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Some Remarks on a Common Fixed Point Theorem in Fuzzy Metric Spaces

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Abstract : We correct some results in a recent paper of this journal.

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1 Main Results

The aim of this paper is to correct some results in [2], Section 3. The terminology and the notations are those in [2].

Theorem 3.2 in the quoted paper states as follows:

Let A, B, S and T be mappings from a fuzzy metric space $(X, M, *)$ into itself satisfying the conditions:

(1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$

(2) $F_{Ax,By}(t) \geq r(F_{Sx,Ty}(t))$ *for all* x, y *in* X, *where* $r : [0,1] \to [0,1]$ *is a continuous function such that* $r(t) > t$ *for each* $0 < t < 1, r(0) = 0$ *and* $r(1) = 1$.

(3) *If there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *such that* $x_n \to x$ *and* $y_n \to y$ *and* $t > 0$, *then* $F_{x_n,y_n}(t) \to F_{x,y}(t)$.

Suppose that

(4) *One of* A, B, S *and* T *is continuous and*

(5) *The pairs* (A, S) *and* (B, T) *are R-weakly commuting on* X.

Then the mappings A, B, T, S *have a unique common fixed point.*

We give an example to show that the above result is not true.

Example 1.1. Let $X = \{0, 1\}$, and

$$M(0,0,t) = M(1,1,t) = 1 \quad (t > 0)$$

$$M(0,1,t) = M(1,0,t) = \begin{cases} 0, & \text{if} \quad t \leq 1 \\ 1, & \text{if} \quad t > 1. \end{cases}$$

Let $A, B, S, T : X \to X, Ax = Bx = \begin{cases} 0, & \text{x=1} \\ 1, & \text{x=0}, \end{cases}$ $Sx = Tx = \begin{cases} 0, & \text{x=0} \\ 1, & \text{x=1}, \end{cases}$ and

$$r : [0,1] \to [0,1], r(t) = \sqrt{t}.$$

Obviously, the mapping r is continuous and $r(t) > t \ \forall t \in (0,1)$.

Next, it is immediate that the space $(X, M, *)$ is a fuzzy metric space under the t-norm Min. Since a sequence $\{x_n\}$ in X converges to $x \in X$ iff there is n_0 such that $x_n = x \ \forall n \geq n_0$, it follows that $(X, M, *)$ is complete and if $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$.

Also, the mappings A, B, T, S are continuous, $A(X) = B(X) = T(X) = S(X) = X$ and A, S and B, T are R-weakly commuting, for $AS = SA = A$.

Finally, since $M(0, 1, t) \in \{0, 1\} \ \forall t > 0$,

$$r(M(Ax, By, t)) \leq M(Sx, Ty, t) \quad (x, y \in X, t > 0)$$

reduces to $M(x, y, t) = r(M(x, y, t)) \quad (x, y \in \{0, 1\}, t > 0)$.

Thus, all the conditions of Theorem 3.2 in [2] are satisfied. Nevertheless, A and S have not any common fixed point.

This happens because the relation $r(l) > l$ (see the proof of Lemma 3.1 in [2]) is not always true, for the strict inequality $r(t) > t$ does not hold if $t = 0$.

We can correct this result by adding to the hypotheses of Theorem 3.2 the condition $M(x, y, t) > 0 \ \forall t > 0$. This condition also ensures the uniqueness of the common fixed point, see [3] (we note that the inequality $r(M(z, z', t)) > M(z, z', t)$ in the proof of Theorem 3.2 in [2] is not true if $M(z, z', t) = 0$).

We also note that Example 3.3 in [2] is not correct, because the considered fuzzy metric space $(X, M, *)$, where

$$X = [2, 20], \ M(x, y, t) = \frac{t}{t + |x - y|} \quad (t > 0), \ a * b = ab$$

is not complete (G-complete, cf. the terminology in [1] and [4]).

Example 1.2. For every $m \in \mathbb{N}$ with $2^k \leq m < 2^{k+1}$, define

$$x_m = 3 + cos\frac{2\pi(m - 2^k)}{2^k}.$$

Then $x_{2^n} = 4$ and $x_{3\cdot 2^n} = 3 + cos\frac{2\pi(3\cdot 2^n - 2^{n+1})}{2^{n+1}} = 3 + cos\pi = 2 \quad (n \in \mathbb{N})$, hence $\{x_n\}$ is not convergent.

On the other hand, if $n \in \mathbb{N}$ is such that $2^k \leq n < 2^{k+1}$, then either $2^k \leq n + 1 < 2^{k+1}$ or $n + 1 = 2^{k+1}$. Correspondingly, $x_{n+1} - x_n$ is either

$$cos\frac{2\pi(n + 1 - 2^k)}{2^k} - cos\frac{2\pi(n - 2^k)}{2^k}$$

or

$$cos\frac{2\pi(2^{k+1} - 2^{k+1})}{2^{k+1}} - cos\frac{2\pi(2^{k+1} - 1 - 2^k)}{2^k} = 1 - cos\frac{2\pi(2^k - 1)}{2^k}.$$

As $|cos\alpha - cos\beta| \leq 2|sin\frac{\alpha-\beta}{2}|$, in the first case $|x_{n+1} - x_n| \leq 2|sin\frac{\pi}{2^k}|$, hence $lim_{n\to\infty}|x_{n+1} - x_n| = 0$. In the second case the limit $lim_{n\to\infty}|x_{n+1} - x_n|$ is obviously 0. Consequently,

$$lim_{n\to\infty}M(x_{n+1}, x_n, t) = lim_{n\to\infty}\frac{t}{t + |x_{n+1} - x_n|} = 1$$

for all $t > 0$, concluding that $\{x_n\}$ is Cauchy.

To exemplify the corrected version of Theorem 3.2, instead of Example 3.3 in [2] one may consider the space (X, M, Min), where

$$X = \{1, 1/2, ..., 1/n, ...\} \cup \{0\},$$

$$M(x, y, 0) = 0,\ M(x, y, t) = \frac{t}{t + |x - y|} \quad (t > 0)$$

and the mappings $r : [0, 1] \to [0, 1]$,

$$r(t) = \sqrt{t},$$

$A, B, T, S : X \to X$,

$$Ax = Bx = 1 \quad (x \in X),$$

$$Sx = Tx = 1 \text{ if } x \text{ is rational}, Sx = Tx = 0 \text{ if } x \text{ is irrational}.$$

Then (see [5], Example) the conditions of Theorem 3.2 in [2], as well as the additional one $M(x, y, t) > 0\ \forall t > 0$ are satisfied (the common fixed point of A, B, S, T is $x = 1$).

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