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On the Almost Sure Convergence Rates for Pairwise Negative Quadrant Dependent Random Variables[1]

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Abstract : For sequences of pairwise negative quadrant dependent random variables, we obtain the almost sure convergence rate $n^{-1/2}(\log n)^{3/2}(\log\log n)^{1/\xi}$ with any $0 < \xi < 2$ of $\sum_{i=1}^{n}(X_i - EX_i)/n \to 0$ *a.s.* by a maximal moment inequality, which improves the relevant results in Wu (2002) and Wang et al. (2008). In addition, the faster convergence rate $n^{-1/2}(\log n)^{1/2}$ is also obtained by the exponential inequality established in this paper, which reaches the available one for independent random variables in terms of Berstein type inequality. Further, we give the corresponding precise asymptotic with respect to the rate $n^{-1/2}(\log n)^{1/2}$.

Keywords : Pairwise negative quadrant dependence, Moment inequality, Exponential inequality, Almost sure convergence rate.
2000 Mathematics Subject Classification : 60F15, 62G20, 60G50

1 Introduction

Definition 1.1 The pair (X, Y) is said to be negative quadrant dependent (NQD) if for any $x, y \in \mathbf{R}$,

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y).$$

A sequence of random variables is said to be pairwise negative quadrant dependent (pairwise NQD) if X_i and X_j are NQD for any $i, j \in \mathbf{N}$ and $i \neq j$.

The definition above was given by Lehmann (1966). Obviously, sequences of pairwise NQD random variables are a family of very wide scope, which includes pairwise independent random variable sequences. Many known types of negative

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dependence such as negative upper (lower) orthant dependence and negative association (NA) etc. have developed on the basis of NQD notation. Among them the negative associated (NA) class with many applications in multivariate analysis is the special case of pairwise NQD sequences.

The limit theorems for pairwise NQD sequences have been investigated by some scholars, such as, Matula (1992) obtained the Kolmogorov strong law of large numbers for pairwise NQD random variable sequences with the same distribution, Wang et al. (1998) investigated the Marcinkiewicz weak law of large numbers, Wu (2002) gave the three series theorem of pairwise NQD sequences and proved the Marcinkiewicz strong law of large numbers, Chen (2005) discussed Kolmogorov-Chung strong law of large numbers for the non-identically distributed pairwise NQD sequences under mild conditions and, Li and Wang (2008) explored the central limit theorem for pairwise NQD random variables by Stein's method. In this paper, by the maximal moment inequality in Lemma 3.2 given later, we obtain the almost sure convergence rate $n^{-1/2}(\log n)^{3/2}(\log\log n)^{1/\xi}$ with any $0<\xi<2$ of $\sum_{i=1}^{n}(X_i-EX_i)/n \to 0$ *a.s.*, which improves the corresponding results in Wu (2002) and Wang et al. (2008). And by establishing an exponential inequality for pairwise NQD sequences, the faster convergence rate $n^{-1/2}(\log n)^{1/2}$ is also derived, which reaches the available one for independent random variables in terms of Berstein type inequality. Further, we give the corresponding precise asymptotic with respect to the rate $n^{-1/2}(\log n)^{1/2}$.

Throughout this paper, we always suppose that C denotes a positive constant which only depends on some given numbers and may vary from one place to another, $[x]$ denotes the integer part of x, $a_n \ll b_n$ means $a_n \le Cb_n$ and $S_n =: \sum_{i=1}^{n} X_i$. And this paper is organized as follows. Section 2 contains our main results. Section 3 contains some lemmas. And the proofs of Theorems 2.1, 2.2 and 2.3 are contained by the Sections 4, 5 and 6, respectively.

2 Main Results

In this section, we show the main results as follow.

Theorem 2.1 Let $\{X_i, i\ge 1\}$ be a pairwise NQD sequence of random variables with $EX_i=0$ and $\sup_{i\ge 1} E|X_i|^v<\infty$ for some $1<v\le 2$. Then, we have for any $0<\xi<2$,

$$S_n/\Big(n(\log\log n)^{2/\xi}\log^3 n\Big)^{1/v} \to 0 \quad a.s. \tag{2.1}$$

In particular, we have

$$S_n/\Big(n(\log\log n)^{2/\xi}\log^3 n\Big)^{1/2} \to 0 \quad a.s. \tag{2.2}$$

for the case $v=2$.

Remark 3.1.1 (1) Wu (2002) presented a strong law of large numbers for pairwise NQD random variables (see Corollary 3 therein). For convenience of comparison, we write the result below.

Theorem A Let $\{X_i, i \geq 1\}$ be a pairwise NQD sequence of random variables with $EX_i = 0$ and $\sup_{i\geq 1} E|X_i|^v \leq C$ for $1 < v \leq 2$. Then

$$\frac{S_n}{n^{1/v} (\log n)^{(3+\delta)/v}} \to 0 \;\; a.s., \tag{2.3}$$

where $\delta > 0$.

In terms of (2.3), the almost sure convergence rate of

$$\sum_{i=1}^{n} (X_i - EX_i)/n \to 0 \;\; a.s. \;\text{ is }\; n^{\frac{1}{v}-1} (\log n)^{\frac{3+\delta}{v}},$$

which is slower than the corresponding one $n^{\frac{1}{v}-1} (\log n)^{\frac{3}{v}} (\log \log n)^{\frac{2}{v\xi}}$ obtained by (2.1) because of $(\log \log n)^{\frac{2}{v\xi}} < (\log n)^{\frac{\delta}{v}}$ for n large enough. In addition, the method of the proof of our result is different from the relevant one of the proof of Theorem A.

(2) For the case of $v = 2$, the convergence rate derived by (2.2) is

$$n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} (\log \log n)^{\frac{1}{\xi}}$$

with $0 < \xi < 2$, which is obviously faster than the relevant one $n^{\frac{1}{\tilde{p}}-1}$ with $1 \leq \tilde{p} < 2$ which Wang et al. (2008) obtained in Theorem 2.3.1.

Theorem 2.2 Let $\{X_i, i \geq 1\}$ be a sequence of strictly stationary and pairwise NQD random variables which satisfies $EX_i = 0$ and $\sup_{i\geq 1} E(e^{\alpha|X_i|}) \leq M < \infty$ for some $\alpha > 1$. Then, we have

$$S_n/(n \log n)^{1/2} \to 0 \;\; a.s. \tag{2.4}$$

Remark 2.2 By Theorem 2.2, we obtain the almost sure convergence rate $n^{-1/2}(\log n)^{1/2}$, which is faster than the corresponding one $n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}} (\log \log n)^{\frac{1}{\xi}}$ with $0 < \xi < 2$ and reaches the available one obtained in terms of Berstein type inequality. The price we pay out, however, is that the moment condition $\sup_{i\geq 1} E(e^{\alpha|X_i|}) \leq M < \infty$ for some $\alpha > 1$ is stronger than the relevant one $\sup_{i\geq 1} E|X_i|^2 < \infty$ in Theorem 2.1.

In the following theorem, the precise asymptotic is obtained with respect to the rate $n^{-1/2}(\log n)^{1/2}$ in Theorem 2.2.

Theorem 2.3 Let $\{X_i, i \geq 1\}$ be a sequence of strictly stationary and pairwise NQD random variables which satisfies $EX_i = 0$ and $0 < EX_1^2 =: \sigma^2 < \infty$. If

$$(\sigma\sqrt{n})^{-1} \sum_{i=1}^{n} EX_i f\left(S_n^{(i)}/(\sigma\sqrt{n})\right) = 0$$

for every bounded absolutely continuous function $f(\cdot)$, where $S_n^{(i)} = \sum_{i=1}^{n} X_i - X_i$, and the Lindeberg condition that

$$\text{for any } \; \varepsilon > 0, \frac{1}{\sigma^2 n} \sum_{i=1}^{n} EX_i^2 I\{|X_i| > \varepsilon\sigma\sqrt{n}\} \to 0 \;\; as \;\; n \to \infty$$

is satisfied. Then for $-1 < \beta < 0$, we obtain

$$\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=1}^{\infty} \frac{(\log n)^{\beta}}{n} P\left(|S_n| \geq \epsilon\sigma\sqrt{n \log n}\right) = \frac{E|N|^{2(\beta+1)}}{\beta+1}, \tag{2.5}$$

where N stands for the standard normal random variable.

3 Some Lemmas

In this section, we give some lemmas which will be frequently used later.

Lemma 3.1 (Lehmann 1966) Let X and Y be NQD random variables, then

(1) $EXY \leq EXEY$,

(2) $P(X > x, Y > y) \leq P(X > x)P(Y > y)$,

(3) If $r(\cdot)$ and $s(\cdot)$ are non-decreasing, then $r(X)$ and $s(Y)$ are still NQD random variables.

Lemma 3.2 Let $\{X_i, i \geq 1\}$ be a pairwise NQD sequence with $EX_i = 0$ and $\sup_{i\geq 1} EX_i^2 < \infty$. Then for any $n \geq 1$, we have

$$E|S_n|^2 \leq n \sup_{i\geq 1} EX_i^2 \tag{3.1}$$

and

$$E \max_{1\leq j\leq n} |S_j|^2 \leq n\left(\log 2n/\log 2\right)^2 \sup_{i\geq 1} EX_i^2 \tag{3.2}$$

Proof By the definition of pairwise negative quadrant dependence, we have

$$\begin{aligned} E|S_n|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} EX_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \operatorname{Cov}(X_i, X_j) \\ &\leq \sum_{i=1}^{n} EX_i^2 \leq n \sup_{i\geq 1} EX_i^2, \end{aligned}$$

which yields (3.1). Combining Theorem 2.4.1 in Stout (1974) and the result stated earlier yields the desired result (3.2). The proof is completed.

Lemma 3.3 (Li and Wang 2008) Let $\{X_i, i \geq 1\}$ be a pairwise NQD sequence with $EX_i = 0$. If

$$B_n^{-1} \sum_{i=1}^{n} EX_i f\left(S_n^{(i)}/B_n\right) = 0$$

for every bounded absolutely continuous function $f(\cdot)$, where $B_n = \sqrt{\sum_{i=1}^{n} EX_i^2}$ and $S_n^{(i)} = \sum_{i=1}^{n} X_i - X_i$, and the Lindeberg condition that

$$\text{for any } \varepsilon > 0, \frac{1}{B_n^2} \sum_{i=1}^{n} EX_i^2 I\{|X_i| > \varepsilon B_n\} \to 0 \quad as \quad n \to \infty$$

is satisfied. Then

$$\frac{S_n}{B_n} \to N(0,\ 1) \ \text{ in distribution as } \ n \to \infty.$$

4 Proof of Theorem 2.1

In this section, we will show the proof of Theorem 2.1 as follows.

Proof of Theorem 2.1 Set $b_n = \big(n(\log\log n)^{2/\xi}\log^3 n\big)^{1/v}$, $X_{i1} = -b_n I(X_i < -b_n) + X_i I(|X_i| \le b_n) + b_n I(X_i > b_n)$ and $S_{j1} = \sum_{i=1}^{j}(X_{i1} - EX_{i1})$. Obviously, we know that the sequence $\{X_{i1}, i \ge 1\}$ is still pairwise NQD by Lemma 3.1. In what follows we will prove first that

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\le j\le n} |S_j| > \varepsilon b_n) < \infty \tag{4.1}$$

for any $\varepsilon > 0$. For this purpose, the first thing we need to do is to show that

$$b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_{i1}\right| \to 0. \tag{4.2}$$

Since

$$\begin{aligned} &E|X_i|I(|X_i| > b_n) + b_n P\left(|X_i| > b_n\right) \\ &\le b_n^{1-v} E|X_i|^v I(|X_i| > b_n) + b_n^{1-v} E|X_i|^v \\ &\ll b_n^{1-v}, \end{aligned}$$

we have, by $EX_i = 0$,

$$\begin{aligned} &b_n^{-1} \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_{i1}\right| \\ &\le b_n^{-1} \sum_{i=1}^{n} \left(E|X_i|I(|X_i| > b_n) + b_n P(X_i < -b_n) + b_n P(X_i > b_n)\right) \\ &\ll n b_n^{-v} \to 0. \end{aligned}$$

Hence, (4.2) holds. From this, it follows that for sufficiently large n

$$\begin{aligned} &P\left(\max_{1\le j\le n} |S_j| > \varepsilon b_n\right) \\ &\le P\left(\max_{1\le i\le n} |X_i| > b_n\right) + P\left(\max_{1\le j\le n} \left|\sum_{i=1}^{j} X_{i1}\right| > \varepsilon b_n\right) \\ &\le P\left(\max_{1\le i\le n} |X_i| > b_n\right) + P\left(\max_{1\le j\le n} |S_{j1}| > \varepsilon b_n - \max_{1\le j\le n} \left|\sum_{i=1}^{j} EX_{i1}\right|\right) \\ &\le \sum_{i=1}^{n} P\left(|X_i| > b_n\right) + P\left(\max_{1\le j\le n} |S_{j1}| > \varepsilon b_n/2\right). \end{aligned} \tag{4.3}$$

Thus, we need only to prove that

$$I := \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(|X_i| > b_n\right) < \infty,$$

$$II := \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le j \le n} |S_{j1}| > \varepsilon b_n/2\right) < \infty. \tag{4.4}$$

By Markov inequality, it follows that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} P\left(|X_i| > b_n\right) \\ &\le \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} b_n^{-v} E|X_i|^v \\ &\ll \sum_{n=1}^{\infty} b_n^{-v} < \infty. \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} II &= \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \le j \le n} |S_{j1}| > \varepsilon b_n/2\right) \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-2} E \max_{1 \le j \le n} |S_{j1}|^2 \\ &\le C \sum_{n=2}^{\infty} n^{-1} n b_n^{-2} \log^2 n \sup_{1 \le i \le n} EX_{i1}^2 \\ &\le C \sum_{n=2}^{\infty} b_n^{-2} \log^2 n \sup_{1 \le i \le n} \left(EX_i^2 I(|X_i| \le b_n) + b_n^2 P(|X_i| > b_n)\right) \\ &\le C \sum_{n=2}^{\infty} b_n^{-2} \log^2 n \sup_{1 \le i \le n} \left(EX_i^2 I(|X_i| \le b_n)\right) + C \sum_{n=2}^{\infty} \left(n(\log\log n)^{2/\xi} \log n\right)^{-1} \\ &\le C \sum_{n=2}^{\infty} b_n^{-v} \log^2 n \sup_{i \ge 1} \left(E|X_i|^v I(|X_i| \le b_n)\right) + C \\ &\le C \sum_{n=2}^{\infty} \left(n(\log\log n)^{2/\xi} \log n\right)^{-1} + C \\ &< \infty. \end{aligned}$$

Combing (4.3) and (4.4) yields the result (4.1). Since

$$\begin{aligned}
&\frac{1}{2}\sum_{\widetilde{i}=1}^{\infty} P\left(\max_{1\le j\le 2^{\widetilde{i}}}|S_j| \ge \varepsilon\left(2^{\widetilde{i}+1}\left(\log\log 2^{\widetilde{i}+1}\right)^{2/\xi}\log^3 2^{\widetilde{i}+1}\right)^{1/v}\right)\\
&\le \sum_{\widetilde{i}=0}^{\infty}\sum_{n=2^{\widetilde{i}}}^{2^{\widetilde{i}+1}-1} n^{-1}P(\max_{1\le j\le n}|S_j|\ge \varepsilon b_n)\\
&= \sum_{n=1}^{\infty} n^{-1}P(\max_{1\le j\le n}|S_j|\ge \varepsilon b_n)\\
&< \infty,
\end{aligned}$$

we can get by Borel-Cantelli lemma,

$$\lim_{\widetilde{i}\to\infty}\frac{\max_{1\le j\le 2^{\widetilde{i}}}|S_j|}{\left(2^{\widetilde{i}+1}\left(\log\log 2^{\widetilde{i}+1}\right)^{2/\xi}\log^3 2^{\widetilde{i}+1}\right)^{1/v}} = 0 \quad a.s.$$

Using

$$\begin{aligned}
\left|\frac{S_n}{b_n}\right| &\le \max_{2^{\widetilde{i}-1}\le n<2^{\widetilde{i}}}\left|\frac{S_n}{b_n}\right|\\
&\le 2^{\frac{1}{v}}\frac{\max_{1\le j\le 2^{\widetilde{i}}}|S_j|}{\left(2^{\widetilde{i}+1}\left(\log\log 2^{\widetilde{i}+1}\right)^{2/\xi}\log^3 2^{\widetilde{i}+1}\right)^{1/v}}\left(\frac{(\widetilde{i}+1)\log^3(\widetilde{i}+1)}{(\widetilde{i}-1)\log^3(\widetilde{i}-1)}\right)^{1/v},
\end{aligned}$$

then $|\frac{S_n}{b_n}| \to 0, a.s.$ Hence,

$$\frac{S_n}{b_n} \to 0, \ a.s. \tag{4.5}$$

This completes the proof.

5 Proof of Theorem 2.2

To prove Theorem 2.2, some notations are needed. Let $c_n, n \ge 1$, be a sequence of nonnegative real numbers such that $c_n \to \infty$. Also, for convenience, we define X_{ni} by $X_{ni} = X_i$ for $1 \le i \le n$ and $X_{ni} = 0$ for $i > n$. Let

$$X_{1,i,n} = -c_n I_{(-\infty,-c_n)}(X_{ni}) + X_{ni}I_{[-c_n,c_n]}(X_{ni}) + c_n I_{(c_n,+\infty)}(X_{ni}),$$

$$X_{2,i,n} = (X_{ni} - c_n)I_{(c_n,+\infty)}(X_{ni}), \quad X_{3,i,n} = (X_{ni} + c_n)I_{(-\infty,-c_n)}(X_{ni}) \tag{5.1}$$

for each $n, i \ge 1$, where I_A represents the characteristic function of the set A. Consider now a sequence of natural numbers p_n such that for each $n \ge 1, p_n < n/2$

and set $r_n = [n/(2p_n)] + 1$. Define then,

$$\begin{aligned} Y_{q,j,n} &= \sum_{i=2(j-1)p_n+1}^{2(j-1)p_n+p_n} \left(X_{q,i,n} - E(X_{q,i,n})\right), Z_{q,j,n} \\ &= \sum_{i=2(j-1)p_n+p_n+1}^{2jp_n} \left(X_{q,i,n} - E(X_{q,i,n})\right), \end{aligned} \tag{5.2}$$

for $q = 1, 2, 3$ and $j = 1, 2, \cdots, r_n$ and

$$S_{q,n,od} = \sum_{j=1}^{r_n} Y_{q,j,n}, S_{q,n,ev} = \sum_{j=1}^{r_n} Z_{q,j,n}. \tag{5.3}$$

Clearly, $n \leq 2r_n p_n < 2n$. Next, we show the following propositions used later.

Proposition 5.1 Let $\{X_i, i \geq 1\}$ be a sequence of pairwise NQD random variables with $\sup_{i\geq 1} EX_i^2 < \infty$. If $0 < 2\lambda p_n c_n \leq 1$ for $\lambda > 0$, then on account of definitions (5.1), (5.2) and (5.3), we have

$$E\left(\exp(\lambda S_{1,n,od})\right) \leq \exp\left(C_1 \lambda^2 n\right), \tag{5.4}$$

$$E\left(\exp(\lambda S_{1,n,ev})\right) \leq \exp\left(C_1 \lambda^2 n\right), \tag{5.5}$$

where $C_1 = \sup_{i\geq 1} EX_i^2 < \infty$.

Proof Since $EY_{1,j,n} = 0$ and $0 < 2\lambda p_n c_n \leq 1$, we have

$$\begin{aligned} E(\exp(\lambda Y_{1,j,n})) &= \sum_{w=0}^{\infty} \frac{E(\lambda Y_{1,j,n})^w}{w!} \\ &= 1 + \sum_{w=2}^{\infty} \frac{E(\lambda Y_{1,j,n})^w}{w!} \\ &\leq 1 + E(\lambda Y_{1,j,n})^2 \sum_{w=2}^{\infty} \frac{1}{w!} \\ &\leq 1 + \lambda^2 EY_{1,j,n}^2 \\ &\leq \exp(\lambda^2 EY_{1,j,n}^2). \end{aligned} \tag{5.6}$$

Therefore, in terms of (5.6), (3.1), $\sup_{i\geq 1} EX_i^2 < \infty$ and $r_n p_n < n$,

$$\begin{aligned}\prod_{j=1}^{r_n} E(\exp(\lambda Y_{1,j,n})) \leq \exp\left(\lambda^2 \sum_{j=1}^{r_n} EY_{1,j,n}^2\right) \\ \leq \exp\left(\lambda^2 \sum_{j=1}^{r_n} p_n \sup_{i\geq 1} \mathrm{VarX_{1,i,n}}\right) \\ \leq \exp\left(\lambda^2 \sum_{j=1}^{r_n} p_n \sup_{i\geq 1} EX_{1,i,n}^2\right) \\ \leq \exp\left(\lambda^2 \sum_{j=1}^{r_n} p_n \sup_{i\geq 1} EX_i^2\right) \\ \leq \exp\left(C_1 \lambda^2 n\right), \end{aligned} \tag{5.7}$$

where $C_1 = \sup_{i\geq 1} EX_i^2 < \infty$. Also, we have

$$E\left(\exp(\lambda S_{1,n,od})\right) \leq \prod_{j=1}^{r_n} E\left(\exp(\lambda Y_{1,j,n})\right) \tag{5.8}$$

by the proof of (8) in Lu and Zhao (2007). Hence, combining (5.7) and (5.8) yields the desired result (5.4). Similarly, we can get (5.5) by the proof above. The proof is completed.

Proposition 5.2 Let $\{X_i, i \geq 1\}$ be a sequence of pairwise NQD random variables satisfying $\sup_{i\geq 1} EX_i^2 < \infty$. If $0 < 2\lambda p_n c_n \leq 1$ for $\lambda > 0$, then for any $\varepsilon > 0$, we have

$$P\left(|S_{1,n,od}| > n\varepsilon/2\right) \leq 2\exp\left\{-\frac{n\varepsilon^2}{16C_1}\right\} \tag{5.9}$$

and

$$P\left(|S_{1,n,ev}| > n\varepsilon/2\right) \leq 2\exp\left\{-\frac{n\varepsilon^2}{16C_1}\right\}. \tag{5.10}$$

Proof Applying Markov inequality and Proposition 5.1, we obtain

$$\begin{aligned} P\left(|S_{1,n,od}| > n\varepsilon/2\right) &= P\left(S_{1,n,od} > n\varepsilon/2\right) + P\left(-S_{1,n,od} > n\varepsilon/2\right) \\ &= P\left(e^{\lambda S_{1,n,od}} > e^{\lambda n\varepsilon/2}\right) + P\left(e^{-\lambda S_{1,n,od}} > e^{\lambda n\varepsilon/2}\right) \\ &\leq 2\exp\left(C_1\lambda^2 n - \lambda n\varepsilon/2\right). \end{aligned}$$

Optimizing the exponent in the last term of this upper-bound, we take $\lambda = \varepsilon/(4C_1)$, so that this exponent becomes equal to $-n\varepsilon^2/(16C_1)$, as desired. The proof is completed.

Taking

$$\varepsilon_n = 4\sqrt{(\alpha C_1 \log n)/n} \quad \text{for some } \alpha > 0, \tag{5.11}$$

then we can get the following result controlling the bounded terms.

Proposition 5.3 Let $\{X_i, i \geq 1\}$ be a sequence of pairwise NQD random variables satisfying $\sup_{i\geq 1} EX_i^2 < \infty$. Suppose that ε_n is as in (5.11) and $p_n \leq \sqrt{C_1 n/(4\alpha c_n^2 \log n)}$. Then we have

$$P\left(\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| > n\varepsilon_n\right) \leq 4\exp(-\alpha \log n). \tag{5.12}$$

Proof It is obvious that

$$P\left(\left|\sum_{i=1}^{n}(X_{1,i,n} - EX_{1,i,n})\right| > n\varepsilon\right) \leq P\left(|S_{1,n,od}| > n\varepsilon/2\right) + P\left(|S_{1,n,ev}| > n\varepsilon/2\right). \tag{5.13}$$

Also, by the proof of Proposition 5.2, the optimizing value of λ is

$$\lambda = \varepsilon_n/(4C_1) = \sqrt{\alpha \log n/(C_1 n)}, \tag{5.14}$$

which implies that $2\lambda p_n c_n \leq 1$ can follow from $p_n \leq \sqrt{C_1 n/(4\alpha c_n^2 \log n)}$. So, combining (5.13) and (5.14), we can get (5.12) by Proposition 5.2. The proof is completed.

To control the unbounded terms, we give the following results.

Proposition 5.4 Let $\{X_i, i \geq 1\}$ be a sequence of pairwise NQD random variables which satisfies $\sup_{i\geq 1} E(e^{tX_i}) \leq M < \infty$ for some $t > 0$. Then,

$$P\left(\left|\sum_{i=1}^{n}(X_{q,i,n} - EX_{q,i,n})\right| > n\varepsilon\right) \leq \frac{2Me^{-tc_n}}{nt^2\varepsilon^2}, \quad q = 2, 3. \tag{5.15}$$

Proof Firstly, let us estimate $EX_{q,i,n}^2$. Without loss of generality, set $q = 2$. We assume $F(x) = P(X_i > x)$. Then, by Markov inequality and $\sup_{i\geq 1} E(e^{tX_i}) \leq M < \infty$ for some $t > 0$, it follows that

$$F(x) \leq e^{-tx} E(e^{tX_i}) \leq Me^{-tx}.$$

Writing the mathematical expectation as a Stieltjes integral and integrating by parts we have

$$\begin{aligned} EX_{2,i,n}^2 &= -\int_{(c_n,+\infty)} (x - c_n)^2 F(dx) \\ &\leq 2M \int_{(c_n,+\infty)} (x - c_n)\, e^{-tx} dx \\ &\leq 2M \frac{e^{-tc_n}}{t^2} \end{aligned} \tag{5.16}$$

by the inequality stated earlier. Hence, using (5.16) and (3.1), we have

$$
\begin{aligned}
&P\left(\left|\sum_{i=1}^{n}(X_{2,i,n}-EX_{2,i,n})\right|>n\varepsilon\right)\\
&\leq \frac{E\left|\sum_{i=1}^{n}(X_{2,i,n}-EX_{2,i,n})\right|^2}{n^2\varepsilon^2}\\
&\leq \frac{2Me^{-tc_n}}{nt^2\varepsilon^2},
\end{aligned}
$$

which completes the proof of the proposition.

Applying Proposition 5.4, we can get immediately the following result by taking values for t, c_n and ε.

Corollary 5.1 Let $\{X_i, i\geq 1\}$ be a sequence of pairwise NQD random variables which satisfies $\sup_{i\geq 1}E(e^{tX_i})\leq M<\infty$ for some $t>0$. Then for $n\geq 3$,

$$
P\left(\left|\sum_{i=1}^{n}(X_{q,i,n}-EX_{q,i,n})\right|>n\varepsilon_n\right)\leq \frac{M}{8C_1\alpha^3\log n}\exp(-\alpha\log n),\quad q=2,3. \tag{5.17}
$$

provided $t=\alpha$, $c_n=\log n$ and $\varepsilon_n=4\sqrt{(\alpha C_1\log n)/n}$.

A combination of Proposition 5.3 with Corollary 5.1 yields

Proposition 5.5 Let $\{X_i, i\geq 1\}$ be a sequence of pairwise NQD random variables which satisfies $EX_i=0$ and $\sup_{i\geq 1}E(e^{\alpha|X_i|})\leq M<\infty$ for some $\alpha>0$. Suppose that ε_n is as in (5.11) and $p_n\leq\sqrt{C_1n/(4\alpha\log^3 n)}$. Then for $n\geq 3$,

$$
P\left(|S_n|>3n\varepsilon_n\right)\leq\left(4+\frac{M}{4C_1\alpha^3\log n}\right)\exp(-\alpha\log n). \tag{5.18}
$$

By Proposition 5.5, we can give

Proof of Theorem 2.2 Taking $p_n=\left[\sqrt{C_1n/(4\alpha\log^3 n)}\right]$, $\varepsilon_n=4\sqrt{(\alpha C_1\log n)/n}$ and $\alpha>1$ in (5.18), the desired result (2.4) can be obtained by Borel-Cantelli lemma.

6 Proof of Theorem 2.3

Without loss of generality, set $\sigma=1$ in what follows. In order to prove Theorem 2.3, the following propositions are needed.

Proposition 6.1 For any $\beta>-1$, we have

$$
\lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}\sum_{n=1}^{\infty}\frac{(\log n)^\beta}{n}P\left(|N|\geq\epsilon\sqrt{\log n}\right)=\frac{E|N|^{2(\beta+1)}}{\beta+1}, \tag{6.1}
$$

where N stands for the standard normal random variable.

Proof Note that $P(|N| \geq x) = 2P(N \geq x)$ for any $x > 0$, it suffices to show that

$$\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=1}^{\infty} \frac{(\log n)^\beta}{n} P\left(N \geq \epsilon\sqrt{\log n}\right) = \frac{E|N|^{2(\beta+1)}}{2(\beta+1)}.$$

It is easy to observe that

$$\begin{aligned} &\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=1}^{\infty} \frac{(\log n)^\beta}{n} P\left(N \geq \epsilon\sqrt{\log n}\right) \\ &\leq \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \int_e^{\infty} \frac{(\log x)^\beta}{x} P\left(N \geq \epsilon\sqrt{\log x}\right) dx \\ &= \frac{E|N|^{2(\beta+1)}}{2(\beta+1)}, \end{aligned}$$

which implies (6.1). The proof is complete.

Proposition 6.2 Under the conditions of Theorem 2.3, we obtain for $-1 < \beta < 0$

$$\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n=1}^{\infty} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon\sqrt{n \log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| = 0. \quad (6.2)$$

Proof Let $J(\epsilon) = \exp\left(\frac{M^{-\frac{1}{\beta}}}{\epsilon^2}\right)$, where $M > 4$ and $0 < \epsilon < 1/4$. Obviously,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon\sqrt{n \log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| \\ &\leq \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \left| P\left(|S_n| \geq \epsilon\sqrt{n \log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| \\ &\quad + \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} P\left(|S_n| \geq \epsilon\sqrt{n \log n}\right) + \sum_{n > J(\epsilon)} \frac{(\log n)^\beta}{n} P\left(|N| \geq \epsilon\sqrt{\log n}\right) \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

thus it is sufficient to prove that

$$\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_1 = 0, \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_2 = 0 \quad \text{and} \quad \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_3 = 0, \quad (6.3)$$

respectively. We consider firstly I_1. Set $\Delta_n = \sup_x \left|P\left(|S_n| \geq x\sqrt{n}\right) - P\left(|N| \geq x\right)\right|$. Noticing Lemma 3.3, we have $\Delta_n \to 0$ as $n \to \infty$. From this result, it follows that

$$\begin{aligned} &\lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} I_1 \\ &\leq \lim_{\epsilon \searrow 0} \epsilon^{2(\beta+1)} \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \Delta_n \\ &\leq \lim_{\epsilon \searrow 0} M^{-\frac{\beta+1}{\beta}} \frac{1}{(\log J(\epsilon))^{\beta+1}} \sum_{n \leq J(\epsilon)} \frac{(\log n)^\beta}{n} \Delta_n \to 0, \end{aligned}$$

which implies that $\lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}I_1 = 0$. Turn to I_2, we have by Lemma 3.2,

$$\begin{aligned}
&\lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}I_2\\
&\le \lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}\sum_{n>J(\epsilon)}\frac{(\log n)^\beta}{n}\frac{ES_n^2}{\epsilon^2 n\log n}\\
&\le \lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}\sum_{n>J(\epsilon)}\frac{(\log n)^\beta}{n}\frac{nEX_1^2}{\epsilon^2 n\log n}\\
&\le C\lim_{\epsilon\searrow 0}\epsilon^{2\beta}\sum_{n>J(\epsilon)}\frac{(\log n)^\beta}{n\log n}\\
&\le C\lim_{\epsilon\searrow 0}\epsilon^{2\beta}\int_{J(\epsilon)}^{\infty}\frac{(\log x)^\beta}{x\log x}dx\\
&= -\frac{C}{\beta}M^{-1}\to 0 \text{ as } M\to\infty,
\end{aligned}$$

uniformly for $0<\epsilon<1/4$. Hence, $\lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}I_2\to 0$ when $M\to\infty$. On the other hand, noting that $M>4$ and $0<\epsilon<1/4$ imply $J(\epsilon)-1\ge\sqrt{J(\epsilon)}$, we can obtain

$$\begin{aligned}
&\lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}I_3\\
&\le \lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}\int_{J(\epsilon)}^{\infty}\frac{(\log x)^\beta}{x}P\left(|N|\ge\epsilon\sqrt{\log x}\right)dx\\
&\le \lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}\int_{J(\epsilon)-1}^{\infty}\frac{(\log x)^\beta}{x}P\left(|N|\ge\epsilon\sqrt{\log x}\right)dx\\
&\le \lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}\int_{\sqrt{J(\epsilon)}}^{\infty}\frac{(\log x)^\beta}{x}P\left(|N|\ge\epsilon\sqrt{\log x}\right)dx\\
&\le C\lim_{\epsilon\searrow 0}\int_{\epsilon\sqrt{M^{-1/\beta}/(2\epsilon^2)}}^{\infty}y^{2\beta+1}P\left(|N|\ge y\right)dy\\
&\le C\int_{\sqrt{M^{-1/\beta}/2}}^{\infty}y^{2\beta+1}P\left(|N|\ge y\right)dy\to 0 \text{ as } M\to\infty,
\end{aligned}$$

uniformly for $0<\epsilon<1/4$. Thus we have $\lim_{\epsilon\searrow 0}\epsilon^{2(\beta+1)}I_3\to 0$ when $M\to\infty$. Combining the earlier results together yields (6.3). The proof is completed.

Now, we can give

Proof of Theorem 2.3 Combining Propositions 6.1 and 6.2 yields the desired result (2.5).

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