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Some Inequalities Related to a New Sequence Space which Include c

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Abstract : The sequence space c_A including the space c has recently been defined in [10] and its some properties have been investigated. In the present paper, we have studied some inequalities related to the this sequence space analogously to those that given in [2, 6, 7].

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1 Introduction

Let ℓ_∞ and c be the spaces of all real bounded and convergent sequences, respectively.

Let us define the matrix $A^r = (a^r_{nk})$ by

$$a^r_{nk} = \begin{cases} \frac{1+r^k}{1+n}, & \text{for } 0 \le k \le n \\ 0, & \text{for } k > n. \end{cases}$$

Using the convergence domain of the matrix A^r, the new sequence spaces a^r_c including the space c has been constructed in [1] and its some properties has been investigated. Using the definition of the sequence space a^r_c some inclusion theorems which like to Knoop' s Core Theorem were proved in [4] and [5].

Let $A = (a_{nk})$ be a lower triangular matrix defined by $a_{nk} = a_n b_k$, where a_n depends only on n and b_k only on k, $0 \le k \le n$. Recently, as a generalization of the sequence space a^r_c, the sequence space c_A has been defined by

$$c_A = \Big\{x = (x_k) : \{t_n(x)\} \in c, t_n(x) = \frac{1}{a_n}\sum_{k=0}^{n} b_k x_k\Big\}$$

and its some topological properties have been studied in [10]. Also; it is shown that c_A is isomorphic to c whenever A is regular. Throughout the paper we consider the regular matrices A. Also, $(c_0)_A$ has been defined by

$$c_A = \Big\{x = (x_k) : \{t_n(x)\} \in c_0\Big\}.$$

Let E be a subset of $\mathbb{N} = \{0, 1, 2, ...\}$. Natural density δ of E is defined by

$$\delta(E) = \lim_n \frac{1}{n} \left| \{k \leq n : k \in E\} \right|,$$

where the vertical bars indicate the number of elements in the enclosed set. A sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every ε, $\delta\{k : |x_k - l| \geq \varepsilon\} = 0$, [8]. Statistical limit superior and inferior of a real sequence was defined in [9].

Let $T = (t_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ a real sequence such that

$$Tx = (T_n(x)) = \left(\sum_k t_{nk} x_k \right) \quad \text{exists for each } n.$$

Then the sequence $Tx = (T_n(x))$ is called T- transform of x. For two sequence spaces X and Y we say that the matrix T map X into Y if Tx exits and belongs to Y for all $x \in E$. By (X, Y), we denote the set of all matrices which map X into Y.

In what follows, for any $\lambda \in \mathbb{R}$, we write $\lambda^- = \max\{-\lambda, 0\}$ and $\lambda^+ = \max\{0, \lambda\}$. Then, it is clear that $\lambda = \lambda^+ - \lambda^-$ and $|\lambda| = \lambda^+ + \lambda^-$.

The aim of this paper is to establish some inequalities like to those that studied in [2, 6, 7] by using the definition of the sequence space c_A.

2 Lemmas

In this section, we present some lemmas which will be useful to the proof of our main results.

Lemma 2.1 [2, Th.1(c)] *Let $\mathcal{A} = (a_{nk}(i))$ be conservative. Then, for some constant $\lambda \geq |\chi|$ and for all $x \in \ell_\infty$,*

$$\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k) x_k \leq \frac{\lambda + \chi}{2} L(x) - \frac{\lambda - \chi}{2} l(x)$$

if and only if

$$\limsup_n \sup_i \sum_k |a_{nk}(i) - a_k| \leq \lambda, \tag{1}$$

where χ is the characteristic of $\mathcal{A}$.

Lemma 2.2 [2, Lemma 1] *Let $\mathcal{A} = (a_{nk}(i))$ be conservative and $\lambda \geq 0$. Then (1) holds if and only if*

$$\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)^+ \leq \frac{\lambda + \chi}{2}$$

and

$$\limsup_n \sup_i \sum_k (a_{nk}(i) - a_k)^- \le \frac{\lambda - \chi}{2}.$$

Lemma 2.3 [2, Lemma 2] *Let* $\|\mathcal{A}\| < \infty$ *and* $\lim_n sup_i a_{nk}(i) = 0$. *Then, there exists a* $y \in \ell_\infty$ *with* $\|y\| \le 1$ *such that*

$$\limsup_n \sup_i \sum_k a_{nk}(i) y_k = \limsup_n \sup_i \sum_k |a_{nk}(i)|.$$

Lemma 2.4 *Let* X *be any sequence space. Then,* $T \in (X, c_A)$ *if and only if* $D \in (X, c)$*, where* D *is defined as in the proof.*

Proof. Let $x \in X$ and consider the equality

$$\frac{1}{a_n} \sum_{i=0}^{n} b_i \sum_{k=0}^{m} t_{ik} x_k = \sum_{k=0}^{m} \frac{1}{a_n} \sum_{i=0}^{n} b_i t_{ik} x_k;\ (m, n \in \mathbb{N})$$

which yields as $m \to \infty$ that

$$\frac{1}{a_n} \sum_{i=0}^{n} b_i (Tx)_i = (Dx)_n;\ (n \in \mathbb{N}),$$

where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \frac{1}{a_n} \sum_{i=0}^{n} b_i t_{ik}, & \text{for } 0 \le k \le n \\ 0, \text{ for} & \text{for } k > n, \end{cases} \tag{2}$$

for all $k, n \in \mathbb{N}$. Therefore, since c_A is isomorphic to c, one can easily see that $T \in (X, c_A)$ if and only if $D \in (X, c)$ and this completes the proof. □

One can easily deduce from Lemma 2.4 that $T \in (c, c_A)$ if and only if

$$\sup_n \sum_k |d_{nk}| < \infty,\ \ \lim_n d_{nk} = \alpha_k$$

for each k and $\lim_n \sum_k d_{nk} = \alpha$. If $T \in (c, c_A)$, the number $\chi(T) = \alpha - \sum_k \alpha_k$ is defined and it is said to be characteristic number of T with respect to A.

Now, we may give our main results.

3 Main Results

Theorem 3.1 *Let* $T \in (c, c_A)$. *Then, for some constant* $\lambda \ge |\chi(T)|$ *and for all* $x \in \ell_\infty$,

$$\limsup_n \sum_k (d_{nk} - \alpha_k) x_k \le \frac{\lambda + \chi(T)}{2} L(x) - \frac{\lambda - \chi(T)}{2} l(x) \tag{3}$$

if and only if

$$\limsup_n \sum_k |d_{nk} - \alpha_k| \leq \lambda, \tag{4}$$

where d_{nk} *is defined by (2),* $L(x) = \limsup x$*, and* $l(x) = \liminf x$*.*

Proof. Let (3) holds. Define the matrix $C = (c_{nk})$ by $c_{nk} = d_{nk} - \alpha_k$ for all $n, k \in \mathbb{N}$. Then, the matrix C satisfies the conditions of Lemma 2.3. Hence, for a $y \in \ell_\infty$ with $\|y\| \leq 1$, we have

$$\limsup_n \sum_k c_{nk} y_k = \limsup_n \sum_k |c_{nk}|. \tag{5}$$

So, from (3), we get that

$$\begin{aligned}\limsup_n \sum_k |c_{nk}| &\leq \frac{\lambda + \chi(T)}{2} L(y) - \frac{\lambda - \chi(T)}{2} l(y) \\ &\leq \Big[\frac{\lambda + \chi(T)}{2} + \frac{\lambda - \chi(T)}{2}\Big] \|y\| \\ &\leq \lambda\end{aligned}$$

which shows the necessity of the condition (4).

Conversely, suppose that (4) holds. For any $x \in \ell_\infty$, we can write $l(x) - \varepsilon \leq x_k \leq L(x) + \varepsilon$ whenever $k \geq k_0$. Now, we can write

$$\sum_k c_{nk} x_k = \sum_{k<k_0} c_{nk} x_k + \sum_{k \geq k_0} c_{nk}^{+} x_k - \sum_{k \geq k_0} c_{nk}^{-} x_k.$$

Hence, from the Lemma 2.2 and the fact that $T \in (c, c_A)$, we get that

$$\begin{aligned}\limsup_n \sum_k c_{nk} x_k &\leq \frac{\lambda + \chi(T)}{2}[L(x) + \varepsilon] - \frac{\lambda - \chi(T)}{2}[l(x) - \varepsilon] \qquad (6) \\ &= \frac{\lambda + \chi(T)}{2} L(x) - \frac{\lambda - \chi(T)}{2} l(x) + \lambda\varepsilon.\end{aligned}$$

Since ε is arbitrary, the proof is completed. □

Theorem 3.2 *Let* $T \in (c, c_A)$*. Then, for some constant* $\lambda \geq |\chi(T)|$ *and for all* $x \in \ell_\infty$*,*

$$\limsup_n \sum_k (d_{nk} - \alpha_k) x_k \leq \frac{\lambda + \chi(T)}{2} \beta(x) + \frac{\lambda - \chi(T)}{2} \alpha(-x) \tag{7}$$

if and only if (4) holds and

$$\lim_n \sum_{k \in E} |d_{nk} - \alpha_k| = 0 \tag{8}$$

for every $E \subset \mathbb{N}$ *with* $\delta(E) = 0$*, where* $\beta(x) = st - \limsup x$ *and* $\alpha(x) = st - \liminf x$*.*

Proof. Let (7) holds. Then, since $\beta(x) \leq L(x)$ and $\alpha(-x) \leq -l(x)$, the necessity of the condition (4) follows from Theorem 3.1.

To show the necessity of the condition (8), for any $E \subset \mathbb{N}$ with $\delta(E) = 0$, define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} d_{nk} - \alpha_k, & \text{for } k \in E \\ 0, & \text{for } k \notin E \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, since $T \in (c, c_A)$, we can write (5) for B. Now; for the same E, let us choose the sequence (y_k) as

$$y_k = \begin{cases} 1, & \text{for } k \in E \\ 0, & \text{for } k \notin E. \end{cases}$$

Then; clearly y is a statistically null sequence and so, $\beta(y) = \alpha(y) = st - \lim y = 0$. Hence, by the assumption and the equation (5), we get that

$$\limsup_n \sum_{k \in E} |b_{nk}| \leq \frac{\lambda + \chi(T)}{2} \beta(y) + \frac{\lambda - \chi(T)}{2} \alpha(-y) = 0$$

which implies (8).

Conversely; suppose that (4) and (8) hold. For any $x \in \ell_\infty$, let us define $E_1 = \{k : x_k > \beta(x) + \varepsilon\}$ and $E_2 = \{k : x_k < \alpha(x) - \varepsilon\}$. Then, $\delta(E_1) = \delta(E_2) = 0$, [9]. Hence the set $E = E_1 \cap E_2$ has also zero density and

$$\alpha(x) - \varepsilon \leq x_k \leq \beta(x) + \varepsilon \tag{9}$$

whenever $k \notin E$. Now; it can be written that

$$\sum_k (d_{nk} - \alpha_k) x_k = \sum_{k \in E} (d_{nk} - \alpha_k) x_k + \sum_{k \notin E} (d_{nk} - \alpha_k)^+ x_k - \sum_{k \notin E} (d_{nk} - \alpha_k)^- x_k.$$

Thus, since (8) implies that the first sum on the right hand-side is zero, we get from a special case of Lemma 2.2 that

$$\limsup_n \sum_k (d_{nk} - a_k) x_k \leq \frac{\lambda + \chi}{2} \beta(x) + \frac{\lambda - \chi}{2} \alpha(-x) + \lambda \varepsilon.$$

Since ε is arbitrary, this step completes the proof. □

To the construction of next theorem, we need to define two new sublinear functionals on ℓ_∞. Let us denote these functionals by $L_A(x)$, $l_A(x)$ and write

$$L_A(x) = \limsup_n \frac{1}{a_n} \sum_{i=0}^{n} b_i t_{ik} x_k, \quad l_A(x) = \liminf_n \frac{1}{a_n} \sum_{i=0}^{n} b_i t_{ik} x_k.$$

Now, we have the following interesting result which will be useful to the proof of next theorem.

Theorem 3.3 *$L_A(x) = W_A(x)$ for all $x \in \ell_\infty$, where $W_A(x) = \inf_{z\in(c_0)_A} L(x+z)$.*

Proof. By the Lemma 1 of [3], it is enough to show that $L_A(x) \leq W_A(x)$ for all $x \in \ell_\infty$.

Since A is regular, we have that $L_A(x) \leq L(x)$ for any $x \in \ell_\infty$. Then, we get

$$\inf_{z\in(c_0)_A} L_A(x+z) \leq \inf_{z\in(c_0)_A} L(x+z) = W_A(x).$$

On the other hand, by the definition of the space $(c_0)_A$, $L_A(z) = 0$ and so,

$$\inf_{z\in(c_0)_A} L_A(x+z) \geq L_A(x) + \inf_{z\in(c_0)_A} L(x+z) = L_A(x).$$

Thus, the result follows. □

Theorem 3.4 *Let $T \in (c, c_A)$. Then, for some constant $\lambda \geq |\chi(T)|$ and for all $x \in \ell_\infty$,*

$$\limsup_n \sum_k (\tilde{t}^n_{mk} - \alpha_k)x_k \leq \frac{\lambda + \chi(T)}{2} L_A(x) + \frac{\lambda - \chi(T)}{2} l_A(-x) \tag{10}$$

if and only if

$$\limsup_n \sum_k |\tilde{t}^n_{mk} - \alpha_k| \leq \lambda, \tag{11}$$

where $\tilde{T}^n = (\tilde{t}^n_{mk})$ is defined by

$$\tilde{t}^n_{mk} = \begin{cases} \triangle_k\left(\frac{t_{nk}}{b_k}\right)\frac{1}{a_k}, & \text{for } 0 \leq k < m, \\ \frac{t_{nm}}{a_m b_m}, & \text{for } k = m, \\ 0, & \text{for } k > n. \end{cases}$$

for all $k, m \in \mathbb{N}$ and for every fixed $n \in \mathbb{N}$.

Proof. Suppose that (10) holds. Then, by the regularity of A, $L_A(x) \leq L(x)$ and $l_A(-x) \leq l(-x)$ for any $x \in \ell_\infty$. Therefore, the necessity (11) follows from a special case of Lemma 2.1.

Conversely, let (11) and $x \in \ell_\infty$. Then, we can write (6) with $\tilde{t}^n_{mk}$ in place of d_{nk}. Thus, by taking infimum for $z \in (c_0)_A$ in (6), we have

$$\begin{aligned}\inf_{z\in(c_0)_A}\left\{\limsup_n \sum_k (\tilde{t}^n_{mk} - \alpha_k)(x_k + z_k)\right\} &\leq \frac{\lambda + \chi(T)}{2} \inf_{z\in(c_0)_A} L_A(x+z) \\ &\quad + \frac{\lambda - \chi(T)}{2} \inf_{z\in(c_0)_A} l_A(-x-z) \\ &= \frac{\lambda + \chi(T)}{2} W_A(x) + \frac{\lambda - \chi(T)}{2} w_A(-x).\end{aligned}$$

On the other hand, by the definition of $(c_0)_A$, $\tilde{T}^n z \in c_0$ and so,

$$\begin{aligned}\inf_{z\in(c_0)_A}\left\{\limsup_n \sum_k (\tilde{t}^n_{mk}-\alpha_k)(x_k+z_k)\right\} &\geq \limsup_n \sum_k (\tilde{t}^n_{mk}-\alpha_k)x_k \\ &\quad + \inf_{z\in(c_0)_A}\limsup_n \sum_k (\tilde{t}^n_{mk}-\alpha_k)z_k \\ &= \limsup_n \sum_k (\tilde{t}^n_{mk}-\alpha_k)x_k.\end{aligned}$$

So, by combining the last inequalities and Theorem 3.3, we complete the proof. □

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