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Some Conditions on Non-Normal Operators which Imply Normality

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Abstract : In this paper, we prove the following assertions:

(i) Let $A, B, X \in \mathbf{B}(\mathcal{H})$ be such that A^* is p-hyponormal or log-hyponormal, B is a dominant and X is invertible. If $XA = BX$, then there is a unitary operator U such that $AU = UB$ and hence A and B are normal.

(ii) Let $T = A + iB \in \mathbf{B}(\mathcal{H})$ be the cartesian decomposition of T with AB is p-hyponormal. If A or B is positive, then T is normal.

(iii) Let $A, V, X \in \mathbf{B}(\mathcal{H})$ be such that V, X are isometries and A^* is p-hyponormal. If $VX = XA$, then A is unitary.

(iv) Let $A, B \in \mathbf{B}(\mathcal{H})$ be such that $A + B \geq \pm X$. Then for every paranormal operator $X \in \mathbf{B}(\mathcal{H})$ we have

$$\|AX + XB\| \geq \|X\|^2.$$

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1 Introduction

Let $\mathcal{H}$ be infinite dimensional complex Hilbert, and let $\mathbf{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acts on $\mathcal{H}$. Let $\|.\|$ denote the spectral norm, and $\langle ., . \rangle$ be an inner product in $\mathcal{H}$. For $T \in \mathbf{B}(\mathcal{H})$, we denote the spectrum and the point spectrum of T by $\sigma(T)$, $\sigma_p(T)$.
An operator $A \in \mathbf{B}(\mathcal{H})$ is called positive if $\langle Ax, x \rangle \geq 0$ for all non-zero vectors $x \in \mathcal{H}$, isometry if $\|Ax\| = \|x\|$ for all a non-zero vector $x \in \mathcal{H}$, unitary if

$A^*A = AA^* = I$, where I is the identity operator, normal if $AA^* = A^*A$, hyponormal if $Q_A \geq 0$, where $Q_A = A^*A - AA^*$. We say that A is M-hyponormal for $M > 0$ if $(A - \lambda I)(A - \lambda I)^* \leq M(A - \lambda I)^*(A - \lambda I)$ for all $\lambda \in \mathbb{C}$, dominant if $ran(A - \lambda I) \subset ran(A - \lambda I)^*$ for all $\lambda \in \mathbb{C}$, where $ran(T)$ is the range of T and normaloid if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T.
In [1], an operator T is called p-hyponormal if $|T|^{2p} \geq |T^*|^{2p}$ for $0 < p \leq 1$, where $|T|$ is the square roots of T^*T, that is, $|T| = (T^*T)^{\frac{1}{2}}$. We also say that T is co-hyponormal , co-p-hyponormal, co-M-hyponormal and co-dominant if T^* is hyponormal, p-hyponormal, M-hyponormal and dominant, respectively.
The well-known Fuglede-Putnam Theorem asserts that if A and B are normal and $AX = XB$ for some operator $X \in \mathbf{B}(\mathcal{H})$, then $A^*X = XB^*$. (See [3]). In past years several authors have extended this theorem for non-normal operators, Yoshino [15], proved that if A^* is M-hyponormal, B is dominant and $CA = BC$ for some $C \in \mathbf{B}(\mathcal{H})$, then $CA^* = B^*C$.
Recently, Uchiyama and Tanahashi [14] proved that if A, B^* are p-hyponormal(resp. log-hyponormal) and $AX = XB$, then $A^*X = XB^*$.

2 Main Results

The next theorems explain what conditions imply normality for p-hyponormal operators.

Theorem 2.1. *Let $A, B, X \in \mathbf{B}(\mathcal{H})$ be such that A^* is a p-hyponormal or a* log-*hyponormal, B is a dominant and X is an invertible. If $XA = BX$, then there is a unitary U such that $AU = UB$ and hence A and B are normal.*

Proof. Since $XA = BX$, it follows from Fuglede-Putnam theorem for p-hyponormal [14, theorem 3] that $B^*X = XA^*$ and so $X^*B = AX^*$.
Now

$$AX^*X = X^*BX = X^*XA.$$

Let $X = UP$ be the polar decomposition of X. Since X is an invertible, it follows that P is invertible and U is unitary. Since $AP^2 = P^2A$ and P is positive, it follows that $AP = PA$. Thus $BUP = UPA$ implies $BUP = UAP$. But P is an invertible, we have $BU = UA$. Therefore A, B are unitary equivalent. So, A is dominant and B^* is p-hyponormal. Hence A, B are normal. □

As a consequence of theorem 2.1, we have immediately

Corollary 2.2. *Let $A, B, X \in \mathbf{B}(\mathcal{H})$ be such that A^* is a p-hyponormal or a* log-*hyponormal, B is a dominant. If X is an invertible positive operator, then $XA = BX$ implies $A = B$.*

Theorem 2.3. *Let $T = A + iB \in \mathbf{B}(\mathcal{H})$ be the cartesian decomposition of T with AB is p-hyponormal. If A or B is positive, then T is normal.*

Proof. Assume first that A is positive. Let $S = AB$, then $SA = AS^*$. Then it follows from Fuglede-Putnam theorem for p-hyponormal [14, corollary 2] that $S^*A = AS$, that is, $BA^2 = A^2B$. But A is positive, then $AB = BA$, i.e., T is normal
Now, if B is positive, then apply the same argument to $-iT = B - iA$. □

Theorem 2.4. *Let $T = A + iB$ be the cartesian decomposition of T. If T^* is hyponormal operator and AB is p-hyponormal operator, then T is normal operator.*

Proof. Let $Q = AB$, then $QA = AQ^* = ABA$. Then by Fuglede-Putnam's theorem for p-hyponormal operators, we have $Q^*A = AQ$, i.e., $BA^2 = A^2B$.
Now

$$(Q + Q^*)A = A(Q + Q^*)$$

and

$$(Q - Q^*)A = A(Q^* - Q).$$

Since T^* is hyponormal, we have

$$TT^* - T^*T = 2i(BA - AB) = 2i(Q^* - Q) \geq 0.$$

Let $Y = 2i(BA - AB)$ then $Y \geq 0$ and $YA = -AY$. Now

$$\begin{aligned} Y^2A &= Y(YA) \\ &= Y(-AY) \\ &= -YAY \\ &= -(-AY)Y \\ &= AY^2. \end{aligned}$$

But Y is positive, then $YA = AY = 0$. Hence, $A(AB - BA) = (AB - BA)A = 0$ implies that $\sigma(AB - BA) = \{0\}$. Therefore $AB - BA$ is quasinilpotent skew-hermitian. Thus $AB - BA = 0$. So T is normal. □

Theorem 2.5. *Let $A, V, X \in \mathbf{B}(\mathcal{H})$ be such that V, X are are isometries and A^* is p-hyponormal. If $VX = XA$, then A is unitary.*

Proof. Since $VX = XA$, then by Fuglede-Putnam theorem [14, corollary 2], we have $V^*X = XA^*$. Now multiplying the first equation by V^*, we get $X = V^*XA$, then $X(I - A^*A) = 0$ implies that $X^*X(I - A^*A) = 0$. Hence $A^*A = I$, so A is an isometry. Therefore A and A^* are p-hyponormal. So A is normal isometry. Hence A is unitary. □

The following theorem show that if $A, B \in \mathbf{B}(\mathcal{H})$ are hyponormal and $A^*B = BA^*$, the sum and product of A and B are hyponormal.

Theorem 2.6. *Let $A, B \in \mathbf{B}(\mathcal{H})$ be such that A, B are hyponormal and $A^*B = BA^*$. Then*
(a) $A + B$ is hyponormal.
(b) AB is hyponormal.

Proof. Since $A^*B = BA^*$, then $B^*A = AB^*$. Now

$$\begin{aligned}(a)\quad (A+B)^*(A+B) - (A+B)(A+B)^* &= (A^*A + A^*B + B^*A + B^*B)\\ &\quad - (AA^* + AB^* + BA^* + BB^*)\\ &= (A^*A - AA^*) + (B^*B - BB^*).\end{aligned}$$

Using the fact that, the sum of two positive operators is positive operator. The result follows.

$$\begin{aligned}(b)\quad (AB)^*(AB) - (AB)(AB)^* &= B^*A^*AB - ABB^*A^*\\ &= B^*A^*AB - B^*AA^*B + B^*AA^*B - ABB^*A^*\\ &= B^*(A^*A - AA^*)B + A(B^*B - BB^*)A^*\\ &= B^*Q_AB + AQ_BA^*,\end{aligned}$$

where $Q_A = A^*A - AA^* \geq 0$ and $Q_B = B^*B_BB^* \geq 0$. The result holds by using the fact that if $X \geq 0$, then $E^*XE \geq 0$ and $EXE^* \geq 0$. □

Recall that [9], an operator T is paranormal operator if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in \mathcal{H}$.

Lemma 2.7. *([5, 6]) If T is paranormal operator, then T is normaloid.*

Theorem 2.8. *Let $P, Q \in \mathbf{B}(\mathcal{H})$. Let $C = PQ - QP$. If P is normaloid, then $\|I - C\| \geq 1$.*

Proof. Since P is normaloid, it follows that $r(P) = \|P\|$. So there is a $\lambda \in \sigma(P)$ such that $|\lambda| = \|P\|$. Hence there is a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(P - \lambda I)x_n \to 0$, the normaloidity of P implies $(P^* - \overline{\lambda} I)x_n \to 0$. Now

$$\|I - C\| \geq |\langle (I - C)x_n, x_n\rangle| = |1 - \langle Cx_n, x_n\rangle| \geq 1 - |\langle Cx_n, x_n\rangle|.$$

The result follows if we show that $\langle Cx_n, x_n\rangle \to 0$.
But

$$\begin{aligned}\langle Cx_n, x_n\rangle &= \langle ((P - \lambda I)Q - Q(P - \lambda I))x_n, x_n\rangle\\ &= \langle Qx_n, (P - \lambda I)^*x_n\rangle - \langle (P - \lambda I)x_n, Q^*x_n\rangle\end{aligned}$$

So

$$|\langle Cx_n, x_n\rangle| \leq \|Q\|(\|(P - \lambda I)^*x_n\| + \|(P - \lambda I)x_n\|) \to 0.$$

□

Theorem 2.9. *Let $A, B \in \mathbf{B}(\mathcal{H})$ be self-adjoint such that $A + B \geq a \geq 0$. Then for every normaloid $X \in \mathbf{B}(\mathcal{H})$ we have*

$$\|AX + XB\| \geq a\|X\|.$$

Proof. Since X is normaloid, it follows that $r(X) = \|X\|$. So there is a $\lambda \in \sigma(X)$ such that $|\lambda| = \|X\|$. Hence there is a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(X - \lambda I)x_n \to 0$, the normaloidity of X implies $(X^* - \overline{\lambda} I)x_n \to 0$.
Now

$$\begin{aligned}\|AX + XB\| &\geq |\langle (AX + XB)x_n, x_n\rangle| \\ &= |\langle A(X - \lambda I)x_n, x_n\rangle + \langle (X - \lambda I)Bx_n, x_n\rangle + \lambda\langle (A + B)x_n, x_n\rangle| \\ &= |\langle (X - \lambda I)x_n, Ax_n\rangle + \langle Bx_n, (X^* - \overline{\lambda})x_n\rangle + \lambda\langle (A + B)x_n, x_n\rangle| \\ &\geq |\lambda||\langle (A + B)x_n, x_n\rangle| - \text{terms which goes to zero as } n \to \infty \\ &\geq |\lambda|a - \text{terms which goes to zero as } n \to \infty.\end{aligned}$$

Hence

$$\|AX + XB\| \geq a\|X\|.$$

□

Theorem 2.10. *([7])Let $A, B \in \mathbf{B}(\mathcal{H})$ be self-adjoint such that $A + B \geq \pm X$. Then for every self-adjoint $X \in \mathbf{B}(\mathcal{H})$ we have*

$$\|AX + XB\| \geq \|X\|^2.$$

Lemma 2.11. *If $A \in \mathbf{B}(\mathcal{H})$ is self-adjoint then $\pm A \leq |A|$*

Proof. Let $A = U|A|$ be the polar decomposition of A. Since A is self-adjoint then $A = U|A| = |A|U^*$ and

$$\begin{aligned}(U|A|U^*)^2 &= U|A|U^*U|A|U^* \\ &= U|A|^2U^* \\ &= A^2 = |A|^2,\end{aligned}$$

and so $U|A|U^* = |A|$.
Now for any $x \in \mathcal{H}$ we have

$$\begin{aligned}|\langle Ax, x\rangle|^2 &= |\langle U|A|x, x\rangle|^2 \\ &= |\langle |A|x, U^*x\rangle|^2 \\ &\leq \langle |A|x, x\rangle \langle |A|U^*x, U^*x\rangle \qquad \text{(by the Generalized Cauchy Schwartz}\end{aligned}$$

inequality)

$$\begin{aligned}&= \langle |A|x, x\rangle \langle U|A|U^*x, x\rangle \\ &= \langle |A|x, x\rangle \langle |A|x, x\rangle \\ &= \langle |A|x, x\rangle^2 .\end{aligned}$$

Hence $|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle$. □

Corollary 2.12. *([7])Let $A, B \in \mathbf{B}(\mathcal{H})$ be self-adjoint such that $A+B \geq |X|$ and $A+B \geq |X^*|$. Then*

$$\max(\|AX + XB\|, \|AX^* + X^*B\|) \geq \|X\|^2.$$

Proof. On $\mathcal{H} \oplus \mathcal{H}$, let $T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ then Y is self-adjoint and $|Y| = \begin{pmatrix} |X^*| & 0 \\ 0 & |X| \end{pmatrix}$. From $A+B \geq |X|$ and $A+B \geq |X^*|$, we obtain that $T+S \geq |Y|$ and hence $T+S \geq \pm Y$ by Lemma 2.11. Now by applying Theorem 2.10 to T, S and Y to get

$$\begin{aligned} \|TY + YS\| &= \left\| \begin{pmatrix} 0 & AX + XB \\ AX^* + X^*B & 0 \end{pmatrix} \right\| \\ &= \max(\|AX + XB\|, \|AX^* + X^*B\|) \\ &\geq \|Y\|^2 \\ &= \|X\|^2. \end{aligned}$$

□

Theorem 2.13. *Let $A, B \in \mathbf{B}(\mathcal{H})$ be self-adjoint such that $A+B \geq a \geq 0$. Then for every normaloid $X \in \mathbf{B}(\mathcal{H})$ we have*

$$\|XAX^* + X^*BX\| \geq a\|X\|^2.$$

Proof. Since X is normaloid, it follows from lemma 2.7 that $r(X) = \|X\|$. So there is a sequence of unit vectors $\{x_n\}$ in $\mathcal{H}$ such that $(X-t)x_n \to 0$, where $|t| = \|X\|$, and so $(X-t)^*x_n \to 0$.
Now

$$\begin{aligned} \|XAX^* + X^*BX\| &\geq |\langle (XAX^* + X^*BX)x_n, x_n \rangle| \\ &= |\langle AX^*x_n, (X-t)^*x_n \rangle + t\langle A(X-t)^*x_n, x_n \rangle + |t|^2 \langle Ax_n, x_n \rangle \\ &+ \langle BXx_n, (X-t)x_n \rangle + \bar{t}\langle B(X-t)x_n, x_n \rangle + |t|^2 \langle Bx_n, x_n \rangle | \\ &\geq a|t|^2 - \text{terms which goes to zero as } n \to \infty. \end{aligned}$$

Letting $n \to \infty$, we get

$$\|XAX^* + X^*BX\| \geq a\|X\|^2.$$

□

We point out here that Theorem 2.13 is not true if the assumption on X that is normaloid is removed. For example, consider

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which act on a two-dimensional Hilbert space.

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