F_2}\mathfrak{B}_n^1\omega_n\|+\|J_{r_{1,n}}^{F_1}\omega_n-J_{r_{1,n+1}}^{F_1}\omega_n\|.
\end{aligned}$$

Now, apply (3.4) to conclude (3.3).

Step 3. We claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
On the other hand, from $u_n^{(N)}=\mathfrak{B}_n^N y_n$ and $u_{n+1}^{(N)}=\mathfrak{B}_{n+1}^N y_{n+1}$, we have

$$\begin{aligned}\|u_{n+1}^{(N)}-u_n^{(N)}\| &= \|\mathfrak{B}_{n+1}^N x_{n+1}-\mathfrak{B}_n^N x_n\| \\ &= \|\mathfrak{B}_{n+1}^N x_{n+1}-\mathfrak{B}_{n+1}^N x_n\|+\|\mathfrak{B}_{n+1}^N x_n-\mathfrak{B}_n^N x_n\| \\ &\leq \|x_{n+1}-x_n\|+\|\mathfrak{B}_{n+1}^N x_n-\mathfrak{B}_n^N x_n\|. \end{aligned} \tag{3.6}$$

Since $S-\alpha_n BS$ is nonexpansive, we have

$$\begin{aligned}&\|\psi_{n+1}-\psi_n\| \\ =\ & \|P_C(Su_{n+1}^{(N)}-\lambda_{n+1}BSu_{n+1}^{(N)})-P_C(Su_n^{(N)}-\lambda_n BSu_n^{(N)})\| \\ \leq\ & \|(Su_{n+1}^{(N)}-\lambda_{n+1}BSu_{n+1}^{(N)})-(Su_n^{(N)}-\lambda_n BSu_n^{(N)})\| \\ =\ & \|(Su_{n+1}^{(N)}-\lambda_{n+1}BSu_{n+1}^{(N)})-(Su_n^{(N)}-\lambda_{n+1}BSu_n^{(N)})+(\lambda_n-\lambda_{n+1})BSu_n^{(N)}\| \\ \leq\ & \|(S-\lambda_{n+1}BS)u_{n+1}^{(N)}-(S-\lambda_{n+1}BS)u_n^{(N)}\|+|\lambda_n-\lambda_{n+1}|\|BSu_n^{(N)}\| \\ \leq\ & \|u_{n+1}^{(N)}-u_n^{(N)}\|+|\lambda_n-\lambda_{n+1}|\|BSu_n^{(N)}\|. \end{aligned} \tag{3.7}$$

Substituting (3.6) into (3.7), we obtain

$$\begin{aligned}&\|\psi_{n+1}-\psi_n\| \\ \leq\ & \|u_{n+1}^{(N)}-u_n^{(N)}\|+|\lambda_n-\lambda_{n+1}|\|BSu_n^{(N)}\| \\ \leq\ & \|x_{n+1}-x_n\|+\|\mathfrak{B}_{n+1}^N x_n-\mathfrak{B}_n^N x_n\|+|\lambda_n-\lambda_{n+1}|\|BSu_n^{(N)}\|. \end{aligned} \tag{3.8}$$

Indeed, define a sequence $\{z_n\}$ by

$$x_{n+1}=\beta_n x_n+(1-\beta_n)z_n,\ \ \forall n\geq 1.$$

Then we have

$$\begin{aligned}&z_{n+1}-z_n \\ =\ & \frac{\alpha_{n+1}\psi_{n+1}+\gamma_{n+1}f(Sx_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n+\psi_n+\gamma_n f(Sx_n)}{1-\beta_n} \\ =\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\psi_{n+1}-\psi_n)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)\psi_n \\ & +\frac{\gamma_{n+1}}{1-\beta_{n+1}}(f(Sx_{n+1})-f(Sx_n))+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)f(Sx_n) \\ =\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\psi_{n+1}-\psi_n)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)\psi_n \\ & +\frac{\gamma_{n+1}}{1-\beta_{n+1}}(f(Sx_{n+1})-f(Sx_n))-\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(Sx_n) \\ =\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\psi_{n+1}-\psi_n)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)(\psi_n-f(Sx_n)) \\ & +\frac{\gamma_{n+1}}{1-\beta_{n+1}}(f(Sx_{n+1})-f(Sx_n)). \end{aligned} \tag{3.9}$$

It follows from (3.9) that

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\psi_{n+1}-\psi_n\| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\big(\|\psi_n\| + \|f(Sx_n)\|\big) \\
& + \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)\alpha\|x_{n+1}-x_n\|. \qquad (3.10)
\end{aligned}$$

Combining (3.8) and (3.10), we obtain

$$\begin{aligned}
&\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big\{\|x_{n+1}-x_n\| + \|\mathfrak{B}_{n+1}^N x_n - \mathfrak{B}_n^N x_n\| \\
& + |\lambda_n - \lambda_{n+1}|\|BSu_n^{(N)}\|\Big\} + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\big(\|\psi_n\| + \|f(Sx_n)\|\big) \\
& + \left(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)\alpha\|x_{n+1}-x_n\| - \|x_{n+1}-x_n\| \\
\leq\ & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big\{\|\mathfrak{B}_{n+1}^N x_n - \mathfrak{B}_n^N x_n\| + |\lambda_n - \lambda_{n+1}|\|BSu_n^{(N)}\|\Big\} \\
& + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right|\big(\|\psi_n\| + \|f(Sx_n)\|\big).
\end{aligned}$$

Since conditions (C1)-(C5), (3.3) and $\{f(Sx_n)\}$, $\{\psi_n\}$ and $\{BSu_n^{(N)}\}$ are bounded, so we have

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|\big) \leq 0.$$

Thus, by Lemma 2.5, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Consequently, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \qquad (3.11)$$

From (3.3), (3.6), (3.7) and conditions in Theorem 3.2 and (3.11), we obtain that

$$\lim_{n\to\infty}\|u_{n+1}^{(N)} - u_n^{(N)}\| = \lim_{n\to\infty}\|\psi_{n+1}-\psi_n\| = 0.$$

Step 4. We claim that $\lim_{n\to\infty}\|S\psi_n - \psi_n\| = 0$.

For any $\tilde{x}\in\Theta$, we see that

$$\begin{aligned}
&\|\psi_n - \tilde{x}\|^2 \\
=\ & \|P_C(Su_n^{(N)} - \lambda_n BSu_n^{(N)}) - P_C(\tilde{x} - \lambda_n B\tilde{x})\|^2 \\
\leq\ & \|(Su_n^{(N)} - \lambda_n BSu_n^{(N)}) - (\tilde{x} - \lambda_n B\tilde{x})\|^2 \\
=\ & \|(Su_n^{(N)} - \tilde{x}) - \lambda_n(BSu_n^{(N)} - B\tilde{x})\|^2 \\
=\ & \|Su_n^{(N)} - \tilde{x}\|^2 - 2\lambda_n\langle Su_n^{(N)} - \tilde{x}, BSu_n^{(N)} - B\tilde{x}\rangle + \lambda_n^2\|BSu_n^{(N)} - B\tilde{x}\|^2 \\
\leq\ & \|u_n^{(N)} - \tilde{x}\|^2 - 2\lambda_n\langle Su_n^{(N)} - \tilde{x}, BSu_n^{(N)} - B\tilde{x}\rangle + \lambda_n^2\|BSu_n^{(N)} - B\tilde{x}\|^2 \\
\leq\ & \|x_n - \tilde{x}\|^2 - 2\lambda_n\xi\|BSu_n^{(N)} - B\tilde{x}\| + \lambda_n^2\|BSu_n^{(N)} - B\tilde{x}\|^2 \\
=\ & \|x_n - \tilde{x}\|^2 + \lambda_n(\lambda_n - 2\xi)\|BSu_n^{(N)} - B\tilde{x}\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &= \|\alpha_n(\psi_n-\tilde{x})+\beta_n(x_n-\tilde{x})+\gamma_n(f(Sx_n)-\tilde{x})\|^2\\ &\le \alpha_n\|\psi_n-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\|f(Sx_n)-\tilde{x}\|^2 \qquad (3.12)\\ &\le \alpha_n\Big\{\|x_n-\tilde{x}\|^2+\lambda_n(\lambda_n-2\xi)\|BSu_n^{(N)}-B\tilde{x}\|^2\Big\}\\ &\quad +\beta_n\|x_n-\tilde{x}\|+\gamma_n\|f(Sx_n)-\tilde{x}\|^2\\ &= (1-\gamma_n)\|x_n-\tilde{x}\|^2+\alpha_n\lambda_n(\lambda_n-2\xi)\|BSu_n^{(N)}-B\tilde{x}\|^2\\ &\quad +\gamma_n\|f(Sx_n)-\tilde{x}\|^2\\ &\le \|x_n-\tilde{x}\|^2+\alpha_n\lambda_n(\lambda_n-2\xi)\|BSu_n^{(N)}-B\tilde{x}\|^2\\ &\quad +\gamma_n\|f(Sx_n)-\tilde{x}\|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\alpha_n a(b-2\xi)\|BSu_n^{(N)}-B\tilde{x}\|^2\\ \le\ &\alpha_n\lambda_n(\lambda_n-2\xi)\|BSu_n^{(N)}-B\tilde{x}\|^2\\ \le\ &\|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2+\gamma_n\|f(Sx_n)-\tilde{x}\|^2\\ \le\ &(\|x_n-\tilde{x}\|-\|x_{n+1}-\tilde{x}\|)(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|)+\gamma_n\|f(Sx_n)-\tilde{x}\|^2\\ \le\ &\|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|)+\gamma_n\|f(Sx_n)-\tilde{x}\|^2.\end{aligned}$$

Since $\liminf_{n\to\infty}\alpha_n>0$, $\gamma_n\to 0$, (3.11) and $\{x_n\}$ and $\{f(Sx_n)\}$ are bounded, we have

$$\lim_{n\to\infty}\|BSu_n^{(N)}-B\tilde{x}\|=0. \qquad (3.13)$$

Moreover, form (2.1) we obtain

$$\begin{aligned}&\|\psi_n-\tilde{x}\|^2\\ =\ &\|P_C(Su_n^{(N)}-\lambda_n BSu_n^{(N)})-P_C(\tilde{x}-\lambda_n B\tilde{x})\|^2\\ \le\ &\big\langle(Su_n^{(N)}-\lambda_n BSu_n^{(N)})-(\tilde{x}-\lambda_n B\tilde{x}),\psi_n-\tilde{x}\big\rangle\\ =\ &\frac{1}{2}\Big\{\|(Su_n^{(N)}-\lambda_n BSu_n^{(N)})-(\tilde{x}-\lambda_n B\tilde{x})\|^2+\|\psi_n-\tilde{x}\|^2\\ &-\|(Su_n^{(N)}-\lambda_n BSu_n^{(N)})-(\tilde{x}-\lambda_n B\tilde{x})-(\psi_n-\tilde{x})\|^2\Big\}\\ \le\ &\frac{1}{2}\Big\{\|u_n^{(N)}-\tilde{x}\|^2+\|\psi_n-\tilde{x}\|^2-\|(Su_n^{(N)}-\psi_n)-\lambda_n(BSu_n^{(N)}-B\tilde{x})\|^2\Big\}\\ \le\ &\frac{1}{2}\Big\{\|x_n-\tilde{x}\|^2+\|\psi_n-\tilde{x}\|^2-\|Su_n^{(N)}-\psi_n\|^2\\ &-\lambda_n^2\|BSu_n^{(N)}-B\tilde{x}\|^2+2\lambda_n\langle Su_n^{(N)}-\psi_n,BSu_n^{(N)}-B\tilde{x}\rangle\Big\},\end{aligned}$$

which yields that

$$\|\psi_n-\tilde{x}\|^2\le\|x_n-\tilde{x}\|^2-\|Su_n^{(N)}-\psi_n\|^2+2\lambda_n\|Su_n^{(N)}-\psi_n\|\|BSu_n^{(N)}-B\tilde{x}\|. \qquad (3.14)$$

Substituting (3.14) into (3.12), we have

$$\begin{aligned}
& \|x_{n+1}-\tilde{x}\|^2 \\
\leq\ & \alpha_n\|\psi_n-\tilde{x}\|^2+\beta_n\|x_n-\tilde{x}\|^2+\gamma_n\|f(Sx_n)-\tilde{x}\|^2 \\
\leq\ & \alpha_n\Big\{\|x_n-\tilde{x}\|^2-\|Su_n^{(N)}-\psi_n\|^2+2\lambda_n\|Su_n^{(N)}-\psi_n\|\|BSu_n^{(N)}-B\tilde{x}\|\Big\} \\
& +\beta_n\|x_n-\tilde{x}\|+\gamma_n\|f(Sx_n)-\tilde{x}\|^2 \\
\leq\ & \|x_n-\tilde{x}\|^2-\alpha_n\|Su_n^{(N)}-\psi_n\|^2+2\alpha_n\lambda_n\|Su_n^{(N)}-\psi_n\|\|BSu_n^{(N)}-B\tilde{x}\| \\
& +\gamma_n\|f(Sx_n)-\tilde{x}\|^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \alpha_n\|Su_n^{(N)}-\psi_n\|^2 \\
\leq\ & \|x_n-\tilde{x}\|^2-\|x_{n+1}-\tilde{x}\|^2 \\
& +2\alpha_n\lambda_n\|Su_n^{(N)}-\psi_n\|\|BSu_n^{(N)}-B\tilde{x}\|+\gamma_n\|f(Sx_n)-\tilde{x}\|^2 \\
\leq\ & \|x_n-x_{n+1}\|(\|x_n-\tilde{x}\|+\|x_{n+1}-\tilde{x}\|) \\
& +2\alpha_n\lambda_n(\|Su_n^{(N)}\|+\|\psi_n\|)\|BSu_n^{(N)}-B\tilde{x}\|+\gamma_n\|f(Sx_n)-\tilde{x}\|^2.
\end{aligned}$$

Since $\liminf_{n\to\infty}\alpha_n>0$, $\gamma_n\to 0$, (3.11), (3.13) and $\{x_n\}$, $\{Su_n^{(N)}\}$, $\{\psi_n\}$ and $\{f(Sx_n)\}$ are bounded, we obtain

$$\lim_{n\to\infty}\|Su_n^{(N)}-\psi_n\|=0. \tag{3.15}$$

For any $\tilde{x}\in\Theta$, note that $J_{r_{k,n}}^{F_k}$ is firmly nonexpansive (Lemma 2.4) for $k\in\{1,2,3,\dots,N\}$, then we have

$$\begin{aligned}
\|\mathfrak{B}_n^k x_n-p\|^2 &= \|J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}x_n-J_{r_{k,n}}^{F_k}\tilde{x}\|^2 \\
&\leq \Big\langle J_{r_{k,n}}^{F_k}\mathfrak{B}_n^{k-1}x_n-J_{r_{k,n}}^{F_k}\tilde{x},\mathfrak{B}_n^{k-1}x_n-\tilde{x}\Big\rangle \\
&= \Big\langle \mathfrak{B}_n^k x_n-\tilde{x},\mathfrak{B}_n^{k-1}x_n-\tilde{x}\Big\rangle \\
&= \frac{1}{2}\Big(\|\mathfrak{B}_n^k x_n-\tilde{x}\|^2+\|\mathfrak{B}_n^{k-1}x_n-\tilde{x}\|^2-\|\mathfrak{B}_n^k x_n-\mathfrak{B}_n^{k-1}x_n\|^2\Big)
\end{aligned}$$

and hence

$$\|\mathfrak{B}_n^k x_n-\tilde{x}\|^2\leq\|\mathfrak{B}_n^{k-1}x_n-\tilde{x}\|^2-\|\mathfrak{B}_n^k x_n-\mathfrak{B}_n^{k-1}x_n\|^2, \quad k=1,2,3,\dots,N$$

which implies that for each $k\in\{1,2,3,\dots,N-1\}$,

$$\begin{aligned}
& \|\mathfrak{B}_n^k x_n-\tilde{x}\|^2 \\
\leq\ & \|\mathfrak{B}_n^0 x_n-\tilde{x}\|^2-\|\mathfrak{B}_n^k x_n-\mathfrak{B}_n^{k-1}x_n\|^2 \\
& -\|\mathfrak{B}_n^{k-1}x_n-\mathfrak{B}_n^{k-2}x_n\|^2-\dots-\|\mathfrak{B}_n^2 x_n-\mathfrak{B}_n^1 x_n\|^2-\|\mathfrak{B}_n^1 x_n-\mathfrak{B}_n^0 x_n\|^2 \\
\leq\ & \|x_n-\tilde{x}\|^2-\|\mathfrak{B}_n^k x_n-\mathfrak{B}_n^{k-1}x_n\|^2.
\end{aligned}$$

Also, we observe that for $\tilde{x} \in \Theta$,

$$\begin{aligned}
& \|x_{n+1} - \tilde{x}\|^2 \\
\leq\ & \alpha_n \|P_C(Su_n^{(N)} - \lambda_n BSu_n^{(N)}) - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2 \\
\leq\ & \alpha_n \|u_n^{(N)} - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2 \\
\leq\ & \alpha_n \|\mathfrak{B}_n^k x_n - \tilde{x}\|^2 + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2 \\
\leq\ & \alpha_n \Big\{ \|x_n - \tilde{x}\|^2 - \|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1} x_n\|^2 \Big\} + \beta_n \|x_n - \tilde{x}\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2 \\
=\ & (1 - \gamma_n) \|x_n - \tilde{x}\|^2 - \alpha_n \|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1} x_n\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2 \\
\leq\ & \|x_n - \tilde{x}\|^2 - \alpha_n \|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1} x_n\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \alpha_n \|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1} x_n\|^2 \\
\leq\ & \|x_n - \tilde{x}\|^2 - \|x_{n+1} - \tilde{x}\|^2 + \gamma_n \|f(Sx_n) - \tilde{x}\|^2 \\
\leq\ & \|x_n - x_{n+1}\| (\|x_n - \tilde{x}\| + \|x_{n+1} - \tilde{x}\|) + \gamma_n \|f(Sx_n) - \tilde{x}\|^2.
\end{aligned}$$

Hence, by $\liminf_{n\to\infty} \alpha_n > 0$, $\gamma_n \to 0$, (3.11) and $\{x_n\}$ and $\{f(Sx_n)\}$ are bounded, so we deduce that

$$\lim_{n\to\infty} \|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1} x_n\| = 0 \ \text{ for any } k = 1, 2, \ldots, N-1, \tag{3.16}$$

that is,

$$\|u_n^{(k)} - u_n^{(k-1)}\| \to 0 \text{ as } \ n \to \infty.$$

Therefore, we have

$$\begin{aligned}
& \|x_n - u_n^{(N)}\| \\
=\ & \|\mathfrak{B}_n^0 x_n - \mathfrak{B}_n^k x_n\| \\
\leq\ & \|\mathfrak{B}_n^0 x_n - \mathfrak{B}_n^1 x_n\| + \|\mathfrak{B}_n^1 x_n - \mathfrak{B}_n^2 x_n\| + \ldots + \|\mathfrak{B}_n^{N-1} x_n - \mathfrak{B}_n^N x_n\|.
\end{aligned}$$

From (3.16), we have

$$\lim_{n\to\infty} \|x_n - u_n^{(N)}\| = 0. \tag{3.17}$$

Since $x_{n+1} = \alpha_n \psi_n + \beta_n x_n + \gamma_n f(Sx_n)$ and $\alpha_n + \beta_n + \gamma_n = 1$, we obtain

$$x_{n+1} - x_n \quad = \quad \alpha_n(\psi_n - x_n) + \gamma_n\big(f(Sx_n) - x_n\big).$$

It follows that

$$\alpha_n \|\psi_n - x_n\| \leq \|x_{n+1} - x_n\| + \gamma_n \|f(Sx_n) - x_n\|.$$

Since $\gamma_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\liminf_{n\to\infty} \alpha_n > 0$, so by the boundedness of $\{f(Sx_n) - x_n\}$, we have

$$\lim_{n\to\infty} \|\psi_n - x_n\| = 0. \tag{3.18}$$

Furthermore, by the triangular inequality, we also have

$$\begin{aligned}\|S\psi_n - \psi_n\| &\leq \|S\psi_n - Su_n^{(N)}\| + \|Su_n^{(N)} - \psi_n\| \\ &\leq \|\psi_n - u_n^{(N)}\| + \|Su_n^{(N)} - \psi_n\| \\ &\leq \|\psi_n - x_n\| + \|x_n - u_n^{(N)}\| + \|Su_n^{(N)} - \psi_n\|.\end{aligned}$$

Applying (3.15), (3.17) and (3.18), we have

$$\lim_{n\to\infty} \|S\psi_n - \psi_n\| = 0. \tag{3.19}$$

Step 5. We claim that the mapping $P_\Theta f$ has a unique fixed point.

Let $Q = P_\Theta$. Since f is a contraction with $\alpha \in (0,1)$, we obtain

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq \alpha\|x - y\|, \ \ \forall x, y \in C.$$

Therefore, Qf is a contraction of C into itself, which implies that there exists a unique element $z \in C$ such that $z = Qf(z) = P_\Theta f(z)$.

Step 6. We claim that $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$, where $z = P_\Theta f(z)$.

To show this inequality, we choose a subsequence $\{\psi_{n_i}\}$ of $\{\psi_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, S\psi_n - z\rangle = \lim_{i\to\infty}\langle f(z) - z, S\psi_{n_i} - z\rangle.$$

Since $\{\psi_n\}$ is bounded, there exists a subsequence $\{\psi_{n_i}\}$ of $\{\psi_n\}$ which converges weakly to $z \in C$. Without loss of generality, we may assume that $\{\psi_{n_i}\} \rightharpoonup z$. From $\|S\psi_n - \psi_n\| \to 0$, we obtain $S\psi_{n_i} \rightharpoonup z$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $\{x_{n_i}\} \rightharpoonup w$. Since $\lim_{n\to\infty}\|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1}x_n\| = 0$ for $k = 1, 2, 3, \ldots, N$, we have $\mathfrak{B}_{n_i}^k x_{n_i} \rightharpoonup w$ for $k = 1, 2, 3, \ldots, N$. Next, we show that $w \in \Theta$, where $\Theta := F(S) \cap \big(\cap_{k=1}^N SEP(F_k)\big) \cap VI(C, B)$.

First, we show that $w \in \cap_{k=1}^N SEP(F_k)$. Since $u_n^{(N)} = \mathfrak{B}_n^k x_n$ for $k = 1, 2, 3, \ldots, N$, we also have

$$F_k(\mathfrak{B}_n^k x_n, y) + \frac{1}{r_n}\langle y - \mathfrak{B}_n^k x_n, \mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1}x_n\rangle \geq 0, \quad \forall y \in C.$$

If follows from (A2) that,

$$\frac{1}{r_n}\langle y - \mathfrak{B}_n^k x_n, \mathfrak{B}_n^k x_n - \mathfrak{B}_n^{k-1}x_n\rangle \geq -F_k(\mathfrak{B}_n^k x_n, y) \geq F_k(y, \mathfrak{B}_n^k x_n)$$

and hence

$$\left\langle y - \mathfrak{B}_{n_i}^k x_{n_i}, \frac{\mathfrak{B}_{n_i}^k x_{n_i} - \mathfrak{B}_{n_i}^{k-1}x_{n_i}}{r_{n_i}}\right\rangle \geq F_k(y, \mathfrak{B}_{n_i}^k x_{n_i}).$$

Since $\frac{\mathfrak{B}_{n_i}^k x_{n_i} - \mathfrak{B}_{n_i}^{k-1}x_{n_i}}{r_{n_i}} \to 0$ and $\mathfrak{B}_{n_i}^k x_{n_i} \rightharpoonup w$, it follows by (A4) that

$$F_k(y, w) \leq 0 \quad \forall y \in C,$$

for each $k = 1, 2, 3, \ldots, N$.

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F_k(y_t, w) \leq 0$. So, from (A1) and (A4) we have

$$0 \quad = \quad F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, w) \leq tF_k(y_t, y)$$

and hence $F_k(y_t, y) \geq 0$. From (A3), we have $F_k(w, y) \geq 0$ for all $y \in C$ and hence $w \in EP(F_k)$ for $k = 1, 2, 3, \ldots, N$, that is, $w \in \cap_{k=1}^N SEP(F_k)$.

Next, we show that $w \in F(S)$. Assume $w \notin F(S)$. Since $\psi_{n_i} \rightharpoonup w$ and $w \neq Sw$, it follows by the Opial's condition (Lemma 2.2) that

$$\begin{aligned}
\liminf_{i\to\infty} \|\psi_{n_i} - w\| &< \liminf_{i\to\infty} \|\psi_{n_i} - Sw\| \\
&\leq \liminf_{i\to\infty} \big\{ \|\psi_{n_i} - S\psi_{n_i}\| + \|S\psi_{n_i} - Sw\| \big\} \\
&\leq \liminf_{i\to\infty} \|\psi_{n_i} - w\|.
\end{aligned}$$

This is a contradiction. So, we get $w \in F(S)$.

Finally, we show that $w \in VI(C, B)$. Define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then, T is maximal monotone. Let $(v, w_1) \in G(T)$. Since $w_1 - Bv \in N_C v$ and $\psi_n \in C$, we have $\langle v - \psi_n, w_1 - Bv \rangle \geq 0$. On the other hand, $\psi_n = P_C(Su_n^{(N)} - \lambda_n BSu_n^{(N)})$, we have

$$\big\langle v - \psi_n, \psi_n - (Su_n^{(N)} - \lambda_n BSu_n^{(N)}) \big\rangle \geq 0,$$

and hence

$$\left\langle v - \psi_n, \frac{\psi_n - Su_n^{(N)}}{\lambda_n} + BSu_n^{(N)} \right\rangle \geq 0.$$

Therefore, we have

$$\begin{aligned}
\langle v - \psi_{n_i}, w_1 \rangle &\geq \langle v - \psi_{n_i}, Bv \rangle \\
&\geq \langle v - \psi_{n_i}, Bv \rangle - \left\langle v - v_{n_i}, \frac{\psi_{n_i} - Su_{n_i}^{(N)}}{\lambda_{n_i}} + BSu_{n_i}^{(N)} \right\rangle \\
&= \left\langle v - v_{n_i}, Bv - By_{n_i} - \frac{\psi_{n_i} - Su_{n_i}^{(N)}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - \psi_{n_i}, Bv - B\psi_{n_i} \rangle + \langle v - \psi_{n_i}, B\psi_{n_i} - BSu_{n_i}^{(N)} \rangle \\
&\quad - \left\langle v - \psi_{n_i}, \frac{\psi_{n_i} - Su_{n_i}^{(N)}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - \psi_{n_i}, B\psi_{n_i} - BSu_{n_i}^{(N)} \rangle - \left\langle v - \psi_{n_i}, \frac{\psi_{n_i} - Su_{n_i}^{(N)}}{\lambda_{n_i}} \right\rangle.
\end{aligned}$$

Hence, we have $\langle v-w, w_1\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C,B)$.
Hence $w \in \Theta$, where $\Theta := F(S) \cap \left(\cap_{k=1}^{N} SEP(F_k)\right) \cap VI(C,B)$. Then we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle &= \limsup_{n\to\infty}\langle f(z)-z, S\psi_n - z\rangle \\ &= \lim_{i\to\infty}\langle f(z)-z, S\psi_{n_i}-z\rangle \\ &= \langle f(z)-z, w-z\rangle \leq 0. \end{aligned} \tag{3.20}$$

Step 7. Finally, we claim that $\{x_n\}$ converges strongly to $z = P_\Theta f(z)$.
Indeed, from (3.1) and Lemma 2.7, we have

$$\begin{aligned} & \|x_{n+1}-z\|^2 \\ = \ & \big\langle \alpha_n\psi_n + \beta_n x_n + \gamma_n f(Sx_n) - z, x_{n+1}-z\big\rangle \\ = \ & \gamma_n\big\langle f(Sx_n)-z, x_{n+1}-z\big\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle \\ & +\alpha_n\langle \psi_n - z, x_{n+1}-z\rangle \\ \leq \ & \frac{1}{2}\beta_n\Big(\|x_n-z\|^2 + \|x_{n+1}-z\|^2\Big) + \frac{1}{2}\alpha_n\Big(\|\psi_n - z\|^2 + \|x_{n+1}-z\|^2\Big) \\ & + \gamma_n\big\langle f(Sx_n)-f(z), x_{n+1}-z\big\rangle + \gamma_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\ \leq \ & \frac{1}{2}(1-\gamma_n)\Big(\|x_n-z\|^2 + \|x_{n+1}-z\|^2\Big) + \frac{1}{2}\gamma_n\Big(\|f(Sx_n)-f(z)\|^2 \\ & +\|x_{n+1}-z\|^2\Big) + \gamma_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \\ \leq \ & \frac{1}{2}\Big(1-\gamma_n\big(1-\alpha^2\big)\Big)\|x_n-z\|^2 + \frac{1}{2}(1-\gamma_n)\|x_{n+1}-z\|^2 \\ & +\frac{1}{2}\alpha_n\|x_{n+1}-z\|^2 + \gamma_n\big\langle f(z)-z, x_{n+1}-z\big\rangle, \end{aligned}$$

which implies that

$$\|x_{n+1}-z\|^2 \leq \Big(1-\gamma_n\big(1-\alpha^2\big)\Big)\|x_n-z\|^2 + 2\gamma_n\langle f(z)-z, x_{n+1}-z\rangle. \tag{3.21}$$

Taking

$$\varrho_n = 2\gamma_n\big\langle f(z)-z, x_{n+1}-z\big\rangle \quad \text{and} \quad l_n = \gamma_n(1-\alpha^2),$$

using (C3) and (3.20), we get $l_n \to 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\to\infty}\frac{\varrho_n}{l_n} \leq 0$. Applying Lemma 2.6 to (3.21), we conclude that $x_n \to z$ in norm. Finally, notice that $\|u_n^{(N)} - z\| = \|\mathfrak{B}_n^k x_n - \mathfrak{B}_n^k z\| \leq \|x_n - z\|$. We also conclude that $u_n^{(N)} \to z$ in norm. This completes the proof. □

Acknowledgement(s) : The author is extremely to the referees for their useful suggestions that improved the content of the paper. This work is supported by Rajamangala University of Technology Rattanakosin Research and Development Institute.

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(Received 23 April 2010)

Chaichana Jaiboon
Department of Mathematics,
Faculty of Liberal Arts,
Rajamangala University of Technology Rattanakosin (RMUTR),
Samphanthawang, Bangkok 10100, THAILAND
e-mail : `chaichana.j@rmutr.ac.th`