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Complete-Lattice Morphisms Compatible with Closure Operators

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Abstract : We introduce and study certain morphisms between complete lattices. We investigate three kinds of compatibility of these morphisms with closure operators on complete lattices.

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1 Introduction, preliminaries

Closure operators that are more general than the Kuratowski ones were studied by many authors and occur in numerous branches of mathematics, in particular in set theory, general topology, geometry, algebra, mathematical logic and category theory - cf. [1]-[13] and [16]. But they are utilized also in many other fields, especially in theoretical computer science - see e.g. [14] and [15]. Closure operators on a set X are, in fact, certain structures (unary operations) on the power set 2^X. And morphisms (continuous maps, closed maps, etc.) between closure spaces are usually defined to be maps for which the induced (lifted) maps between the corresponding power sets fulfill certain given axioms. Thus, when working with closure operators on a set X, we work with subsets rather than with points of X. In other words, we work with the complete lattice (Boolean algebra) 2^X. It is therefore natural to generalize the usual approach to closure operators and to consider them on an arbitrary complete lattice. Of course, we can define closure operators even on ordered or preordered sets as it is often done in general algebra. But in this paper we will study closure operators on just complete lattices because such closure operators behave similarly to the usual closure operators on sets. We will define and study certain morphisms between complete lattices. These morphisms will then be discussed with respect to closure operators on complete

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lattices. We will investigate the behavior of the morphisms with respect to regular, continuous and closed maps between complete lattices with a closure operator.

A *closure operator* on a complete lattice $L = (L, \leq)$ with the least element 0 is a map $u : L \to L$ which is

(0) grounded (i.e., u(0)=0),

(1) extensive (i.e., $\forall x \in L: \ x \leq u(x)$),

(2) monotone (i.e., $\forall x, y \in L: \ x \leq y \Rightarrow u(x) \leq u(y)$),

(3) idempotent (i.e., $\forall x \in L: \ u(u(x)) = x$).

The pair (L, u) where L is a complete lattice and u is a closure operator on L is called a *closure system*. An element $x \in L$ is said to be *closed* if it is a fixed point of u, i.e., if $u(x) = x$.

2 Complete-lattice morphisms

¿From now on, $L = (L, \leq)$ and $L' = (L', \leq')$ will be complete lattices with the least elements denoted by 0 and 0' and the joins denoted by $\bigvee$ (or $\vee$) and $\bigvee'$ (or $\vee'$) respectively.

Definition 2.1. Let $f : L \to L'$ be a map. The map $f^{-1} : L' \to L$ given by $f^{-1}(x') = \bigvee\{x;\ f(x) = x'\}$ whenever $x' \in L'$ is called the *inversion* of f. An element $x' \in L'$ is said to be *invertible* if $f^{-1}(x') > 0$.

Thus, x' is not invertible if $x = 0$ is the only element with $f(x) = x'$ or if there is no such an element x.

Definition 2.2. A map $f : L \to L'$ is said to be a *morphism* if the following four conditions are satisfied whenever $x, y \in L$ and $x', y' \in L'$:

M0: $f(x) = 0' \Rightarrow x = 0$ (*conservativity*);

M1: $x \leq y \Rightarrow f(x) \leq' f(y)$ (*monotonicity*);

M2: $f(f^{-1}(y')) \leq y'$ whenever y' is invertible (*inverse consistency*);

M3: $x' \leq' y' \Rightarrow f^{-1}(x') \leq f^{-1}(y')$ (*inverse monotonicity*).

If f is monotone and y' invertible, then from $f(y) = y'$ it follows that $y \leq f^{-1}(y')$ and thus $y' = f(y) \leq' f(f^{-1}(y'))$. Therefore, M1 and M2 imply that $f(f^{-1}(y')) = y'$ for all invertible elements $y' \in L'$. Thus, a monotone map $f : L \to L'$ is inverse consistent if and only if, for every invertible element $y' \in L'$, there exists a greatest element $y \in L$ with $f(y) = y'$. The inverse monotonicity implies that, whenever $y' \in L'$ is invertible and $y' \leq' z'$, z' is invertible too. And conversely, x' is not invertible whenever there is a non-invertible element y' with $x' \leq' y'$.

If f is a morphism, then $f(x) \leq' y'$ is equivalent to $x \leq f^{-1}(y')$ whenever y' is invertible. Thus, f and f^{-1} constitute a covariant Galois connection (cf. [14]) between L and the set of all invertible elements of L'.

Example 2.3. Figure 1 illustrates 5 maps $f_k : 2^X \to 2^{X'}$, $k = 1, ..., 5$, where $X = \{a, b, c\}$, $X' = \{x, y, z\}$ for $k = 1, 2, 3$ and $X' = \{x, y\}$ for $k = 4, 5$. Both power sets 2^X and $2^{X'}$ are considered to be complete lattices with respect to set inclusion.

$Y \in 2^X$	$f_1(y)$	$f_2(y)$	$f_3(y)$	$f_4(y)$	$f_5(y)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\{xy\}$	$\{xz\}$	$\{xy\}$	$\{x\}$	$\emptyset$
$\{b\}$	$\{x\}$	$\{yz\}$	$\{xy\}$	$\{y\}$	$\{x\}$
$\{c\}$	$\{xz\}$	$\{yz\}$	$\{xy\}$	$\{x\}$	$\{y\}$
$\{ab\}$	$\{xyz\}$	$\{xyz\}$	$\{xy\}$	$\{xy\}$	$\{x\}$
$\{ac\}$	$\{xyz\}$	$\{xy\}$	$\{xy\}$	$\{x\}$	$\{y\}$
$\{bc\}$	$\{xyz\}$	$\{yz\}$	$\{xyz\}$	$\{xy\}$	$\{xy\}$
$\{abc\}$	$\{xyz\}$	$\{xyz\}$	$\{xyz\}$	$\{xy\}$	$\{xy\}$
	$M0$	$M0$	$M0$	$M0$	$\neg M0$
	$M1$	$\neg M1$	$M1$	$M1$	$M1$
	$M2$	$M2$	$\neg M2$	$M2$	$M2$
	$M3$	$M3$	$M3$	$\neg M3$	$M3$

Figure 1: Five maps between power sets

The leftmost column denotes the 8 elements of 2^X. The other columns denote the images of each of the elements under f_k, $k = 1, ..., 5$. Only f_1 is a morphism. One can verify that it satisfies M0, M1, M2 and M3 by exhaustive examination. Clearly, f_2 is not monotone because $f_2(\{a\}) \not\subseteq f_2(\{a, c\})$. The map f_3 is not inverse consistent because $f_3^{-1}(\{x, y\}) = \{a, b, c\}$ but $f_3(\{a, b, c\}) = \{x, y, z\}$. Further, f_4 is not inverse monotone because $f_4^{-1}(\{x\}) = \{ac\} \not\subseteq \emptyset = f_4^{-1}(\{x, z\})$. Finally, f_5 is not conservative because $f(\{a\}) = \emptyset$. But one can easily see that each of the maps f_k, $k = 2, ..., 5$, satisfies the remaining three of the four conditions M0, M1, M2 and M3. Therefore, the conditions M0, M1, M2 and M3 are independent.

Observe that inverse monotonicity of f implies $f^{-1}(x') = \bigvee\{x;\ f(x) \leq x'\}$ for every $x' \in X'$ because $f(x) \leq x' \Rightarrow f^{-1}(f(x)) \leq f^{-1}(x') \Rightarrow x \leq f^{-1}(x')$. Note also that the condition $f(0) = 0'$ is not required to be fulfilled by a morphism f. Indeed, such a condition would be quite restrictive because it is not satisfied by numerous fundamental examples of morphisms - see e.g. the following example.

Example 2.4. Let $f : [0, 1] \to [0, 1]$ be a non-decreasing continuous (real) function with $f(x) > 0$ for all $x \in (0, 1]$. Consider the natural linear order $\leq$ on $[0, 1]$, so that $([0, 1], \leq)$ is a complete lattice. It can easily be seen that $f : ([0, 1], \leq) \to ([0, 1], \leq)$ is a morphism.

A map $f : L \to L'$ is said to be *additive* if we have $f(y' \vee z') = f(y') \vee f(z')$ whenever $y', z' \in L'$. Clearly, each additive map is monotone. Of course, a morphism need not be additive. For example, the morphism f_1 in Fig.1 of Example 2.3 is not additive because $f_1(\{a\}) \cup f_1(\{b\}) = \{x, y\} \subset \{x, y, z\} = f_1(\{a, b\})$. On the other hand, the morphisms f from Example 2.4 is additive.

Proposition 2.5. *Let $f : L \to L'$ be an additive, conservative and inverse consistent map. Then the following conditions are equivalent:*

(a) f is a morphism,

(b) f is inverse monotone,

(c) if $y', z' \in L'$ are elements with $y' \leq' z'$ and y' invertible, then z' is invertible.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is trivial.
$(b) \Rightarrow (c)$: Let f be inverse monotone and let $y', z' \in L'$, $y' \leq' z'$, $f^{-1}(y') \neq 0$. Then $0 \neq f^{-1}(y') \leq f^{-1}(z')$, hence z' is invertible.
$(c) \Rightarrow (b)$: Let f fulfil the condition (c) and let $y', z' \in L'$ be subsets with $y' \leq z'$. If y' is not invertible, i.e., $f^{-1}(y') = 0$, then $f^{-1}(y') \leq f^{-1}(z')$ trivially holds. Suppose that y is invertible, i.e., that $f^{-1}(y') > 0$. Then, since f is additive and both y' and z' are invertible, we have $f(f^{-1}(y') \vee f^{-1}(z')) = y' \vee' z' = z'$. Consequently, $f^{-1}(y') \vee f^{-1}(z') \leq f^{-1}(z')$, which means that $f^{-1}(y') \leq f^{-1}(z')$. Thus, f is inverse monotone. □ □

Lemma 2.6. *Let $L'' = (L'', \leq'')$ be a complete lattice, let $f : L \to L'$ and $g : L' \to L''$ be inverse monotone maps and let g be moreover inverse consistent. Then $(f \circ g)^{-1}(y'') = f^{-1}(g^{-1}(y''))$ for each $y'' \in L''$.*

Proof. Let $y'' \in L''$ and $y \in L$ be elements with $g(f(y)) = y''$. Then $f(y) \leq' g^{-1}(g(f(y))) \leq' g^{-1}(y'')$ because g is inverse monotone. Now, since also f is inverse monotone, we get $y \leq f^{-1}(f(y)) \leq f^{-1}(g^{-1}(y''))$. Consequently, $(g \circ f)^{-1}(y'') = \bigvee\{y \in L;\ g(f(y)) = y''\} \leq f^{-1}(g^{-1}(y''))$. Further, for an arbitrary subset $z \in L$, $f(z) = g^{-1}(y'')$ implies $g(f(z) = g(g^{-1}(y'')) = y''$ by the inverse consistency of g. Using this fact we obtain $f^{-1}(g^{-1}(y'')) = \bigvee\{z \in L;\ f(y) = g^{-1}(y'')\} \leq \bigvee\{z \in L;\ g(f(z)) = y''\} = (g \circ f)^{-1}(y'')$. Thus, the equality $(g \circ f)^{-1}(y'') = f^{-1}(g^{-1}(y''))$ holds. □ □

Proposition 2.7. *Let $L'' = (L'', \leq'')$ be a complete lattice and let $f : L \to L'$ and $g : L' \to L''$ be morphisms. Then the composition $f \circ g : L \to L''$ is a morphism, too.*

Proof. Obviously, the conservativity and monotonicity of maps f, g imply the conservativity and monotonicity of $f \circ g$.
To prove the inverse consistency of $g \circ f$, let $y'' \in L''$ be an invertible element (w.r.t. $g \circ f$). Then there is an element $y \in L$, $y > 0$, with $g(f(y)) = y''$. Then $f(y) \leq' g^{-1}(g(f(y))) = g^{-1}(y'')$ and so y'' is invertible (w.r.t. g) because f is conservative. Further, as f is inverse monotone, we have $y \leq f^{-1}(f(y)) \leq f^{-1}(g^{-1}(y''))$. Therefore, $g^{-1}(y'')$ is invertible (w.r.t. f). Now we get $g(f(g \circ f)^{-1}(y'')) = g(f(f^{-1}(g^{-1}(y'')))) = g(g^{-1}(y'')) = y''$ by Lemma 2.6 and by the inverse consistency of f and g. We have shown that $f \circ g$ is inverse consistent.

Finally, let $x'', y'' \in L''$, $x'' \leq'' y''$. Then $g^{-1}(x'') \leq' g^{-1}(y'')$ as g is inverse monotone, and so $f^{-1}(g^{-1}(x'')) \leq f^{-1}(g^{-1}(y''))$ as f is inverse monotone. Thus, $(g \circ f)^{-1}(x'') \leq (f \circ g)^{-1}(y'')$ by Lemma 2.6. Hence, $g \circ f$ is inverse monotone and the proof is complete. □ □

Since the identity maps are clearly morphisms, the previous Proposition implies that complete lattices and morphisms constitute a category. But it is not the aim of the present paper to study this category.

Example 2.8. Let X, X' be sets and $f : X \to X'$ be a map. Then f induces a map $f^+ : (2^X, \subseteq) \to (2^{X'}, \subseteq)$ given by $f^+(A) = \{f(x);\ x \in A\}$ whenever $A \in 2^X$. Clearly, f^+ fulfills condition M0-M2. Further, we may define a map $f^- : 2^{X'} \to 2^X$ by putting $f^-(A') = \{x \in X;\ f(x) \in A'\}$ whenever $A' \in 2^{X'}$. Observe that f^+ is inverse monotone (and hence a morphism) if and only if f is a surjection. Then $(f^+)^{-1}(A') = f^-(A')$. It can easily be seen that, for a given map $g : 2^X \to 2^{X'}$, there is a map $f : X \to X'$ with $g = f^+$ if and only if g is completely additive (which means that $g(\bigcup\{B;\ B \in \mathcal{B}\}) = \bigcup\{g(B);\ B \in \mathcal{B}\}$ whenever $\mathcal{B} \subseteq 2^X$) and preserves singletons (which means that $g(A)$ is a singleton whenever $A \subseteq X$ is a singleton).

3 Regular, continuous and closed morphisms

Definition 3.1. Let (L, u) and (L', u') be closure systems. A map $f : L \to L'$ is said to be

a) *regular* if, for every $x \in L$, $f(u(x)) = f(x)$ whenever $f(x) \in L'$ is closed;
b) *continuous* if, for every $x \in L$, $f(u(x)) \leq' u'(f(x))$;
c) *closed* if $f(x) \in L'$ is closed whenever $x \in L$ is closed.

Theorem 3.2. *Let (L, u) and (L', u') be closure systems and $f : L \to L'$ a morphism. Then the following conditions are equivalent:*

(1) f is regular,
(2) f is continuous,
(3) $\forall y' \in L' : u(f^{-1}(y')) \leq f^{-1}(u'(y'))$,
(4) $\forall y' \in L' :\ y' closed \Rightarrow f^{-1}(y')$ closed.

Proof. (2) $\Rightarrow$ (1) : Let f be continuous and let $y \in L$ be an element with $f(y)$ closed. Then $f(u(y)) \leq' u'(f(y))$ implies $f(u(y)) \leq' f(y)$. As $y \leq u(y)$, we have $f(y) \leq' f(u(y))$ by monotonicity. Therefore, $f(u(y)) = f(y)$. Thus, f is regular.

(1) $\Rightarrow$ (4) : Let f be regular and let $y' \in L'$ be closed. If y' is not invertible, then $f^{-1}(y') = 0$ is closed. Suppose that y' is invertible and let $y = f^{-1}(y')$. Since $f(f^{-1}(y')) = y'$ and f is regular, we have $f(u(f^{-1}(y'))) = f(f^{-1}(y')) = y'$. Thus, $u(f^{-1}(y')) \leq f^{-1}(f(u(f^{-1}(y') \leq f^{-1}(y')$ and, therefore, $f^{-1}(y')$ is closed.

(4) $\Rightarrow$ (2) : Suppose that, for every $y' \in L'$, $f^{-1}(y')$ is closed whenever y' is closed. Let $x \in L$ be an element. If $x = 0$, then we clearly have $f(u(x)) \leq' u'(f(x))$. Let $x > 0$. As $f(x)$ is invertible, $u'(f(x))$ is invertible too because

$f^{-1}(u'(f(x))) \geq f^{-1}(f(x)) \geq x > 0$ (by the inverse monotonicity). Since $u'(f(x))$ is closed, $f^{-1}(u'(f(x)))$ is closed too. We have $x \leq f^{-1}(f(x)) \leq f^{-1}(u'(f(x)))$ (by inverse monotonicity). Consequently, $u(x) \leq f^{-1}(u'(f(x)))$, which gives $f(u(x)) \leq' f(f^{-1}(u'(f(x)))) = u'(f(x))$ Thus, f is continuous.

(2) $\Rightarrow$ (3) : Let f be continuous and let $y' \in L'$. If y' is not invertible, then $u(f^{-1}(y')) = 0$ so that $u(f^{-1}(y')) \leq f^{-1}(u'(y'))$ trivially holds. Let y' be invertible. Then $f(u(f^{-1}(y'))) \leq' u'(f(f^{-1}(y'))) = u'(y')$. Hence, $u(f^{-1}(y')) \leq f^{-1}(f(u(f^{-1}(y)))) \leq f^{-1}(u'(y'))$.

(3) $\Rightarrow$ (4) : Suppose that $u(f^{-1}(y')) \leq f^{-1}(u'(y'))$ for every $y' \in L'$ and let $y' \in L'$ be a closed element. Then $u(f^{-1}(y')) \leq f^{-1}(y')$, which means that $f^{-1}(y')$ is closed. □ □

Example 3.3. A *closure space* is a pair (X, u) where X is a set and u is a closure operator on the complete lattice 2^X. A closure spaces (X, u) such that u is additive (i.e., $u(A \cup B) = u(A) \cup u(B)$ whenever $A, B \in 2^X$) is called a *topological space*. Given closure or topological spaces (X, u) and (X', u'), a map $f : X \to X'$ is said to be *continuous* (resp. *closed*) if the map $f^+ : 2^X \to 2^{X'}$ is continuous (resp. closed) (cf. [10]). It is well known that, when replacing f^{-1} by f^- in conditions (3) and (4) of the previous statement, each of the two conditions is equivalent to the continuity of f. Of course, regularity of f^+ is in general weaker than continuity of f^+ (f^+ need not be a morphism). But, if f is a surjection, then f^+ is regular if and only if f^+ is continuous (by Example 2.8 and Theorem 3.2).

Proposition 3.4. *Let (L, u), (L', u') and (L'', u'') be closure systems and let $f : L \to L'$, $g : L' \to L''$ be maps, g monotone. If both f and g are continuous, then so is $g \circ f$.*

Proof. We have $f(u(x)) \leq' u'(f(x))$ for every $x \in L$ and $g(u'(y)) \leq'' u''(g(y))$ for every $y \in L'$. Consequently, as g is monotone, $g(f(u(x))) \leq'' g(u'(f(x))) \leq'' u''(g(f(x)))$. This means that $g \circ f$ is continuous. □ □

Proposition 3.5. *Let (L, u), (L', u') and (L'', u'') be closure systems and let $f : L \to L'$, $g : L' \to L''$ be morphisms. If both f and g are regular, then so is $g \circ f$.*

Proof. The statement follows from Theorem 3.2 and Proposition 3.4. □ □

Clearly, an identity map between closure systems is both continuous and regular. Thus, by Propositions 2.7, 3.4 and 3.5, closure systems with continuous monotone maps form a category and so do closure systems with regular morphisms. By Theorem 3.2, the latter category is a subcategory of the former one.

Proposition 3.6. *Let (L, u), (L', u') be closure systems and let $f : L \to L'$ be a monotone map. Then f is closed if and only if $u'(f(x)) \leq' f(u(x))$ for all $x \in L$.*

Proof. Let f be closed and let $x \in X$. By monotonicity, $x \leq u(x)$ implies $f(x) \leq' f(u(x))$. But, because $u(x)$ is closed and also f is closed, $u'(f(x)) \leq' f(u(x))$.
Conversely, let all elements $x \in L$ fulfill $u'(f(x)) \leq' f(u(x))$ and let $y \in L$ be a

closed element. Then $u'(f(y)) \leq' f(y)$. But, we also have $f(y) \leq' u'(f(y))$, so that equality holds. □ □

Now, Proposition 3.6 and Definition 3.1b) clearly give

Corollary 3.7. *Let (L, u), (L', u') be closure systems and let $f : L \to L'$ be a monotone map. Then f is closed and continuous if and only if $f(u(x)) = u'(f(x))$ for all $x \in L$.*

Corollary 3.8. *Let (L, u), (L', u') be closure systems and let $f : L \to L'$ be a morphism. Then f is closed and regular if and only if $f(u(x)) = u'(f(x))$ for all $x \in L$.*

Proof. The statement follows from Corollary 3.7 and Theorem 3.2. □

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