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# A New Representation Formula for the Factorial Function

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**Abstract :** We discuss in this paper a double inequality related with the factorial function due to J. Sandor and L. Debnath [*On certain inequalities involving the constant e and their applications* J. Math. Anal. Appl. 249 (2000) 569-582]. We establish here an asymptotic exapansion, leading to a new accurate approximation formula which provides all exact digits of $n!$, for every $n \leq 28$.

**Keywords :** Factorial function, Stirling's formula, Burnside's formula, Speed of convergence.
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## 1 Introduction

J. Sandor and L. Debnath proved in [11] the following double inequality related to the Stirling's formula:

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n}} < n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1}}. \tag{1.1}$$

In [3], S. Guo rediscovered this formula and other similar estimations was further established.

In the recent paper [2], N. Batir refined and extended inequalities (1.1) to the form

$$\alpha_n = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} < n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}} = \beta_n, \tag{1.2}$$

with the sharp constants $\alpha = 1-2\pi e^{-2}$ and $\beta = 1/6$. It is also verified by numerical methods that the approximation with $\beta = 1/6$,

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}}$$

is very performant, for example it is stronger than the Burnside's formula (see, *e.g.*, [1]):

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \gamma_n.$$

The sequence $\beta_n$ defined in (1.2) is known to be one of the most accurate approximations of large factorials, having a simple form. In fact, one can show that $n! < \beta_n < \gamma_n$.

## 2 Main Results

We are interested in finding a performant approximation formula of the form

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\lambda_n\sqrt{n}} \tag{2.1}$$

by studying the behaviour of the sequence

$$\lambda_n = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{n!\sqrt{n}},$$

which satisfies the approximation (2.1) with equality.

For the sequence $a_n = \ln \lambda_n$, we will use an elementary lemma and the Maple software to give the following iterated speed of convergence:

$$\lim_{n\to\infty} n^2\left(n^2\left(n^2\left(n^2\left(n^2\left(n^2\left(na_n+\frac{1}{12}\right)-\frac{1}{360}\right)+\frac{1}{1260}\right)-\frac{1}{1680}\right)+\frac{1}{1188}\right)-\frac{691}{360360}\right)=\frac{-1}{156}. \tag{2.2}$$

From here it results that there exists a sequence $(\theta_n)_{n\geq 1}$, convergent to 1, such that

$$n^2\left(n^2\left(n^2\left(n^2\left(n^2\left(n^2\left(na_n+\frac{1}{12}\right)-\frac{1}{360}\right)+\frac{1}{1260}\right)-\frac{1}{1680}\right)+\frac{1}{1188}\right)-\frac{691}{360360}\right)=\frac{-\theta_n}{156},$$

then, by succesive computations, we deduce that

$$a_n = -\frac{1}{12n}+\frac{1}{360n^3}-\frac{1}{1260n^5}+\frac{1}{1680n^7}-\frac{1}{1188n^9}+\frac{691}{360360n^{11}}-\frac{\theta_n}{156n^{13}}.$$

By exponentiating and by replacing in (2.1), we obtain the approximation formula:

$$n! = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n\exp\left(-\frac{1}{6n}+\frac{1}{180n^3}-\frac{1}{630n^5}+\frac{1}{840n^7}-\frac{1}{594n^9}+\frac{691}{180180n^{11}}-\frac{\theta_n}{78n^{13}}\right)}}. \tag{2.3}$$

If we make use of the approximation $\exp t \approx t+1$, near the origin, then we obtain

$$n! = \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n - \frac{1}{6} + \frac{1}{180n^2} - \frac{1}{630n^4} + \frac{1}{840n^6} - \frac{1}{594n^8} + \frac{691}{180180n^{10}} - \frac{\theta_n}{78n^{12}}}}, \tag{2.4}$$

which also proves the sharpness of the constant $\beta = 1/6$ in the double inequality (1.2). Better approximations can be obtained from (2.3) using $\exp t \approx 1 + t + t^2/2! + t^3/3! + \cdots$ instead $\exp t \approx 1 + t$.

Furthermore, by replacing $\theta_n$ by 1 in equalities (2.3)-(2.4), we obtain the approximation formula

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n - \frac{1}{6} + \frac{1}{180n^2} - \frac{1}{630n^4} + \frac{1}{840n^6} - \frac{1}{594n^8} + \frac{691}{180180n^{10}} - \frac{1}{78n^{12}}}} = \tau_n \tag{2.5}$$

and much performant approximation formula

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n \exp\left(-\frac{1}{6n} + \frac{1}{180n^3} - \frac{1}{630n^5} + \frac{1}{840n^7} - \frac{1}{594n^9} + \frac{691}{180180n^{11}} - \frac{1}{78n^{13}}\right)}} = \kappa_n. \tag{2.6}$$

For example, by considering only the first terms from (2.5), we have already obtain a stronger result than (1.2), namely

$$n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n - \frac{1}{6} + \frac{1}{180n^2}}} < \beta_n. \tag{2.7}$$

The approximation $\kappa_n$ from (2.6) is very accurate, as we can see from the following table:

| $n!$ | $\kappa_n$ |
|---|---|
| 5 | $5! + 9.4364 \times 10^{-11}$ |
| 7 | $7! + 2.7996 \times 10^{-11}$ |
| 10 | $10! + 1.0118 \times 10^{-10}$ |
| 15 | $15! + 8.5941 \times 10^{-8}$ |
| 20 | $20! + 2.1613 \times 10^{-3}$ |
| 23 | $23! + 2.8327$ |
| 26 | $26! + 7042.1$ |
| 28 | $28! + 1.7538 \times 10^{6}$ |

If we remember that the last digits of the factorials are zeroes (28! ends with six zeroes), then it results from this table that the formula $\kappa_n$ gives all exact digits of the factorials, for every $n \leq 28$.

# 3 The Proofs

The speed of convergence (2.2) will be calculated using the following

**Lemma 3.1 (Stolz-Cesaro).** *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers convergent to zero, $(b_n)_{n\geq 1}$ strictly decreasing, such that exists the limit*

$$l = \lim_{n\to\infty} \frac{a_n - a_{n+1}}{b_n - b_{n+1}}.$$

*Then*

$$l = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

A similar form of this lemma was recently used by Mortici [4]-[10] to accelerate some convergences or to construct asymptotic series.

Now, we compute succesively the limits from (2.2) using the Lemma 3.1 and the identity

$$a_n - a_{n-1} = \left(n - \frac{1}{2}\right) \ln \frac{n}{n-1} - 1.$$

First,

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{\frac{1}{n} - \frac{1}{n-1}} =$$

$$= \lim_{n\to\infty} \frac{\left(n - \frac{1}{2}\right) \ln \frac{n}{n-1} - 1}{\frac{1}{n} - \frac{1}{n-1}} = -\frac{1}{12}.$$

Then

$$\lim_{n\to\infty} n^2 \left(na_n + \frac{1}{12}\right) = \lim_{n\to\infty} \frac{na_n + \frac{1}{12}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{a_n + \frac{1}{12n}}{\frac{1}{n^3}} =$$

$$= \lim_{n\to\infty} \frac{\left(a_n + \frac{1}{12n}\right) - \left(a_{n-1} + \frac{1}{12(n-1)}\right)}{\frac{1}{n^3} - \frac{1}{(n-1)^3}} =$$

$$= \lim_{n\to\infty} \frac{\left(n - \frac{1}{2}\right) \ln \frac{n}{n-1} - 1 + \frac{1}{12n} - \frac{1}{12(n-1)}}{\frac{1}{n^3} - \frac{1}{(n-1)^3}} = \frac{1}{360}. \tag{3.1}$$

Hence

$$\lim_{n\to\infty} n^2 \left(n^2 \left(na_n + \frac{1}{12}\right) - \frac{1}{360}\right) = \lim_{n\to\infty} \frac{n^2 \left(na_n + \frac{1}{12}\right) - \frac{1}{360}}{\frac{1}{n^2}} =$$

$$= \lim_{n\to\infty} \frac{a_n + \frac{1}{12n} - \frac{1}{360n^3}}{\frac{1}{n^5}} =$$

$$= \lim_{n\to\infty} \frac{\left(a_n + \frac{1}{12n} - \frac{1}{360n^3}\right) - \left(a_{n-1} + \frac{1}{12(n-1)} - \frac{1}{360(n-1)^3}\right)}{\frac{1}{n^5} - \frac{1}{(n-1)^5}} =$$

$$\lim_{n\to\infty} \frac{\left(n - \frac{1}{2}\right) \ln \frac{n}{n-1} - 1 + \frac{1}{12n} - \frac{1}{360n^3} - \left(\frac{1}{12(n-1)} - \frac{1}{360(n-1)^3}\right)}{\frac{1}{n^5} - \frac{1}{(n-1)^5}} = -\frac{1}{1260}. \tag{3.2}$$

The other iterated limits from (2.2) can be obtained similarly. Here, we calculated the limits (3.1)-(3.2) and their higher order counterparts using the Maple software.

It seems that our ideas and Lemma 3.1 can have many other practical applications. In this way, elementary methods for asymptotic analysis can be developed, which will replace the well known methods using convexity or variation of some involved auxiliary functions.

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