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Fine Spectrum of the Generalized Difference Operator Δ_v on Sequence Space l_1

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Abstract : The purpose of this paper is to determine spectrum and fine spectrum of the operator Δ_v on sequence space l_1. The operator Δ_v on l_1 is defined by $\Delta_v x = (v_n x_n - v_{n-1}x_{n-1})_{n=0}^{\infty}$ with $x_{-1} = 0$, where $x = (x_n) \in l_1$ and $v = (v_k)$ is either constant or strictly decreasing sequence of positive real numbers satisfying certain conditions. In this paper we have obtained the results on spectrum and point spectrum for the operator Δ_v over the sequence space l_1. Further, the results on continuous spectrum, residual spectrum and fine spectrum of the operator Δ_v on space l_1 are also derived

Keywords : Spectrum of an operator, Generalized difference operator, Sequence spaces.
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1 Introduction

Let $v = (v_k)$ be either constant or strictly decreasing sequence of positive real numbers satisfying

$$\lim_{k\to\infty} v_k = L > 0 \quad \text{and} \tag{1.1}$$

$$\sup_k v_k \leq 2L. \tag{1.2}$$

We introduce the operator Δ_v on sequence space l_1 as follows;
$\Delta_v : l_1 \to l_1$ is defined by,

$$\Delta_v x = \Delta_v(x_n) = (v_n x_n - v_{n-1}x_{n-1})_{n=0}^{\infty} \text{ with } x_{-1} = 0, \text{ where } x \in l_1.$$

It is easy to verify that the operator Δ_v can be represented by the matrix

$$\Delta_v = \begin{pmatrix} v_0 & 0 & 0 & \dots \\ -v_0 & v_1 & 0 & \dots \\ 0 & -v_1 & v_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The fine spectrum of the Cesaro operator on sequence space l_p is studied by Gonzalez [5], where $1 < p < \infty$. The spectrum of the Cesaro operator on sequence spaces bv_0 and bv is also investigated by Okutoyi [9] and Okutoyi [10], respectively. Spectrum and fine spectrum of the difference operator Δ over sequence spaces l_1 and bv is determined by K. Kayaduman and H. Furkan [7]. The fine spectra of the difference operator Δ over sequence space l_p is determined by Akhmedov and Basar [1], where $1 \leq p < \infty$. Furthermore, the fine spectrum of the operator $B(r,s)$ on the sequence spaces l_1 and bv is examined by H. Furkan, H. Bilgic and K. Kayaduman [3]. Recently, H. Bilgic and H. Furkan [2] studied the spectrum and fine spectrum for the operaor $B(r,s,t)$ over sequence spaces l_1 and bv.

In this paper we determine spectrum, point spectrum, continuous spectrum and residual spectrum of the operator Δ_v on sequence space l_1. The results of this paper not only generalize the corresponding results of [7] but also give results for some more operators.

2 Preliminaries and notation

Let X and Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. The set of all bounded linear operators on X into itself is denoted by $B(X)$. The adjoint $T^\times : X^\star \to X^\star$ of T is defined by

$$\left(T^\times \phi\right)(x) = \phi\,(Tx) \text{ for all } \phi \in X^\star \text{ and } x \in X.$$

Clearly, $T^\times$ is a bounded linear operator on the dual space $X^\star$.

Let $X \neq \{\mathbf{0}\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subseteq X$. With T, we associate the operator $T_\alpha = (T - \alpha I)$, where α is a complex number and I is the identity operator on $D(T)$. The inverse of T_α (if exists) is denoted by T_α^{-1} and call it the resolvent operator of T. Many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. We are interested in the set of all α in the complex plane such that T_α^{-1} exists/ T_α^{-1} is bounded/ domain of T_α^{-1} is dense in X. We need some definitions and known results which will be used in the sequel.

Definition 2.1. ([6], pp. 371) Let $X \neq \{\mathbf{0}\}$ be a complex normed space and $T : D(T) \to X$ be a linear operator with domain $D(T) \subset X$. A *regular* value of T is a complex number α such that
(R1) T_α^{-1} exists,
(R2) T_α^{-1} is bounded,
(R3) T_α^{-1} is defined on a set which is dense in X.

Resolvent set $\rho(T,X)$ of T is the set of all regular values α of T. Its complement $\sigma(T,X) = \mathbb{C}\setminus\rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets namely point spectrum, continuous spectrum and residual spectrum as follows:

Point spectrum $\sigma_p(T,X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} does not exist. The element of $\sigma_p(T,X)$ is called *eigenvalue* of T.

Continuous spectrum $\sigma_c(T,X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} exists and satisfies (R3) but not (R2), i.e., range of T_α is dense in X and T_α^{-1} is unbounded.

Residual spectrum $\sigma_r(T,X)$ is the set of all $\alpha \in \mathbb{C}$ such that T_α^{-1} exists but do not satisfy (R3), i.e., domain of T_α^{-1} is not dense in X. The condition (R2) may or may not holds good.

Goldberg's classification of operator T_α (see [4], pp. 58): Let X be a Banach space and $T_\alpha \in B(X)$, where α is a complex number. Again, let $R(T_\alpha)$ and T_α^{-1} be denote the range and inverse of the operator T_α, respectively. Then following possibilities may occur;
(A) $R(T_\alpha) = X$,
(B) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(C) $\overline{R(T_\alpha)} \neq X$,
and
(1) T_α is injective and T_α^{-1} is continuous,
(2) T_α is injective and T_α^{-1} is discontinuous,
(3) T_α is not injective.

Remark 2.1. Combining (A), (B), (C) and (1), (2), (3); we get nine different states. These are labeled by $A_1,\ A_2,\ A_3,\ B_1,\ B_2,\ B_3,\ C_1,\ C_2$ and C_3. We use $\alpha \in B_2\sigma(T,X)$ means the operator $T_\alpha \in B_2$, i.e., $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$ and T_α is injective but T_α^{-1} is discontinuous. Similarly others.

Remark 2.2. If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then α belongs to the resolvent set $\rho(T,X)$ of T on X. The other classification gives rise to the fine spectrum of T.

Definition 2.2. ([8], pp. 220-221) Let λ, μ be two nonempty subsets of the space w of all real or complex sequences and $A = (a_{nk})$ an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$. For every $x = (x_k) \in \lambda$ and every integer n we write

$$A_n(x) = \sum_k a_{nk}x_k,$$

where the sum without limits is always taken from $k = 0$ to $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. Infinite matrix $A \in (\lambda,\ \mu)$ if and only if $Ax \in \mu$ whenever $x \in \lambda$.

Lemma 2.1. ([11], pp. 126) The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(l_1)$ from l_1 to itself if and only if the supremum of l_1 norms of the columns of A is bounded.

Lemma 2.2. ([4], pp. 59) T has a dense range if and only if $T^{\times}$ is one to one, where $T^{\times}$ denotes the adjoint operator of the operator T.

Lemma 2.3. ([4], pp. 60) The adjoint operator $T^{\times}$ of T is onto if and only if T has a bounded inverse.

3 Spectrum and point spectrum of the operator Δ_v on sequence space l_1

In this section we obtain spectrum and point spectrum of the operator Δ_v on sequence space l_1. Throughout this paper, the sequence $v = (v_k)$ satisfy conditions (1.1) and (1.2).

Theorem 3.1. The operator $\Delta_v : l_1 \to l_1$ is a bounded linear operator and

$$\|\Delta_v\|_{(l_1,l_1)} = 2 \sup_k (v_k).$$

Proof. Proof is simple. So we omit. □

Theorem 3.2. The spectrum of Δ_v on sequence space l_1 is given by

$$\sigma(\Delta_v, l_1) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}.$$

Proof. The proof of this theorem is divided into two parts. In the first part, we show that $\sigma(\Delta_v, l_1) \subseteq \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}$ or equivalent to show that

$$\alpha \in \mathbb{C} \text{ with } \left|1 - \frac{\alpha}{L}\right| > 1 \text{ implies } \alpha \notin \sigma(\Delta_v, l_1), \text{ i.e., } \alpha \in \rho(\Delta_v, l_1).$$

In the second part, we establish the reverse inequality, i.e.,

$$\left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \subseteq \sigma(\Delta_v, l_1).$$

Let $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| > 1$. Clearly, $\alpha = L$ as well as $\alpha = v_k$ for any k do not satisfied. So $\alpha \neq L$ and $\alpha \neq v_k$ for each $k \in \mathbb{N}_0$. Consequently, $(\Delta_v - \alpha I) = (a_{nk})$ as a triangle and hence has an inverse $(\Delta_v - \alpha I)^{-1} = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} \frac{1}{(v_0-\alpha)} & 0 & 0 & \cdots \\ \frac{v_0}{(v_0-\alpha)(v_1-\alpha)} & \frac{1}{(v_1-\alpha)} & 0 & \cdots \\ \frac{v_0 v_1}{(v_0-\alpha)(v_1-\alpha)(v_2-\alpha)} & \frac{v_1}{(v_1-\alpha)(v_2-\alpha)} & \frac{1}{(v_2-\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By Lemma 2.1, the operator $(\Delta_v - \alpha I)^{-1} \in (l_1, l_1)$ if $\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty$. In order to show $\sup_k \sum_{n=0}^{\infty} |b_{nk}| < \infty$, first we prove that the series $\sum_{n=0}^{\infty} |b_{nk}|$ is convergent for each $k \in \mathbb{N}_0$.

Let $S_k = \sum_{n=0}^{\infty} |b_{nk}|$. Then the series

$$
\begin{aligned}
S_0 &= \sum_{n=0}^{\infty} |b_{no}| \\
&= \left| \frac{1}{v_0 - \alpha} \right| + \sum_{n=1}^{\infty} \left| \frac{v_0 v_1 \cdots v_{n-1}}{(v_0 - \alpha)(v_1 - \alpha) \cdots (v_n - \alpha)} \right| \qquad (3.1)
\end{aligned}
$$

is convergent because

$$
\lim_{n\to\infty} \left| \frac{b_{n+1,0}}{b_{n0}} \right| = \lim_{n\to\infty} \left| \frac{v_n}{v_{n+1} - \alpha} \right| = \frac{1}{\left|1 - \frac{\alpha}{L}\right|} < 1.
$$

Similarly, we can show that the series $S_k = \sum_{n=0}^{\infty} |b_{nk}|$ is convergent for any $k = 1, 2, 3, \cdots$.

Now we claim that $\sup_k S_k$ is finite. We have

$$
S_k = \frac{1}{|v_k - \alpha|} + \frac{|v_k|}{|v_k - \alpha|\,|v_{k+1} - \alpha|} + \cdots . \qquad (3.2)
$$

Let $\beta = \lim_{k\to\infty} \left| \frac{v_k}{v_{k+1} - \alpha} \right|$. Since modulus function is continuous, so

$$
\beta = \left| \frac{L}{L - \alpha} \right|, \qquad (3.3)
$$

which shows that $0 < \beta < 1$ and gives

$$
\lim_{k\to\infty} \left| \frac{1}{v_k - \alpha} \right| = \lim_{k\to\infty} \left(\left| \frac{v_{k-1}}{v_k - \alpha} \right| \left| \frac{1}{v_{k-1}} \right| \right) = \frac{\beta}{L}. \qquad (3.4)
$$

Taking limit both sides of equation (3.2) and using equations (3.3) and (3.4), we get

$$
\lim_{k\to\infty} S_k = \frac{\beta}{L} \left(\frac{1}{1 - \beta} \right) < \infty.
$$

Since (S_k) is a sequence of positive real numbers and $\lim_{k\to\infty} S_k < \infty$, so $\sup_k S_k < \infty$. Thus,

$$
(\Delta_v - \alpha I)^{-1} \in B(l_1) \text{ for } \alpha \in \mathbb{C} \text{ with } \left| 1 - \frac{\alpha}{L} \right| > 1. \qquad (3.5)
$$

Next, we show that domain of the operator $(\Delta_v - \alpha I)^{-1}$ is dense in l_1 equivalent to say that range of the operator $(\Delta_v - \alpha I)$ is dense in l_1, which follows immediately as the operator $(\Delta_v - \alpha I)$ is onto. Hence we have

$$\sigma\left(\Delta_v, l_1\right) \subseteq \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}. \tag{3.6}$$

Conversely, it is required to show

$$\left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \subseteq \sigma(\Delta_v, l_1). \tag{3.7}$$

First we prove inclusion (3.7) under the assumption that $\alpha \neq L$ as well as $\alpha \neq v_k$ for each $k \in \mathbb{N}_0$, i.e., one of the conditions of Definition 2.1 fails. Let $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| \leq 1$. Clearly, $(\Delta_v - \alpha I)$ is a triangle and hence $(\Delta_v - \alpha I)^{-1}$ exists. So condition (R1) is satisfied but condition (R2) fails as can be seen below:

Suppose $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| < 1$. Then by equation (3.1), the series S_0 is divergent because

$$\lim_{n\to\infty}\left|\frac{b_{n+1,0}}{b_{n0}}\right| = \lim_{n\to\infty}\left|\frac{v_n}{v_{n+1} - \alpha}\right| = \frac{1}{\left|1 - \frac{\alpha}{L}\right|} > 1.$$

So $\sup_k S_k$ is unbounded. Hence

$$\left(\Delta_v - \alpha I\right)^{-1} \notin B\left(l_1\right) \text{ for } \alpha \in \mathbb{C} \text{ with } \left|1 - \frac{\alpha}{L}\right| < 1. \tag{3.8}$$

Next, we consider $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| = 1$, i.e., $|L - \alpha| = L$ which implies $|v_n - \alpha| \leq |v_n|$ for each n, therefore $\dfrac{1}{|v_n|} \leq \dfrac{1}{|v_n - \alpha|}$ for each n. Using this inequality and equation (3.1), the series $S_0 \geq \sum_{n=0}^{\infty} \dfrac{1}{v_n}$ is divergent due to the fact that $v_n > 0$ for all n and $\lim_{n\to\infty} \dfrac{1}{v_n} = \dfrac{1}{L} \neq 0$. Thus, $\sup_k S_k$ is unbounded. Hence

$$\left(\Delta_v - \alpha I\right)^{-1} \notin B\left(l_1\right) \text{ for } \alpha \in \mathbb{C} \text{ with } \left|1 - \frac{\alpha}{L}\right| = 1. \tag{3.9}$$

Finally, we prove the inclusion (3.7) under the assumption that $\alpha = L$ as well as $\alpha = v_k$ for all $k \in \mathbb{N}_0$. We have

$$\left(\Delta_v - v_k I\right)x = \begin{pmatrix} (v_0 - v_k)x_0 \\ -v_0x_0 + (v_1 - v_k)x_1 \\ \vdots \\ -v_{k-1}x_{k-1} \\ -v_kx_k + (v_{k+1} - v_k)x_{k+1} \\ \vdots \end{pmatrix}.$$

Case(i): If (v_k) is a constant sequence, say $v_k = L$ for all $k \in \mathbb{N}_0$, then

$$(\Delta_v - v_k I)\, x = \mathbf{0} \qquad \Rightarrow x_0 = 0,\; x_1 = 0,\; x_2 = 0, \cdots.$$

This shows that the operator $(\Delta_v - \alpha I)$ is one to one, but $R\,(\Delta_v - \alpha I)$ is not dense in l_1. So condition (R3) fails. Hence $L \in \sigma\,(\Delta_v, l_1)$.

Case(ii): If (v_k) is strictly decreasing sequence, then for fixed k,

$$(\Delta_v - v_k I)\, x = \mathbf{0}$$

$$\Rightarrow x_0 = 0,\; x_1 = 0,\; \cdots,\; x_{k-1} = 0,\; x_{n+1} = \left(\frac{v_n}{v_{n+1} - v_k}\right) x_n \text{ for all } n \geq k.$$

This shows that $(\Delta_v - v_k I)$ is not injective. So condition (R1) fails. Hence $v_k \in \sigma\,(\Delta_v, l_1)$ for all $k \in \mathbb{N}_0$.

Again, if $\alpha = L$, then $|v_n - \alpha| < |v_n|$ for each n, i.e., $\frac{1}{|v_n|} < \frac{1}{|v_n - \alpha|}$ for each n. Using this inequality and equation (3.1), the series $S_0 > \sum_{n=0}^{\infty} \frac{1}{v_n}$ is divergent due to fact that $v_n > 0$ for all n and $\lim_{n\to\infty} \frac{1}{v_n} = \frac{1}{L} \neq 0$. Thus, $\sup_k S_k$ is unbounded. So condition (R2) fails. Hence

$$(\Delta_v - \alpha I)^{-1} \notin B(l_1) \text{ for } \alpha = L. \tag{3.10}$$

So $L \in \sigma\,(\Delta_v, l_1)$. Thus, in this case also $v_k \in \sigma\,(\Delta_v, l_1)$ for all $k \in \mathbb{N}_0$ and $L \in \sigma\,(\Delta_v, l_1)$. Hence we have

$$\left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \subseteq \sigma\,(\Delta_v, l_1)\,. \tag{3.11}$$

From inclusions (3.6) and (3.11), we get

$$\sigma(\Delta_v, l_1) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}. \qquad \square$$

Theorem 3.3. Point spectrum of the operator Δ_v over l_1 is given by

$$\sigma_p\,(\Delta_v, l_1) = \begin{cases} \emptyset, \text{ if } (v_k) \text{ is a constant sequence.} \\ \\ \{v_0, v_1, v_2, \cdots\}, \text{ if } (v_k) \text{ is a strictly decreasing sequence.} \end{cases}$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Suppose (v_k) is a constant sequence, say $v_k = L$ for all $k \in \mathbb{N}_0$. Consider $\Delta_v x = \alpha x$ for $x \neq \mathbf{0} = (0, 0, \cdots)$ in l_1, which gives

$$\left.\begin{aligned} v_0 x_0 &= \alpha x_0 \\ -v_0 x_0 + v_1 x_1 &= \alpha x_1 \\ -v_1 x_1 + v_2 x_2 &= \alpha x_2 \\ &\vdots \\ -v_{k-1} x_{k-1} + v_k x_k &= \alpha x_k \\ &\vdots \end{aligned}\right\} \tag{3.12}$$

Let x_t be the first non-zero entry of the sequence $x = (x_n)$, so we get $-Lx_{t-1} + Lx_t = \alpha x_t$, which implies $\alpha = L$ and from the equation

$$-Lx_t + Lx_{t+1} = \alpha x_{t+1},$$

we get $x_t = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p(\Delta_v, l_1) = \emptyset.$$

Case(ii): Suppose (v_k) is a strictly decreasing sequence. Consider $\Delta_v x = \alpha x$ for $x \neq \mathbf{0} = (0, 0, \cdots)$ in l_1, which gives system of equations (3.12).
If $\alpha = v_0$, then

$$\begin{aligned} x_k &= \left(\frac{v_{k-1}}{v_k - v_0}\right) x_{k-1} \quad \text{for all } k \geq 1 \\ &= \left[\frac{v_{k-1}v_{k-2}\cdots v_0}{(v_k - v_0)(v_{k-1} - v_0)\cdots(v_1 - v_0)}\right] x_0 \quad \text{for all } k \geq 1. \end{aligned}$$

If we take $x_0 \neq 0$, then get non-zero solution of $(\Delta_v - v_0 I)\, x = \mathbf{0}$.
Similarly, if $\alpha = v_k$ for all $k \geq 1$, then $x_{k-1} = 0,\ x_{k-2} = 0,\ \cdots,\ x_0 = 0$ and

$$\begin{aligned} x_{n+1} &= \left(\frac{v_n}{v_{n+1} - v_k}\right) x_n \quad \text{for all } n \geq k \\ &= \left[\frac{v_n v_{n-1}\cdots v_k}{(v_{n+1} - v_k)(v_n - v_k)\cdots(v_{k+1} - v_k)}\right] x_k \quad \text{for all } n \geq k. \end{aligned}$$

If we take $x_k \neq 0$, then get non-zero solution of $(\Delta_v - v_k I)\, x = \mathbf{0}$. Hence

$$\sigma_p\left(\Delta_v, l_1\right) = \{v_0, v_1, v_2, \cdots\}.$$

□

4 Residual and continuous spectrum of the operator Δ_v on sequence space l_1

We need result of point spectrum of the operator $\Delta_v^\times$ on $l_1^\star$ for obtaining residual and continuous spectrum. So first we determine point spectrum of the dual operator $\Delta_v^\times$ of Δ_v on space $l_1^\star$.

Let $T : l_1 \to l_1$ be a bounded linear operator having matrix representation A and the dual space of l_1 denoted by $l_1^\star$. Then the adjoint operator $T^\times : l_1^\star \to l_1^\star$ is defined by the transpose of the matrix A.

Theorem 4.1. Point spectrum of the operator $\Delta_v^\times$ over $l_1^\star$ is

$$\sigma_p(\Delta_v^\times, l_1^\star) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}.$$

Proof. Suppose $\Delta_v^{\times} f = \alpha f$ for $\mathbf{0} \neq f \in l_1^{\star} \cong l_\infty$, where

$$\Delta_v^{\times} = \begin{pmatrix} v_0 & -v_0 & 0 & \dots \\ 0 & v_1 & -v_1 & \dots \\ 0 & 0 & v_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix}.$$

This gives

$$f_k = \left[\frac{(v_{k-1} - \alpha)(v_{k-2} - \alpha) \cdots (v_0 - \alpha)}{v_{k-1} v_{k-2} \cdots v_0} \right] f_0 \quad \text{for all } k \geq 1.$$

Hence

$$|f_k| = \left| \frac{(v_{k-1} - \alpha)(v_{k-2} - \alpha) \cdots (v_0 - \alpha)}{v_{k-1} v_{k-2} \cdots v_0} \right| |f_0| \quad \text{for all } k \geq 1. \tag{4.1}$$

But

$$\begin{aligned} |v_{k-1} - \alpha| &\leq (v_{k-1} - L) + |L - \alpha| \\ \Rightarrow \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| &\leq 1 \text{ for all } k \geq 1 \text{ provided } \left|1 - \frac{\alpha}{L}\right| \leq 1. \end{aligned}$$

Using equation (4.1), we get

$$|f_k| \leq |f_0| \text{ for all } k \geq 1. \text{ So } \sup_k |f_k| < \infty.$$

Hence

$$\left|1 - \frac{\alpha}{L}\right| \leq 1 \quad \Rightarrow \quad \sup_k |f_k| < \infty.$$

Converse follows from the fact that

$$\begin{aligned} \sup_k |f_k| < \infty \quad &\Rightarrow \quad \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| \leq 1 \text{ for all } k \geq m, \\ &\qquad\qquad \text{where } m \text{ is a positive integer.} \\ &\Rightarrow \quad \lim_{k \to \infty} \left| \frac{v_{k-1} - \alpha}{v_{k-1}} \right| \leq 1 \\ &\Rightarrow \quad \left|1 - \frac{\alpha}{L}\right| \leq 1. \end{aligned}$$

Hence

$$\sup_k |f_k| < \infty \quad \Rightarrow \quad \left|1 - \frac{\alpha}{L}\right| \leq 1.$$

This means that $f \in l_1^{\star}$ if and only if $f_0 \neq 0$ and $\left|1 - \frac{\alpha}{L}\right| \leq 1$. Hence

$$\sigma_p(\Delta_v^{\times}, l_1^{\star}) = \left\{ \alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1 \right\}.$$

□

Theorem 4.2. Residual spectrum $\sigma_r(\Delta_v, l_1)$ of the operator Δ_v over l_1 is

$$\sigma_r(\Delta_v, l_1) = \begin{cases} \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}, \text{ if } (v_k) \text{ is a constant sequence.} \\ \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \setminus \{v_0, v_1, v_2, \cdots\}, \text{ if } \\ (v_k) \text{ is a strictly decreasing sequence.} \end{cases}$$

Proof. The proof of this theorem is divided into two cases.
Case(i): Let (v_k) be a constant sequence. For $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| \leq 1$, the operator $(\Delta_v - \alpha I)$ is a triangle except $\alpha = L$ and consequently, the operator $(\Delta_v - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_v - \alpha I)$ is one to one for $\alpha = L$ and hence has an inverse.

But by Theorem 4.1, the operator $(\Delta_v - \alpha I)^{\times} = \Delta_v^{\times} - \alpha I$ is not one to one for $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| \leq 1$. Hence by Lemma 2.2, the range of the operator $(\Delta_v - \alpha I)$ is not dense in l_1. Thus,

$$\sigma_r(\Delta_v, l_1) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}.$$

Case(ii): Let (v_k) be a strictly decreasing sequence with $\lim_{k\to\infty} v_k = L$. For $\alpha \in \mathbb{C}$ such that $\left|1 - \frac{\alpha}{L}\right| \leq 1$, the operator $(\Delta_v - \alpha I)$ is a triangle except $\alpha = v_k$ for all $k \in \mathbb{N}_0$ and consequently, the operator $(\Delta_v - \alpha I)$ has an inverse. Further by Theorem 3.3, the operator $(\Delta_v - v_k I)$ is not one to one and hence $(\Delta_v - v_k I)^{-1}$ does not exists for all $k \in \mathbb{N}_0$.

On the basis of argument as given in case(i), it is easy to verify that the range of the operator $(\Delta_v - \alpha I)$ is not dense in l_1. Thus,

$$\sigma_r(\Delta_v, l_1) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\} \setminus \{v_0, v_1, v_2, \cdots\}. \qquad \square$$

Theorem 4.3. Continuous spectrum $\sigma_c(\Delta_v, l_1)$ of the operator Δ_v over l_1 is $\sigma_c(\Delta_v, l_1) = \emptyset$.

Proof. It is known that $\sigma_p(\Delta_v, l_1)$, $\sigma_r(\Delta_v, l_1)$ and $\sigma_c(\Delta_v, l_1)$ are pairwise disjoint sets and union of these sets is $\sigma(\Delta_v, l_1)$. But by Theorems 3.2, 3.3 and 4.2; we get

$$\sigma(\Delta_v, l_1) = \sigma_p(\Delta_v, l_1) \cup \sigma_r(\Delta_v, l_1).$$

Therefore, $\sigma_c(\Delta_v, l_1) = \emptyset$. $\square$

5 Fine spectrum of the operator Δ_v on sequence space l_1

Theorem 5.1. If α satisfies $\left|1 - \frac{\alpha}{L}\right| > 1$, then $(\Delta_v - \alpha I) \in A_1$.

Proof. It is required to show that the operator $(\Delta_v - \alpha I)$ is bijective and has a continuous inverse for $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| > 1$. Since $\alpha \neq L$ and $\alpha \neq v_k$ for each $k \in \mathbb{N}_0$, therefore $(\Delta_v - \alpha I)$ is a triangle. Hence it has an inverse. The inverse of the operator $(\Delta_v - \alpha I)$ is continuous for $\alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| > 1$ by statement (3.5). Also the equation

$$(\Delta_v - \alpha I)\, x = y \quad \text{gives} \quad x = (\Delta_v - \alpha I)^{-1}\, y,$$
$$\text{i.e.,} \quad x_n = \left((\Delta_v - \alpha I)^{-1} y\right)_n,\ n \in \mathbb{N}_0.$$

Thus for every $y \in l_1$, we can find $x \in l_1$ such that

$$(\Delta_v - \alpha I)x = y, \text{ since } (\Delta_v - \alpha I)^{-1} \in (l_1, l_1).$$

This shows that operator $(\Delta_v - \alpha I)$ is onto and hence $(\Delta_v - \alpha I) \in A_1$. □

Theorem 5.2. Let (v_k) be a constant sequence, say $v_k = L$ for all $k \in \mathbb{N}_0$. Then $L \in C_1\sigma(\Delta_v, l_1)$.

Proof. We have

$$\sigma_r\,(\Delta_v, l_1) = \left\{\alpha \in \mathbb{C} : \left|1 - \frac{\alpha}{L}\right| \leq 1\right\}.$$

Clearly, $L \in \sigma_r\,(\Delta_v, l_1)$. It is sufficient to show that the operator $(\Delta_v - LI)^{-1}$ is continuous. By Lemma 2.3, it is enough to show that $(\Delta_v - LI)^{\times}$ is onto, i.e., for given $y = (y_n) \in l_\infty$, we have to find $x = (x_n) \in l_\infty$ such that $(\Delta_v - LI)^{\times}\, x = y$. Now $(\Delta_v - LI)^{\times}\, x = y$, i.e.,

$$\begin{aligned} -Lx_1 &= y_0 \\ -Lx_2 &= y_1 \\ &\vdots \\ -Lx_i &= y_{i-1} \\ &\vdots \end{aligned}$$

Thus, $-Lx_n = y_{n-1}$ for all $n \geq 1$, which implies $\sup_n |x_n| < \infty$, since $y \in l_\infty$ and $L \neq 0$. This shows that operator $(\Delta_v - LI)^{\times}$ is onto and hence $L \in C_1\sigma\,(\Delta_v, l_1)$. □

Theorem 5.3. Let (v_k) be a constant sequence, say $v_k = L$ for all $k \in \mathbb{N}_0$ and $\alpha \neq L$, $\alpha \in \sigma_r\,(\Delta_v, l_1)$. Then $\alpha \in C_2\sigma\,(\Delta_v, l_1)$.

Proof. It is sufficient to show that the operator $(\Delta_v - \alpha I)^{-1}$ is discontinuous for $\alpha \neq L$ and $\alpha \in \sigma_r\,(\Delta_v, l_1)$. The operator $(\Delta_v - \alpha I)^{-1}$ is discontinuous by statements (3.8) and (3.9) for $L \neq \alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| \leq 1$. □

Theorem 5.4. Let (v_k) be a strictly decreasing sequence of positive real numbers and $\alpha \in \sigma_r(\Delta_v, l_1)$. Then $\alpha \in C_2\sigma\left(\Delta_v, l_1\right)$.

Proof. It is sufficient to show that the operator $(\Delta_v - \alpha I)^{-1}$ is discontinuous for $\alpha \in \sigma_r\left(\Delta_v, l_1\right)$. The operator $\left(\Delta_v - \alpha I\right)^{-1}$ is discontinuous by statements (3.8), (3.9) and (3.10) for $v_k \neq \alpha \in \mathbb{C}$ with $\left|1 - \frac{\alpha}{L}\right| \leq 1$. □

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