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Hyers-Ulam Stability of Linear Differential Equations $y'' = \lambda^2 y$

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Abstract : The aim of this paper is to prove the stability in the sense of Hyers-Ulam of differential equation of second order $y'' = \lambda^2 y$.

Keywords : Hyers-Ulam stability, Differential equation.
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1 Introduction and preliminaries

In 1940, S. M. Ulam [18] posed the following problem concerning the stability of functional equations: Give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by D. H. Hyers [2] when G_1 and G_2 are Banach spaces and the result of Hyers was generalized by Th. M. Rassias (see [14]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [3], [4], [5], [13] and [14]).

C. Alsina and R. Ger [1] remarked that the differential equation $y' = y$ has the Hyers-Ulam stability. More explicitly, they proved that if a differentiable function $y : I \to R$ satisfies $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \leq 3\varepsilon$ for every $t \in I$.

The above result of C. Alsina and R. Ger has been generalized by T. Miura, S.-E. Takahasi and H. Choda [12], by T. Miura [9], and also by S.-E. Takahasi, T. Miura and S. Miyajima [16]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while C. Alsina and R. Ger investigated the differential equation $y'(t) = y(t)$.

Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized by T. Miura, S. Miyajima and S. -E. Takahasi [11], by S.-E. Takahasi, H. Takagi, T. Miura and S. Miyajima [17], and also by S.-M. Jung ([4], [5], [8]). They dealt with the nonhomogeneous linear differential equation of first order

$$y' + p(t)y + q(t) = 0.$$

S.-M. Jung [8] studied the generalized Hyers-Ulam stability of differential equations of the form $ty'(t) + \alpha y(t) + \beta t^r x_0 = 0$. Recently, G. Wang, M. Zhou and L. Sun [19] discussed the Hyers-Ulam stability of the first-order nonhomogeneous linear differential equation.

Motivated by the works of [16] and [19], in this paper, we will investigate the Hyers-Ulam stability of the following linear differential equations of second order:

$$y'' = \lambda^2 y \tag{1.1}$$

where $y \in C^2(I) = C^2(a, b), -\infty < a < b < +\infty, \lambda > 0$.

We say that Eq. (1.1) has the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: for every $\varepsilon > 0, y \in C^2(I)$, if

$$|y'' - \lambda^2 y| \leq \varepsilon,$$

then there exists some $z \in C^2(I)$ satisfying

$$z'' - \lambda^2 z = 0$$

such that $|y(x) - z(x)| \leq K\varepsilon$. We call such K a Hyers-Ulam stability constant for Eq. (1.1).

2 Main Results

Now, the main result of this work is given in the following theorem.

Theorem 2.1. *If a twice continuously differentiable function $y : I \to R$ satisfies the differential inequality*

$$|y'' - \lambda^2 y| \leq \varepsilon$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $v : I \to R$ of the Eq. (1) such that

$$|y(x) - v(x)| \leq K\varepsilon$$

Where $K > 0$ is a constant.

Proof. Let $\varepsilon > 0$ and $y : I \to R$ be a twice continuously differentiable function such that

$$|y'' - \lambda^2 y| \leq \varepsilon$$

We will show that there exists a constant K independent of ε and v such that $|y - v| \leq K\varepsilon$ for some $v \in C^2(I)$ satisfying $v'' - \lambda^2 v = 0$.

If we set

$$g(x) = y'(x) - \lambda y(x),$$

then

$$g'(x) = y''(x) - \lambda y'(x)$$

thus

$$|g'(x) + \lambda g(x)|$$
$$= |y''(x) - \lambda y'(x) + \lambda(y'(x) - \lambda y(x))|$$
$$= |y'' - \lambda^2 y| \leq \varepsilon$$

Equivalently, g satisfies

$$-\epsilon \leq g'(x) + \lambda g(x) \leq \epsilon$$

Multiplying the formula by the function $e^{\lambda(x-a)}$, we obtain

$$-\epsilon e^{\lambda(x-a)} \leq g'(x)e^{\lambda(x-a)} - \lambda g(x)e^{\lambda(x-a)} \leq \epsilon e^{\lambda(x-a)}$$

For the case $0 < \lambda \leq 1$, there exists $M > 0$ such that $M\lambda > 1$, so without loss of generality, we may assume that $\lambda > 1$, thus

$$-\lambda\epsilon e^{\lambda(x-a)} \leq g'(x)e^{\lambda(x-a)} - \lambda g(x)e^{\lambda(x-a)} \leq \lambda\epsilon e^{\lambda(x-a)} \tag{2.1}$$

For some fixed $c \in (a, b)$ with $g(c) < \infty$ and any $x \in (c, b)$, integrating (2.1) from c to x, we get

$$-\varepsilon(e^{\lambda(x-a)} - e^{\lambda(c-a)}) \leq g(x)e^{\lambda(x-a)} - g(c)e^{\lambda(c-a)} \leq \varepsilon(e^{\lambda(x-a)} - e^{\lambda(c-a)})$$

so

$$-\varepsilon e^{\lambda(x-a)} \leq g(x)e^{\lambda(x-a)} - (g(c) - \varepsilon)e^{\lambda(c-a)} \leq \varepsilon e^{\lambda(x-a)}$$

Multiplying the formula by the function $e^{-\lambda(x-a)}$, we get

$$-\varepsilon \leq g(x) - (g(c) - \varepsilon)e^{\lambda(c-a)}e^{-\lambda(x-a)} \leq \varepsilon$$

$$-\varepsilon \leq g(x) - (g(c) - \varepsilon)e^{\lambda(c-x)} \leq \varepsilon$$

Let $z(x) = (g(c) - \varepsilon)e^{\lambda(c-x)}$, then $z(x)$ satisfies

$$z'(x) + \lambda z(x) = 0$$

and

$$|g(x) - z(x)| \leq \varepsilon$$

For any $x \in (x, c)$, the proof is very similar to the above, so we omit it.

Since $g(x) = y'(x) - \lambda y(x)$, we have

$$-\epsilon \leq y'(x) - \lambda y(x) - z(x) \leq \epsilon \tag{2.2}$$

By an argument similar to the above, we can show that there exists $u(x) = (g(c) - \epsilon)e^{\lambda(x-c)} - e^{\lambda(x-a)} \int_x^b z(s)e^{-\lambda(s-a)}ds$ such that

$$|y(x) - u(x)| \leq \epsilon$$

and $u \in C^2(I)$ satisfying

$$u'(x) - \lambda u(x) - z(x) = 0$$

so

$$z(x) = u'(x) - \lambda u(x)$$

by

$$z'(x) + \lambda z(x) = 0$$

We obtain

$$u''(x) - \lambda u'(x) + \lambda(u'(x) - \lambda u(x)) = 0$$

Hence

$$u''(x) - \lambda^2 u(x) = 0$$

which completes the proof. □

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