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The convolution product of $\delta^{(k)}(|x|-c) * \delta^{(l)}(|x|-c)$

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Abstract : In this article, we give a sense the Fourier transform of $\delta^{(k)}(r-c)$ and consequently we obtain a expasion type Taylor of $\delta^{(k)}(|x|-c)$ and the mains results is the convolution of $\delta^{(k+2j-1)}(|x|-c) * \delta^{(n+2l-1)}(|x|-c)$.

1 Introduction

Let $x = (x_1, ..., x_n)$ a point of $n-$dimensional Euclidean space in $\mathbb{R}^n$. Let $\delta(r-c)$ be the singular generalized function which corresponds to a uniform mass distribution of unit density on the sphere O_c of radius c centered at the origin, when

$$r = |x| = \sqrt{x_1^2 + ... + x_n^2} \tag{1.1}$$

In other words

$$< \delta(r-c), \varphi >= \int_{O_c} \varphi(x)dx \ , \qquad ([1], \text{ page } 198)$$

for all φ in D, where D is the set of c^∞ functions with compact support. (1.2)

Considering that $\delta(r-c)$ has bounded support, the Fourier transform of $\delta(r-c)$ is given by the following formulae

$$\begin{aligned} \{\delta(r-c)\} &=< \delta(r-c), e^{i<x,\sigma>} >= \\ &= \int_{U_a} e^{i<x,\sigma>} dx \qquad ([1], \text{ page198}) \end{aligned} \tag{1.3}$$

where $< x, \sigma >= x_1\sigma_1 + ... + x_n\sigma_n$, $U_a = O_c$ is the sphere of radius $a = c$.

From ([1],page198), formula (1.2), we have,

$$\{\delta(r-c)\} = \frac{\Omega_{n-1} c^{\frac{n}{2}} \rho^{1-\frac{n}{2}}}{N_n} J_{\frac{n-2}{2}}(c.\rho) \tag{1.4}$$

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where

$$N_n = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \quad , \tag{1.5}$$

$$\Omega_{n-1} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \quad , \tag{1.6}$$

$J_\lambda(z)$ is the Bessel functions the first kind,

$$J_\lambda(z) = (\frac{z}{2})^\lambda \sum_{i\geq 0} \frac{(-1)^i(\frac{z}{2})^{2i}}{i!\Gamma(\lambda+i+1)} \tag{1.7}$$

and

$$\rho = |\sigma| = \sqrt{\sigma_1^2 + ... + \sigma_n^2}. \tag{1.8}$$

In this article, we give a sense the Fourier transform of $\delta^{(k)}(r-c)$ and consequently we obtain a expasion type Taylor of $\delta^{(k)}(|x|-c)$ and the mains results is the convolution of

$$\delta^{(k+2j-1)}(|x|-c) * \delta^{(n+2l-1)}(|x|-c). \tag{1.9}$$

To obtain our results, we need the following formulae:

$$\begin{aligned} < \delta^{(k)}(r-c), \varphi >= \int \delta^{(k)}(r-c)\varphi dx = \\ = \tfrac{(-1)^k}{c^{n-1}} \int_{O_c} [\tfrac{\partial^k}{\partial r^k}(\varphi r^{n-1})]_{r=c} dO_c \end{aligned} \tag{1.10}$$

([1],page231,formula(1.10)) where dO_c is the Euclidean element of area of it.

2 The Fourier transform of $\delta^{(k)}(r-c)$

Using the formula (1.3) the Fourier transform of $\delta^{(k)}(r-c)$ is given by following formula

$$\begin{aligned} \{\delta^{(k)}(r-c)\} =< \delta(r-c), e^{i<x,\sigma>} >= \\ = \textstyle\int_{O_c} \delta^{(k)}(r-c)e^{i<x,\sigma>}dx. \end{aligned} \tag{2.1}$$

Now using the definition (1.10) and making spherical coordinates ($r = |x|, \rho = |\sigma|$), where θ is the angle between the x and σ vectors) this becomes

$$\begin{aligned} \{\delta^{(k)}(r-c)\} = \tfrac{(-1)^k}{c^{n-1}} \int [\tfrac{\partial^k}{\partial r^k}(e^{ir\rho\cos\theta}r^{n-1})]_{r=c} c^{n-1} \sin^{n-2}\theta d\theta d\omega = \\ = (-1)^k \Omega_{n-1} \textstyle\int_0^\pi [\tfrac{\partial^k}{\partial r^k}(e^{ir\rho\cos\theta}r^{n-1})]_{r=c} \sin^{n-2}\theta d\theta \end{aligned} \tag{2.2}$$

where Ω_{n-1} is defined by (1.6).

Now using the integral representation of the function $J_\nu(x)$:

$$J_\nu(x) = \frac{1}{2^\nu \sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{+ix\cos\theta} x^\nu \sin^{2v}\theta d\theta \tag{2.3}$$

([2],formula(9,2,16),page409,([3]),formula7,page953),

where $J_\nu(x)$ is defined by (1.7) we have,

$$\{\delta(|x|-c)\} = \frac{(-1)^k 2^{\frac{n-2}{2}}[\pi\Gamma(\frac{n-1}{2})]}{\rho^{\frac{n-2}{2}}}[\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}} J_{\frac{n-2}{2}}(r\rho))]_{r=c} \tag{2.4}$$

Using(1.7)and considering the formula

$$z(z-1)...(z-k+1) = \frac{\Gamma(z+1)}{\Gamma(z-k+1)} \quad \text{([4],page344)for } k=1,2,.. \tag{2.5}$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx \tag{2.6}$$

we have,

$$[\frac{\partial^k}{\partial r^k}(r^{\frac{n}{2}} J_{\frac{n-1}{2}}(r\rho))]_{r=c} = \sum_{j\geq 0} \frac{(-1)^j (\frac{\rho}{2})^{\frac{n}{2}-1+2j}\Gamma(n+2j)}{j!\Gamma(\frac{n}{2}+j)\Gamma(n+2j-k)} c^{n-1+2j-k}. \tag{2.7}$$

From (2.4) and (2.7) we obtain the formula

$$\{\delta^{(k)}(|x|-c)\} = (-1)^k 2^{\frac{n}{2}} \pi^{\frac{n}{2}} \sum_{j\geq 0} \frac{(-1)^j (\rho^2)^j \Gamma(n+2j)}{j!\Gamma(\frac{n}{2}+j)\Gamma(n+2j-k) 2^{\frac{n}{2}+2j-1}} c^{n-1+2j-k}. \tag{2.8}$$

In particular by getting $k=0$ in (2.8) we have

$$\{\delta(|x|-c)\} = \frac{c^{n-1} 2^{\frac{n}{2}} \pi^{\frac{n}{2}}}{2^{\frac{n}{2}-1}} \sum_{j\geq 0} \left(\frac{(\frac{\rho c}{2})^{2j+\frac{n}{2}-1}}{j!\Gamma(\frac{n}{2}+j)} \right) \left(\frac{\rho c}{2}\right)^{1-\frac{n}{2}}. \tag{2.9}$$

From (2.9) and using (1.7) we obtain

$$\{\delta(|x|-c)\} = c^{\frac{n}{2}} 2^{\frac{n}{2}} \pi^{\frac{n}{2}} \rho^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\rho c). \tag{2.10}$$

The formula (2.10) is the formula (1.4) and appears in([1],page198).

We observe that putting $k = n+2s-1$ in (2.8) where $s = 0,1,2...$ we obtain the following formula

$$\begin{aligned} &\{\delta^{(n+2s-1)}(|x|-c)\} = (-1)^{n+2s-1} 2^{\frac{n}{2}} \pi^{\frac{n}{2}}. \\ &\cdot \textstyle\sum_{j\geq 0} \frac{(-1)^j (\rho^2)^j \Gamma(n+2j)}{j!\Gamma(\frac{n}{2}+j)\Gamma(2j-2s+1) 2^{\frac{n}{2}+2j-1}} c^{2j-2s}. \end{aligned} \tag{2.11}$$

Using that

$$\frac{1}{\Gamma(2j-2s+1)} = 0 \text{ if } j < s, \tag{2.12}$$

from (2.11) we obtain

$$\begin{aligned} \{\delta^{(n+2s-1)}(|x|-c)\} &= (-1)^{n+2s-1} 2^{\frac{n}{2}} \pi^{\frac{n}{2}}. \\ &\cdot \textstyle\sum_{j\geq 0} \frac{(-1)^j (\rho^2)^j \Gamma(n+2j)}{j!\Gamma(\frac{n}{2}+j)\Gamma(2j-2s+1) 2^{\frac{n}{2}+2j-1}} c^{2j-2s} = \\ &= (-1)^{n+2s-1} 2^{\frac{n}{2}} \pi^{\frac{n}{2}}. \\ &\cdot \textstyle\sum_{j\geq 0} \frac{(-1)^{l+s} (\rho^2)^{l+s} \Gamma(n+2l+2s)}{(l+s)!\Gamma(\frac{n}{2}+l+s)\Gamma(2l+1) 2^{\frac{n}{2}+2(l+s)-1}} c^{2l} \end{aligned} \tag{2.13}$$

3 The Fourier transform of $(|x|-c)_+^\lambda$

Let $x = (x_1, ..., x_n)$ be a point of the $n-$dimensional Euclidean space and let be $(|x|-a)_+^\lambda$ the distribution defined by

$$(|x|-a)_+^\lambda = \begin{cases} (|x|-a)^\lambda \text{ if } |x|-a \geq 0 \\ \\ 0 \text{ if } |x|-a < 0 \end{cases} \tag{3.1}$$

The Fourier transform of $(|x|-a)_+^\lambda$ by definition is given by the following formula

$$\{(|x|-a)_+^\lambda\} = \int_{\mathbb{R}^n} e^{-i<\sigma,x>} (|x|-a)_+^\lambda dx \tag{3.2}$$

where

$$<\sigma, x> = \sigma_1 x_1 + ... + \sigma_n x_n \tag{3.3}$$

Using the formulae

$$(1-z)^\lambda = \sum_{\nu\geq 0} (-1)^\nu \binom{\lambda}{\nu} z^\nu \quad \text{if } |z| < 1 \tag{3.4}$$

where

$$\binom{\lambda}{\nu} = \frac{\Gamma(\lambda+1)}{v!\Gamma(\lambda-\nu+1)} = \frac{(-1)^v \Gamma(-\lambda+v)}{v!\Gamma(-\lambda)} \tag{3.5}$$

and taking into account spherical coordinate $r = |x|, \rho = |\sigma|$ and θ the angle between the x and σ vector this be comeswhere

$$\begin{aligned} \{(|x|-a)_+^\lambda\} &= \textstyle\sum_{j\geq 0} (-1)^j a^j \binom{\lambda}{j} \int_{\mathbb{R}^n} e^{-i<\sigma,x>} |x|^\lambda dx = \\ &= \textstyle\sum_{j\geq 0} (-1)^j a^j \binom{\lambda}{j} \Omega_{n-1}. \\ &\textstyle\int_0^\infty r^{\lambda-j+n-1} \{\int_0^\pi e^{ir|\sigma|\cos\theta} \sin^{n-2}\theta d\theta\} dr \end{aligned} \tag{3.6}$$

where Ω_{n-1} is defined by (1.6).

Now from(3.6)and using the formula(2.3), we have

$$\{(|x|-a)_+^\lambda\} = 2\pi^{\frac{n-1}{2}} 2^{\frac{n-2}{2}} \pi^{\frac{n}{2}} |\sigma|^{-\frac{n}{2}+1} \sum_{j\geq 0} (-1)^j a^j \binom{\lambda}{j} \int_0^\infty r^{\lambda-j+\frac{n}{2}} J_{\frac{n-2}{2}}(r, |\sigma|) dr. \tag{3.7}$$

Now using the formula

$$\int_0^\infty x^\mu J_v(cx)dx = \frac{2^\mu c^{-\mu-1}\Gamma(\frac{1}{2}+\frac{\mu}{2}+\frac{v}{2})}{\Gamma(\frac{1}{2}-\frac{\mu}{2}+\frac{v}{2})} \tag{3.8}$$

$Re\ \nu - 1 < Re\ \mu < \frac{1}{2}$ ([3],formula14,page 684).

We have

$$\{(|x|-a)_+^\lambda\} = 2^{\frac{n}{2}} \pi^{\frac{n}{2}} 2^{\lambda+\frac{n}{2}} \sum_{j\geq 0} (-1)^j a^j 2^{-j} \binom{\lambda}{j} \frac{\Gamma(\frac{\lambda+n-j}{2})}{\Gamma(-\frac{\lambda}{2}+\frac{j}{2})} |\sigma|^{-\lambda+j-n} \quad . \tag{3.9}$$

From (3.9) and using (3.6) we arrive at the following formula

$$\{(|x|-a)_+^\lambda\} = 2^{\frac{n}{2}} \pi^{\frac{n}{2}} \sum_{j\geq 0} \frac{\Gamma(-\lambda+j)}{j!\Gamma(-1)} 2^{\lambda+\frac{n}{2}-j} a^j \frac{\Gamma(\frac{\lambda+n-j}{2})}{\Gamma(-\frac{\lambda}{2}+\frac{j}{2})} |\sigma|^{-\lambda+j-n} \tag{3.10}$$

under the condition $\lambda \neq -n+j-2t, t = 0, 1, 2, ...$

Putting $a = 0$ in (3.10) we obtain the Fourier transform of $(|x|)^\lambda$,

$$\{(|x|)^\lambda\}^\wedge = \frac{2^{\lambda+n}\pi^{\frac{n}{2}}\Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda}{2})} |\sigma|^{-\lambda-n} \tag{3.11}$$

under the condition $\lambda \neq -n-2t, t = 0, 1, 2, ...$

The formula (3.11) appears in([1] page194 formula (1.2)).

4 The Residue of $\{(|x|-a)_+^\lambda\}$

Considering that the gamma function $\Gamma(z)$ defined by (2.6) has singularities at $z = 0, -1, -2, ...$ then

$$\Gamma(\frac{\lambda+n-j}{2}) \tag{4.1}$$

has singularities at $\lambda = -n-2t+j, t = 0, 1, 2, ..$

On the other hand the function

$$|x|^\lambda = r^\lambda \tag{4.2}$$

where λ is a complex number, has singularities at $\lambda = -n, n-2, ..$ In the points $\lambda = -n-2t+j$ for $t = 0, 1, 2, ...$ and $j = 0, 1, 2, ...$the equation (3.10) has singularities in the factor $\Gamma(\frac{\lambda+n-j}{2})$ but not in $|\sigma|^{-\lambda-n+j}$. Therefore the residue of $\{(|x|-a)_+^\lambda\}$ at $\lambda = -n-2t+j$ depend of the factor $\Gamma(\frac{\lambda+n-j}{2})$.

From (3.10) and considering (4.1) we have

$$\underset{\lambda=-n-2s}{Res}\{(|x|-a)_+^\lambda\} = 2^{\frac{n}{2}}\pi^{\frac{n}{2}}\sum_{j\geq 0}\frac{\Gamma(n+2s+j)}{j!\Gamma(n+2s)2^{-n-2s+\frac{n}{2}-j}}. \tag{4.3}$$

$$.\frac{a^j\left|\sigma^{2s+j}\right|}{\Gamma(\frac{n}{2}+s+\frac{j}{2})}.\left[\lim_{\lambda\to -n-2s}(\lambda+n+2s)\Gamma(\frac{\lambda+n-j}{2})\right].$$

Now considering the Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2}) \tag{4.4}$$

and the formula

$$\frac{\Gamma(z)}{\Gamma(z-t)} = \frac{(-1)^t\Gamma(-z+t+1)}{\Gamma(-z+1)} \tag{4.5}$$

([4],page 4)

and putting $\beta = \lambda + n$ we have,

$$\lim_{\lambda\to -n-2s}(\lambda+n+2s)\Gamma(\frac{\lambda+n-j}{2}) = \lim_{\lambda\to -n-2s}\frac{(\lambda+n+2s)\Gamma(\lambda+n-j)}{2^{\lambda+n-j-1}\pi^{-\frac{1}{2}}\Gamma(\frac{\lambda+n-j}{2}+\frac{1}{2})} =$$

$$= \frac{1}{2^{-2s-j-1}\pi^{-\frac{1}{2}}\Gamma(\frac{-2s-j}{2}+\frac{1}{2})}.\lim_{\beta\to -2s}(\beta+2s)\Gamma(\beta-j). \tag{4.6}$$

We observe that using (4.5)

$$\lim_{\beta\to -2s}(\beta+2s)\Gamma(\beta-j) = \lim_{\beta\to -2s}\frac{(\beta+2s)\Gamma(\beta)\Gamma(1-\beta)}{(-1)^i\Gamma(-\beta+j+1)} = \tag{4.7}$$

$$= \frac{\Gamma(1+2s)}{(-1)^i\Gamma(2s+j+1)}.\lim_{\beta\to -2s}(\beta+2s)\Gamma(\beta)$$

and

$$\lim_{\beta\to -2s}(\beta+2s)\Gamma(\beta) = \lim_{\alpha\to 0}\alpha\Gamma(\alpha-2s) =$$

$$= \lim_{\alpha\to 0}\frac{\alpha\Gamma(1-\alpha)\Gamma(\alpha)}{(-1)^{2s}\Gamma(-\alpha+2s+1)} = \tag{4.8}$$

$$= \lim_{\alpha\to 0}\frac{\Gamma(\alpha+1)\Gamma(1-\alpha)}{(-1)^{2s}\Gamma(-\alpha+2s+1)} = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2s+1)} =$$

$$= \frac{1}{(2s)!}$$

The formula (4.8) also can be obtain using the formula (4.4).

From (4.6), (4.7) and (4.8) we obtain

$$\lim_{\lambda\to -n-2s}(\lambda+n+2s)\Gamma(\frac{\lambda+n-j}{2}) = \frac{\Gamma(1+2s)}{(-1)^j\Gamma(2s+j+1)}. \tag{4.9}$$

$$.\frac{1}{(2s)!}.\frac{1}{2^{-2s-j-1}\pi^{-\frac{1}{2}}\Gamma(-s-\frac{j}{2}+\frac{1}{2})}.$$

On the other hand, using the formula

$$\Gamma(\frac{1}{2}-z)\Gamma(\frac{1}{2}+z)=\frac{\pi}{\cos z\pi} \qquad \text{([4],page 3.)} \tag{4.10}$$

we have

$$\frac{1}{\Gamma(\frac{1}{2}-(\frac{j}{2}+s))}=\frac{\Gamma(\frac{1}{2}+s+\frac{j}{2})}{\pi}\cos(s+\frac{j}{2})\pi=\begin{cases} 0 \text{ if } j \text{ is odd} \\ \frac{\Gamma(\frac{1}{2}+s+\frac{j}{2})(-1)^s\cos\frac{j\pi}{2}}{\pi} \text{ if } j \text{ is even.}\end{cases} \tag{4.11}$$

From (4.9) and (4.11) we obtain

$$\lim_{\lambda\to -n-2s}(\lambda+n+2s)\Gamma(\frac{\lambda+n-j}{2})=\begin{cases} 0 \text{ if } j \text{ is odd} \\ \frac{\Gamma(\frac{1}{2}+s+\frac{j}{2})(-1)^s\cos\frac{j\pi}{2}}{(-1)^j\ 2^{-2s-j-1}\pi^{-\frac{1}{2}}\Gamma(2s+j+1)} \text{ if } j \text{ is even}\end{cases} \tag{4.12}$$

By letting $j=2m, m=0,1,2,...$ in (4.3) and using (4.12) we have the following formula

$$\underset{\lambda=-n-2s}{Res}\left\{(|x|-a)_+^\lambda\right\}^\wedge = \tag{4.13}$$

$$=\frac{2\pi^{\frac{n}{2}}}{\pi^{\frac{1}{2}}}\sum_{m\geq 0}\frac{\Gamma(n+2s+2m)a^{2m}\Gamma(m+s+\frac{1}{2})(-1)^s(-1)^m(|\sigma|^2)^{s+m}}{(2m)!\Gamma(n+2s)\Gamma(2s+2m+1)\Gamma(\frac{n}{2}+s+m)}.$$

Using the Legendare´s formula (see formula 38) the formula (4.13) can be rewritten in the following form

$$\underset{\lambda=-n-2s}{Res}\left\{(|x|-a)_+^\lambda\right\}^\wedge = \tag{4.14}$$

$$=\frac{2\pi^{\frac{n}{2}}}{\pi^{\frac{1}{2}}}\sum_{m\geq 0}\frac{\Gamma(n+2s+2m)a^{2m}(-1)^s(-1)^m(|\sigma|^2)^{s+m}}{(2m)!\Gamma(n+2s)2^{2s+2m}\pi^{-\frac{1}{2}}(s+m)!\Gamma(\frac{n}{2}+s+m)}.$$

5 The Residue of $(|x|-a)_+^\lambda$

Let $(|x|-a)_+^\lambda$ be the distribution family defined by

$$<(|x|-a)_+^\lambda,\varphi>=\int_{|x|-a\geq 0}(|x|-a)^\lambda\varphi(x)dx \tag{5.1}$$

for all φ in D (space Schwartz of the test functions).

Let us go over to spherical coordinate in (5.1), writting it in the form

$$<(|x|-a)_+^\lambda,\varphi>=\int_{r-a\geq 0}(r-a)^\lambda\left\{\int_{0_a}\varphi(rw)d0_a\right\}dr=$$

$$=\int_{r-a\geq 0}(r-a)^\lambda\left\{\frac{1}{a^{n-1}}\int_{0_a}[\varphi(rw)r^{n-1}]_{r=a}d0_a\right\}dr= \tag{5.2}$$

$$=\int_{r-a\geq 0}(r-a)^\lambda\psi(r)dr$$

where

$$\psi(r) = \frac{1}{a^{n-1}} \int_{0_a} [\varphi(rw) r^{n-1}]_{r=a} d0_a \quad , \tag{5.3}$$

By putting $\rho = r - a$ in (5.3) we have

$$< (|x| - a)_+^\lambda, \varphi >= \int_0^\infty \rho^\lambda \psi_1(\rho + a) d\rho \tag{5.4}$$

where

$$\psi_1(\rho + a) = \psi(r). \tag{5.5}$$

Using the formula

$$\underset{\lambda=-k,k=1..}{Res} < x_+^\lambda, \varphi >= \frac{\varphi^{(k-1)}(0)}{(k-1)!} \tag{5.6}$$

([1],page49),from (5.4) we have,

$$\begin{aligned}
\underset{\lambda=-j}{Res} < (|x| - a)_+^\lambda, \varphi > &= \tfrac{1}{(j-1)!} \left[\tfrac{\partial^{j-1}}{\partial \rho^{j-1}} \{\psi_1(\rho + c)\} \right]_{\rho=0} = \\
&= \tfrac{1}{(j-1)!} \left[\tfrac{\partial^{s-1}}{\partial r^{s-1}} \{\psi(r)\} \right]_{r=a} = \\
&= \tfrac{1}{(j-1)!} \tfrac{\partial^{j-1}}{\partial r^{j-1}} \{ \tfrac{1}{a^{n-1}} \textstyle\int_{0_a} [\varphi r^{n-1}]_{r=a} dO_a \} = \\
&= \tfrac{1}{(j-1)!} \tfrac{1}{a^{n-1}} \textstyle\int_{0_a} \tfrac{\partial^{j-1}}{\partial r^{j-1}} [\varphi r^{n-1}]_{r=a} dO_a.
\end{aligned} \tag{5.7}$$

Now using the formula (1.10) we have,

$$\underset{\lambda=-j}{Res} < (|x| - a)_+^\lambda, \varphi >= \frac{(-1)^{j-1} < \delta^{(j-1)}(r-a), \varphi >}{(j-1)!}. \tag{5.8}$$

From (5.8) we arrive at the following formula

$$\underset{\lambda=-j}{Res} (|x| - a)_+^\lambda = \frac{(-1)^{j-1}}{(j-1)!} \delta^{(j-1)}(r-a)_{j=1,2,...} \quad . \tag{5.9}$$

In particular if $j = n + 2s, s = 0, 1, ..$

$$\underset{\lambda=-n-2s}{Res} (|x| - a)_+^\lambda = \frac{(-1)^{n+2s-1}}{(n+2s-1)!} \delta^{(n+2s-1)}(|x| - a) \quad . \tag{5.10}$$

On the other hand,considering(3.1) and using the formulae(3.4)and(3.5) we have

$$\begin{aligned}
\underset{\lambda=-n-2s}{Res} (|x| - a)_+^\lambda &= \lim_{\lambda \to -n-2s} (\lambda + n + 2s) \textstyle\sum_{j \geq 0} (-1)^j \binom{\lambda}{j} |x|^{\lambda - j} a^j = \\
&= \lim_{\lambda \to -n-2s} (\lambda + n + 2s) \{ \textstyle\sum_{m \geq 0} (-1)^{2m+1} \binom{\lambda}{2m+1} |x|^{\lambda - 2m - 1} a^{2m+1} + \\
&\quad + \textstyle\sum_{m \geq 0} (-1)^{2m} \binom{\lambda}{2m} |x|^{\lambda - 2m} a^{2m} \}
\end{aligned} \tag{5.11}$$

From (5.10),considering that $r^{\lambda}=|x|^{\lambda}$ has simple poles at $\lambda=-n,-n-2,-n-4,...$

([1] page 99) and the formula(5.11) we have,

$$\underset{\lambda=-n-2s}{Res}(|x|-a)_+^{\lambda}=\sum_{m\geq 0}\left[\tfrac{(-1)^{2m}\Gamma(n+2s+2m)}{(2m)!\Gamma(n+2s)}.a^{2m}\right]. \\ .\lim_{\lambda\to -n-2s}(\lambda+n+2s)\,|x|^{\lambda-2m}. \tag{5.12}$$

On the other hand, using the formulae

$$\delta^{(2j)}(r)=\frac{(2j)!}{\Omega_n}\lim_{\lambda\to -n-2j}(\lambda+n+2j)r^{\lambda}=\frac{(2j)!}{\Omega_n}\underset{\lambda=-n-2j}{Res}\,r^{\lambda}\ \text{([5]],page 792)} \tag{5.13}$$

and

$$\delta^{(2j)}(r)=\frac{(2j)!\Gamma(\frac{n}{2})}{2^{2j}j!\Gamma(\frac{n}{2}+j)}\Delta^j\delta\ \ \text{([5],page 793)} \tag{5.14}$$

where

$$\Omega_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{5.15}$$

and

$$\Delta^j=\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right)^j \tag{5.16}$$

we have

$$\lim_{\lambda\to -n-2s}(\lambda+n+2s)\,|x|^{\lambda-2m}=\lim_{\gamma\to -n-2s-2m}(\gamma+2m+n+2s)\,|x|^{\gamma}= \\ =\underset{\gamma=-n-2(s+m)}{Res}|x|^{\gamma}=\tfrac{2\pi^{\frac{n}{2}}\Delta^{j+m}\delta}{2^{2(s+m)}(s+m)!\Gamma(\frac{n}{2}+s+m)}. \tag{5.17}$$

From(5.12)and using (5.17) we obtain the formula,

$$\underset{\lambda=-n-2s}{Res}(|x|-a)_+^{\lambda}=\sum_{m\geq 0}\tfrac{(-1)^{2m}\Gamma(n+2s+2m)}{(2m)!\Gamma(n+2s)}. \\ .\tfrac{a^{2m}2\pi^{\frac{n}{2}}\Delta^{s+m}\delta}{2^{2(s+m)}(s+m)!\Gamma(\frac{n}{2}+s+m)}. \tag{5.18}$$

From(5.18) and considering the Fourier transform of Laplacian operator itereted s times $\Delta^s\delta$,

$$(\Delta^s\delta)^{\wedge}=(-1)^j(\sigma^2)^j \tag{5.19}$$

where

$$\sigma^2=\sigma_1^2+...+\sigma_n^2 \tag{5.20}$$

([1],page201) we obtain the Fourier transform of $\underset{\lambda=-n-2s}{Res}(|x|-a)_+^\lambda$,

$$\left\{\underset{\lambda=-n-2s}{Res}(|x|-a)_+^\lambda\right\}^\wedge = \sum_{m\leq 0}\frac{\Gamma(n+2)(-2m)}{(2m)!\Gamma(n+2s)}\cdot \cdot\frac{a^{2m}2\pi^{-\frac{n}{2}}(-1)^{s+m}(\left|\sigma^2\right|)^{s+m}}{2^{2(s+m)}(s+m)!\Gamma(\frac{n}{2}+s+m)} \tag{5.21}$$

From(4.14)and(5.21),we arrive at the following propierty

$$\left\{\underset{\lambda=-n-2s}{Res}(|x|-a)_+^\lambda\right\}^\wedge = \underset{\lambda=-n-2s}{Res}\{(|x|-a)_+^\lambda\}^\wedge . \tag{5.22}$$

We observe by putting $j=n+2s$ and $a=c$ in(5.9)and using(5.22),we obtain the same formula given by (2.13) :

$$\begin{aligned}\left\{\delta^{(n+2s-1)!}(|x|-c)\right\}^\Lambda &= \left\{\frac{(n+2s-1)!}{(-1)^{(n+2s-1)}}\underset{\lambda=-n-2s}{Res}(|x|-c)_+^\lambda\right\}^\Lambda = \\ &= \frac{k!}{(-1)^k}\cdot\sum_{m\geq 0}\frac{(-1)^{2m}\Gamma(n+2s+2m)}{\Gamma(n+2s)}\frac{a^{2m}2\pi^{\frac{n}{2}}(-1)^{s+m}(\left|\sigma^2\right|)^{s+m}}{2^{2(s+m)}(s+m)!\Gamma(\frac{n}{2}+s+m)} \\ &= \frac{1}{(-1)^{(n+2s-1)}}\sum_{m\geq 0}\frac{(-1)^{2m}\Gamma(n+2s+2m)a^{2m}2\pi^{\frac{n}{2}}(-1)^{s+m}(\left|\sigma^2\right|)^{s+m}}{(2m)!2^{2(s+m)}(s+m)!\Gamma(\frac{n}{2}+s+m)}.\end{aligned} \tag{5.23}$$

6 The expansion of $\delta^{(k)}(|x|-a)$

From (2.8), considering the formula(5.18)and using the theorem of unicity for the Fourier transform we obtain the following formula

$$\delta^{(k)}(|x|-c) = (-1)^k 2^{\frac{n}{2}}\pi^{\frac{n}{2}}\sum_{j\geq 0}\frac{\Gamma(n+2j)c^{n-1+2j-k}}{2^{\frac{n}{2}+2j-1}j!\Gamma(\frac{n}{2}+j)\Gamma(n+2j-k)}\Delta^j\delta \tag{6.1}$$

From (6.2) and taking into account the formula (5.16) we obtain the following formula

$$\delta^{(k)}(|x|-c) = \frac{(-1)^k 2^{\frac{n}{2}}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})2^{\frac{n}{2}-1}}\sum_{j\geq 0}\frac{\Gamma(n+2j)c^{n-1+2j-k}}{(2j)!\Gamma(n+2j-k)}\delta^{(2j)}(|x|) . \tag{6.2}$$

By putting $k=n+2s-1, s=0,1,2,..$ in(6.2) and considering that

$$\frac{1}{\Gamma(1+2j-2s)} = 0 \quad \text{if} \quad j<s \tag{6.3}$$

we have,

$$\begin{aligned}\delta^{(n+2s-1)}(|x|-c) &= 2\frac{(-1)^{n+2s-1}2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\sum_{j\geq s}\frac{\Gamma(n+2j)c^{2j-2s}}{(2j)!\Gamma(2j-2s+1)}\delta^{(2j)}(|x|) = \\ &= 2\frac{(-1)^{n+2s-1}2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\sum_{l\geq 0}\frac{\Gamma(n+2l+2s)c^{2l}}{(2(l+s))!\Gamma(2l+1)}\delta^{(2(l+s))}(|x|) .\end{aligned} \tag{6.4}$$

Now using the formula(5.16) and the following formula

$$\delta^{(n+2l-1)}(|x|)=a_{l,n}\Delta^l\delta(x) \qquad \text{([6],formula(1,2,4)and (1,2,5))} \tag{6.5}$$

where

$$a_{l,n}=\frac{(n+2l-1)!(-1)^{n+2l-1}2.\pi^{\frac{n}{2}}}{2^{2l}l!\Gamma(\frac{n}{2}+l)} \quad l=0,1,2,.. \tag{6.6}$$

we have

$$\delta^{(n+2l-1)}(|x|)=\tfrac{(n+2l-1)!(-1)^{n+2l-1}2.\pi^{\frac{n}{2}}}{(2l)!\Gamma(\frac{n}{2})}. \\ .\delta^{(2l)}(|x|). \tag{6.7}$$

From(6.4) and(6.7) we arrive at the following formula

$$\begin{gathered}\delta^{(n+2s-1)}(|x|-c)=\tfrac{2(-1)^{n+2s-1}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\sum_{l\geq 0}\tfrac{c^{2l}}{\Gamma(2l+1)}. \\ .\left[\tfrac{\Gamma(\frac{n}{2})\delta^{(n+2(l+s)-1)}(|x|)}{(-1)^{n+2(l+s)-1}\pi^{\frac{n}{2}}}\right]= \\ =\tfrac{(-1)^{n+2s-1}\pi^{\frac{n}{2}}}{(-1)^{n+2(l+s)-1}\pi^{\frac{n}{2}}}.\sum_{l\geq 0}\tfrac{(-c)^{2l}}{(2l)!}\delta^{(n+2(l+s)-1)}(|x|)= \\ =\sum_{l\geq 0}\tfrac{(-c)^{2l}}{(2l)!}\delta^{(n+2(l+s)-1)}(|x|)\end{gathered} \tag{6.8}$$

7 The convolution product of $\delta^{(k)}(|x|-c)*\delta^{(l)}(|x|-c)$

To obtain our results, we need the following formulae

$$\delta^{(2j)}(|x|)*\delta^{(2s)}(|x|)=b_{j,s,n}\delta^{(2(j+s))}(|x|) \quad \text{([5],page793,formula(33)).} \tag{7.1}$$

and

$$\delta^{(n+2s-1)}(|x|)*\delta^{(n+2j-1)}(|x|)=d_{s,j,n}\delta^{(n+2(j+s)-1)}(|x|) \qquad \text{([6],page143).} \tag{7.2}$$

where

$$b_{j,s,n}=\frac{(j+s)!(2s)!(2j)!\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+j+s)}{j!s!(2(j+s))!\Gamma(\frac{n}{2}+s)\Gamma(\frac{n}{2}+j)} \tag{7.3}$$

$$d_{s,j,n}=(-1)^{n-1}\pi^{\frac{n-1}{2}}2^n\frac{(s+j)!}{s!j!}\frac{\Gamma(\frac{n}{2}+s+\frac{1}{2})\Gamma(\frac{n}{2}+j+\frac{1}{2})}{\Gamma(\frac{n}{2}+j+s+\frac{1}{2})} \quad . \tag{7.4}$$

Using the Legendre´s duplication formula

$$\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2}) \quad , \tag{7.5}$$

$b_{j,s,n}$ can be rewritten in the following form

$$b_{j,s,n} = \pi^{-\frac{1}{2}} \frac{\Gamma(s+\frac{1}{2})\Gamma(j+\frac{1}{2})}{\Gamma(s+j+\frac{1}{2})} . \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+s+j)}{\Gamma(\frac{n}{2}+s)\Gamma(\frac{n}{2}+j)} \quad . \tag{7.6}$$

We observe that getting $j = s = 0$ in(7.1) and considering(7.3) we have,

$$\delta(|x|) * \delta(|x|) = b_{0,0,n}\delta(|x|) = \delta(|x|). \tag{7.7}$$

Now from(6.8)and using(7.2)and(7.4) we obtain

$$\begin{aligned} &\delta^{(n+2k-1)}(|x|-c) * \delta^{(n+2l-1)}(|x|-c) = \\ &= \textstyle\sum_{i\geq 0} \frac{(-c)^{2i}}{(2i)!}\delta^{(n-1+2(i+k))}(|x|) * \sum_{\nu\geq 0} \frac{(-c)^{2\nu}}{(2\nu)!}\delta^{(n-1+2(\nu+l))}(|x|) = \\ &= \textstyle\sum_{i\geq 0}\sum_{\nu\geq 0} \frac{(-c)^{2(i+\nu)}}{(2\nu)!(2i)!}\left\{\delta^{(n-1+2(i+k))}(|x|) * \delta^{(n-1+2(\nu+l))}(|x|)\right\} = \\ &= (-1)^{n-1}\pi^{\frac{n-1}{2}}2^n \textstyle\sum_{t\geq 0} \frac{(t+k+l)!(-1)^{2t}}{\Gamma(\frac{n}{2}+k+t+l+\frac{1}{2})} B_{t,k,l,n}\delta^{(n-1+2(t+k+l))}(|x|) \end{aligned} \tag{7.8}$$

where

$$B_{t,k,l,n} = \sum_{\nu\geq 0}^{t} \frac{\Gamma(\frac{n}{2}+l+v+\frac{1}{2})\Gamma(\frac{n}{2}+k+t-v+\frac{1}{2})}{(2\nu)!(2(t-v))!(k+t-v)!(l+v)!} \tag{7.9}$$

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