Thai Journal of Mathematics
Volume 8 (2010) Number 2 : 263–273

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Warped Product Pseudo-Slant Submanifold of Trans-Sasakian Manifolds

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Abstract : In this paper, we have obtained the Integrability condition of pseudo-slant submanifold in trans-Sasakian manifold. Also, we have studied the warped and doubly warped product pseudo-slant submanifold of trans-Sasakian manifold.

Keywords : Warped Product, Doubly Warped Product, Pseudo-Slant Submanifold, Trans-Sasakian manifolds
2000 Mathematics Subject Classification : 53C40; 53B25; 53C25

1 Introduction

Bishop and O' Neill [9] introduced the concept of warped products. B.Y.Chen [5] extended the work of Bishop and O' Neill and studied the warped product CR-submanifold of Kaehler manifolds and many more[5],[6]. After B.Y.Chen many other authors extended these results in different settings [3],[8].

Our aim in this paper is to study the warped and doubly warped product pseudo-slant submanifold of trans-Sasakian manifold. This paper is organized as, section 2 is devoted to preliminaries. In section 3, some basic results for pseudo-slant submanifolds are given and also, we have obtained integrability condition of pseudo-slant submanifold of trans-Sasakian manifold. Section 4 deals with the basic results of warped and doubly warped product submanifolds. In section 5, we have studied warped product pseudo-slant submanifold of trans-Sasakian manifold. In section 6, we have studied the pseudo-slant warped product submanifold of trans-Sasakian manifold. After the main results we have provide an example of pseudo-slant warped product submanifold of Kenmotsu manifold. In section 7 and 8 we have obtained doubly warped product pseudo-slant and pseudo-slant doubly warped product submanifolds of trans-Sasakian manifold.

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2 Preliminaries

A $(2n+1)-$dimensional Riemannian manifold $(\bar{M}, g)$ is said to be a trans-Sasakian manifold if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the following:

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \phi(\xi) = 0, \eta \circ \phi = 0 \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi) \tag{2}$$

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{3}$$

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{4}$$

for any $X, Y \in T\bar{M}$. In this case

$$g(\phi X, Y) = -g(X, \phi Y) \tag{5}$$

where α and β are smooth function on $\bar{M}$ and $\bar{\nabla}$ denote the Riemannian connection with respect to the Riemannian metric g. If α (respectively β) is zero then $\bar{M}$ is called β-Kenmotsu (respectively α-Sasakian). If α and β are both zero then the manifold $\bar{M}$ becomes Cosymplectic.

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^{\perp}M$ be the Lie algebra of vector fields tangential to M and normal to M respectively and ∇ be the induced Levi-Civita connections on M, then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{6}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{7}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{8}$$

For any $x \in M$ and $X \in T_xM$, we write

$$\phi X = TX + NX \tag{9}$$

where $TX \in T_xM$ and $NX \in T^{\perp}_x M$. Similarly, for $V \in T^{\perp}_x M$, we have

$$\phi V = tV + nV \tag{10}$$

where tV (resp. nV) is the tangential component (resp. normal component) of ϕV.

From (5) and (9), it is easy to observe that for each $x \in M$, and $X, Y \in T_xM$

$$g(TX, Y) = -g(X, TY). \tag{11}$$

For any $X, Y \in TM$ on using (4) and (6) we have the following

$$(a) \ \ \nabla_X \xi = -\alpha TX + \beta(X - \eta(X)\xi) \quad (b) \ \ h(X, \xi) = -\alpha NX, \tag{12}$$

3 Pseudo-Slant submanifolds

A.Carriazo [1] defined and studied Bi-slant immersion in almost Hermitian manifold and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifold. Recently, V.A.Khan and M.A.Khan [10] studied the pseudo-slant submanifold of a Sasakian manifold and found some basic results.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a slant submanifold if for any $x \in M$ and $X \in T_xM$, linearly independent to ξ, the angle between ϕX and T_xM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of M in $\bar{M}$. If $\theta = 0$ the submanifold is invariant submanifold if $\theta = \pi/2$ then it is called anti-invariant submanifold if $\theta \neq 0, \pi/2$ then it is called proper slant submanifold.

We say that M is a pseudo-slant submanifold of an almost contact metric manifold $\bar{M}$, if there exist two orthogonal distributions $D^\perp$ and D_θ on M such that

(i) $T_M = D^\perp \oplus D_\theta \oplus < \xi >$.

(ii) The distribution $D^\perp$ is anti-invariant *i.e.*, $\phi D^\perp \subseteq T^\perp M$.

(iii) The distribution D_θ is slant with slant angle $\theta \neq \pi/2$

from the definition it is clear that if $\theta = 0$, the pseudo-slant submanifold become semi-invariant submanifold.

Suppose M to be a pseudo-slant submanifold of an almost contact metric manifold $\bar{M}$. Then, for any $X \in TM$, put

$$X = P_1X + P_2X + \eta(X)\xi \tag{13}$$

where $P_i = (i = 1, 2)$ are projection maps on the distributions $D^\perp$ and D_θ. Now operating ϕ on both sides of equation (13)

$$\phi X = NP_1X + TP_2X + NP_2X \tag{14}$$

it is easy to see that

$$TX = TP_2X, \quad NX = NP_1X + NP_2X$$

and,

$$\phi P_1X = NP_1X, \quad TP_1X = 0, \tag{15}$$

$$TP_2X \in D_\theta. \tag{16}$$

If μ is the invariant subspace of the normal bundle $T^\perp M$ then, in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \mu \oplus ND^\perp \oplus ND_\theta. \tag{17}$$

As $D^\perp$ and D_θ are orthogonal distribution on M, $g(Z, X) = 0$ for each $Z \in D^\perp$ and $X \in D_\theta$. Thus, by equation (9) and (5), we may write

$$g(NZ, NX) = g(\phi Z, \phi X) = g(Z, X) = 0.$$

That means the distributions $ND^\perp$ and ND_θ are mutually perpendicular. Infect, the decomposition (17) is an orthogonal direct decomposition.

For a pseudo-slant submanifold of a trans-Sasakian manifold the following lemmas play an important role in working out the integrability conditions of the distributions involved in this setting.

Lemma 3.1. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$, then*

$$A_{\phi Y}X = A_{\phi X}Y \tag{18}$$

for all $X, Y \in D^\perp$.

Proof. For any X, Y in $D^\perp$ and Z in TM, using (8),(5),(4) and (6) we find that

$$\begin{aligned} g(A_{\phi Y}X, Z) &= -g(\phi\bar{\nabla}_Z X, Y) \\ &= -g(\bar{\nabla}_Z \phi X - (\bar{\nabla}_z \phi)X, Y). \end{aligned}$$

on applying equations (3) and (7) the above equation yields

$$g(A_{\phi Y}X, Z) = g(A_{\phi X}Y, Z)$$

the results follows from the above equation. □

Lemma 3.2. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$, $\alpha \neq 0$, then for any $X, Y \in D^\perp \oplus D_\theta$*

$$g([X, Y], \xi) = 2\alpha g(TX, Y) \tag{19}$$

the proof of equation (19) is straightforward and may be obtained by using (12)(a).

Proposition 3.3. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then, anti-invariant distribution $D^{\perp}$ is integrable.*

Proof. For any $X, Y \in D^{\perp}$ and $Z \in D_{\theta}$, by (13)

$$g([X,Y], TP_2Z) = -g(\phi[X,Y], P_2Z)$$

Now, using (3) and (7), we find

$$g([X,Y], TP_2Z) = g(A_{\phi Y}X - A_{\phi X}Y, P_2Z)$$

Now, the integrability of the distribution $D^{\perp}$ follows on using equation (18) and (19). □

Lemma 3.4. *Let M be a pseudo-slant submanifold of trans-Sasakian manifold $\bar{M}$, with $\alpha \neq 0$, then the slant distribution D_{θ} is not integrable.*

By the definition of pseudo-slant submanifold and in view of equation (19) the results follows.

Proposition 3.5. *Let M be a pseudo-slant submanifold of $\bar{M}$, with $\alpha \neq 0$, then the distribution $D_{\theta} \oplus < \xi >$ is integrable if and only if*

$$h(Z,TW) - h(W,TZ) + \nabla^{\perp}_{X} NW - \nabla^{\perp}_{W} NZ$$

lies in ND_{θ} for each $Z, W \in D_{\theta} \oplus < \xi >$.

Proof. Making use of equations (9),(3),(6) and (7), we obtain

$$g(N[Z,W], NX) = g(h(Z,TW) - h(W,TZ) + \nabla^{\perp}_{X} NW - \nabla^{\perp}_{W} NZ, NX)$$

for each $X \in D^{\perp}$ and $Z, W \in D_{\theta}$. The Result follows on using the fact that $ND^{\perp}$ and ND_{θ} are mutually perpendicular. □

Now we have the following consequence of above result.

Corollary 3.6. *Let M be a pseudo-slant submanifold of a β-Kenmotsu manifold or Cosymplectic manifold, then the slant distribution D_{θ} is integrable iff*

$$h(Z,TW) - h(W,TZ) + \nabla_{X}{}^{\perp} NW - \nabla_{W}{}^{\perp} NZ$$

lies in ND_{θ} for each $Z, W \in D_{\theta}$.

4 Warped Product and Doubly warped product submanifolds

The study of warped product submanifold was initiated by R.L. Bishop and B.O'Neill [9]. They defined as follows

Definition. Let (B, g_B) and F, g_F be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and f a positive differentiable function on B. The warped product $B \times_f F$ of B and F is the Riemannian manifold $(B \times F, g)$, where

$$g = g_B + f^2 g_F. \tag{20}$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $B \times f$ on B and F, respectively.

The following lemma provides some basic formulas on warped product submanifolds.

Lemma 4.1. *Let $M = B \times_f F$ be warped product manifold. If $X, Y \in TB$ and $V, W \in TF$ then*

(i) $\nabla_X Y \in TB$

(ii) $\nabla_X V = \nabla_V X = X(lnf)V$

(iii) $nor(\nabla_V W) = -\frac{g(V,W)}{f}\nabla f$

where $nor(\nabla_V W)$ is the component of $\nabla_V W$ in TB and ∇f is the gradient vector field of the warping function f.

From (ii) of above lemma we can see that

$$\nabla_U V = \nabla_V U = (Ulnf)V \tag{21}$$

for any vector fields U tangent to B and V tangent to F.

If the manifolds N_θ and $N_\perp$ are slant and anti-invariant submanifolds respectively of trans-Sasakian manifold $\bar{M}$, then their warped products are
(a) $N_\perp \times_f N_\theta$,
(b) $N_\theta \times_f N_\perp$.

Note (i) In the sequel, we call the warped product submanifold (a) as warped product pseudo-slant submanifold and the warped product (b) as pseudo-slant warped product submanifold.

B.Y.Chen [5],[6] extended the work of Bishop and O' Neill and studied the warped product CR-submanifold of Kaehler manifolds. B. Sahin [3] proved the non-existence theorem for warped product CR-submanifolds. Doubly warped product manifolds were introduced as generalization of warped product manifold by B. Unel [4]. A doubly warped product (M, g) is a product manifold of the form $M =_f B \times_b F$ with the metric $g = f^2 g_B \oplus b^2 g_F$, where $b : B \longrightarrow (0, \infty)$ and $f : F \longrightarrow (0, \infty)$ are smooth maps and g_B, g_F are the metric on the Riemannian manifolds B and F respectively. If either $b = 1$ or $f = 1$, but not both, then we obtain a (*single*) warped product. If both $b = 1$ and $f = 1$, then we have a product manifold. If neither b nor f is constant, then we have a non trivial doubly warped product.

If $X \in T(B)$ and $Z \in T(F)$, then the Levi-Civita connection is

$$\nabla_X Z = Z(lnf)X + X(lnb)Z. \tag{22}$$

5 Warped product Pseudo-slant submanifold of trans-Sasakian manifold

Throughout this section, we assume that $\bar{M}$ is a trans-Sasakian manifolds, with $\alpha \neq 0$ and $M = N_\perp \times_f N_\theta$ be a warped product pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Such submanifolds are always tangent to the structure vector field ξ. We distinguish 2 cases

(i) ξ tangent to $N_\perp$,
(ii) ξ tangent to N_θ.

Note (ii). By lemma (3.4), in case of trans-Sasakian manifold with $\alpha \neq 0$, The slant distribution D_θ is not integrable, due to this there does not exist slant submanifold N_θ, but $D_\theta \oplus < \xi >$ is integrable, so in view of this remark we cannot take ξ tangential to $N_\perp$, hence there is only possibility of case (ii).

In view of above note we have the following result.

Theorem 5.1. *Let $\bar{M}$ be a trans-Sasakian manifold, $\alpha \neq 0$, then there do not exist warped product submanifolds $M = N_\perp \times_f N_\theta$ in $\bar{M}$ such that $N_\perp$ is anti-invariant submanifold and N_θ is a proper slant submanifold of $\bar{M}$, ξ tangent to N_θ.*

Proof. By equation (21),

$$\nabla_X Z = \nabla_Z X = (Zlnf)X \tag{23}$$

for any vector fields $X \in N_\theta$ and $Z \in N_\perp$.

Now, for $\xi \in N_\theta$

$$\nabla_Z \xi = (Zlnf)\xi \tag{24}$$

Also, by equation (3) and (6), we have

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi) \tag{25}$$

$$\nabla_Z \xi + h(Z, \xi) = -\alpha FZ + \beta Z \tag{26}$$

from (26), we get

$$\nabla_Z \xi = \beta Z \tag{27}$$

$$h(Z, \xi) = -\alpha FZ \tag{28}$$

Thus (24) and (27) imply $(Zlnf) = 0$, for all $Z \in N_\perp$. Which shows that f is constant, thus proof is complete. □

6 Pseudo-slant Warped product submanifolds of trans-Sasakian manifold

In this case we will study pseudo-slant warped product submanifolds $N_\theta \times_f N_\perp$, there are also two cases.

(i) ξ tangent to $N_\perp$,
(ii) ξ tangent to N_θ.

In view of note ii, we cannot take ξ tangential to $N_\perp$, So, the remaining case is case (ii).

Theorem 6.1. *Let $\bar{M}$ be trans-Sasakian manifold, with $\alpha \neq 0$, then there exit $M = N_\theta \times_f N_\perp$ pseudo-slant warped product submanifold, such that N_θ is a proper slant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of $\bar{M}$.*

Proof. For any vector fields $X \in N_\theta$ and $Z \in N_\perp$, using equation(21), for $\xi \in N_\theta$ we have

$$\nabla_Z \xi = (\xi lnf)Z. \tag{29}$$

Now, by structure equation(3) and using(6), we have

$$\nabla_Z \xi = \beta Z \tag{30}$$

$$h(Z, \xi) = -\alpha FZ \tag{31}$$

These equation imply, $\xi lnf = \beta$ for all $Z \in N_\perp$. Therefore in this case warped product do exist. □

In particular, we can obtain an example of pseudo-slant submanifold in the setting of Kenmotsu manifold as follows.

Example 6.2. *Consider the complex space C^4 with the Usual Kaehler Structure and real global coordinates $(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4)$. Let $\bar{M} = R \times_f C^4$ be the warped product between the real line R and C^4, where the warping function e^z, where z being the global coordinates in R, then $\bar{M}$ is a Kenmotsu manifold [8]. Now defining orthogonal basis*

$e_1 = \partial/\partial x^1$, $e_2 = \partial/\partial y^3$, $e_3 = \cos\theta\ \partial/\partial y^4 - \sin\theta\ \partial/\partial x^4$, $e_4 = \cos\theta\ \partial/\partial x^4 + \sin\theta\ \partial/\partial y^4$ *and* $e_5 = \partial/\partial z$

Distribution $D_\theta =< e_3, e_4 >$ and $D^\perp =< e_5, e_1, e_2 >$ are integrable and denoted by N_θ and $N_\perp$, then $M = N_\perp \times_f N_\theta$ is a pseudo-slant warped product submanifold isometrically immersed in $\bar{M}$, here the warping function is $f = e^z$

7 Doubly warped product Pseudo-slant submanifold of trans-Sasakian manifold

In this section, we will study doubly warped product pseudo-slant submanifold of trans-Sasakian manifold, as earlier in view of note ii, there is only case in which we can take ξ tangential to N_θ, in this case we have the following result.

Theorem 7.1. *Let $\bar{M}$ be trans-Sasakian manifold, with $\alpha \neq 0$, then there do not exist doubly warped product submanifolds $M = {}_{f_2}N_\perp \times {}_{f_1}N_\theta$ in $\bar{M}$, such that $N_\perp$ is anti-invariant and N_θ is proper slant submanifold of $\bar{M}$, ξ tangential to N_θ.*

Proof. Let $M = {}_{f_2}N_\perp \times {}_{f_1}N_\theta$ be doubly warped product pseudo-slant submanifold of trans-Sasakian manifold $\bar{M}$, ξ tangent to N_θ then, for any $Z \in TN_\perp$

$$\nabla_Z \xi = Z(lnf_1)\xi + \xi(lnf_2)Z \tag{32}$$

Also, by structure equation(3) and (6), we have

$$-\alpha\phi Z + \beta(Z - \eta(Z)\xi) = \bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi) \tag{33}$$

$$-\alpha FZ + \beta Z = \nabla_Z \xi + h(Z, \xi) \tag{34}$$

This means that

$$\nabla_Z \xi = \beta Z \tag{35}$$

$$h(Z, \xi) = -\alpha FZ. \tag{36}$$

Using equation (32) and (35), we get

$$Z(lnf_1)\xi + \xi(lnf_2)Z = \beta Z \tag{37}$$

By the orthogonality of two distribution, we get

$$Z(lnf_1) = 0 \tag{38}$$

$$\xi(lnf_2) = \beta \tag{39}$$

(38) yields, f_1 is constant. So, there does not exist doubly warped product pseudo-slant submanifold of the form ${}_{f_2}N_\perp \times {}_{f_1}N_\theta$, with ξ tangent to N_θ. □

8 Pseudo-slant Doubly warped product submanifold in trans-Sasakian manifold

This section is devoted to the study of pseudo-slant Doubly warped product submanifold in trans-Sasakian manifold. In this section, we have obtained the nonexistence of pseudo-slant warped product submanifold, the main result of this section is given as follows

Theorem 8.1. *There is no proper pseudo-slant doubly warped product submanifolds in trans-Sasakian manifolds, with $\alpha \neq 0$.*

Proof. Let $M =_{f_1} N_\theta \times_{f_2} N_\perp$ be a pseudo-slant Doubly warped product submanifold in trans-Sasakian manifold $\bar{M}$, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds respectively.
Due to note (ii), we can not take ξ tangential to $N_\perp$, So taking ξ tangential to N_θ, Now, for any $X \in N_\theta$ and $Z \in N_\perp$, by equation (3) and (6), we have

$$\nabla_Z \xi = \beta Z \tag{40}$$

Also, from (22), we get

$$\nabla_Z \xi = Z(lnf_1)\xi + \xi(lnf_2)Z. \tag{41}$$

It follows from (40) and (41)

$$Z(lnf_1) = 0, \tag{42}$$

$$\xi(lnf_2) = \beta \tag{43}$$

From (43),$Z(lnf_1) = 0$, shows that f is constant. Therefore there does not exist pseudo-slant doubly warped product submanifold in trans-Sasakian manifold.

□

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(Received 2 April 2009)

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