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Graded Modules which Satisfy the Gr-Radical Formula

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Abstract : Let G be a monoid with identity e, and R be a graded commutative ring. Here we study the graded modules which satisfy the Gr- radical formula. The main part of this work is devoted to extending some results from McCasland modules to Gr-McCasland modules.

Keywords : Graded rings, Gr-McCasland modules, Graded radical formula
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1 Introduction

Let G be an arbitary monoid with identity e. A ring R with non-zero identity is G-graded if it has a direct sum decomposition (as an additive group) $R = \bigoplus_{g\in G} R_g$ such that for all $g, h \in G$, $R_gR_h \subseteq R_{gh}$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$, $R_gM_h \subseteq M_{gh}$. An element of R_g or M_g is said to be a homogeneous element. If $x \in M$, then x can be written uniquely as $x = \sum_{g\in G} x_g$, where x_g is the homogeneous component of x in M_g. A submodule $N \subseteq M$, where M is graded, is called G-graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Clearly, 0 is a graded submodule of M. Also, we write $h(R) = \bigcup_{g\in G} R_g$ and $h(M) = \bigcup_{g\in G} M_g$. Throughout this paper R is a commutative G-graded ring with identity.

Let R be a G-graded ring. A graded ideal I of R is said to be a graded prime ideal if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. The graded radical of I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. A proper graded submodule N of a graded module M is called graded prime if $rm \in N$, then $m \in N$ or $r \in (N : M) = \{r \in R : rM \subseteq N\}$, where $r \in h(R)$, $m \in h(M)$ (note that

$(N : M)$ is graded by [2, Lemma 2.1]). A graded submodule N of a graded R-module M is called graded maximal submodule if $N \neq M$ and there is no graded submodule K of M such that $N \nsubseteq K \subsetneqq M$. A graded R-module M is called graded finitely generated if $M = \sum_{i=1}^{n} Rx_{g_i}$, where $x_{g_i} \in h(M)$ $(1 \leq i \leq n)$. A graded R-module M is called a graded multiplicative module (denoted by Gr-multiplicative) if for every graded submodule N of M, $N = IM$ for some graded ideal I of R. In this case, it is clear that every graded module which is multiplicative is a Gr-multiplicative module and $N = (N : M)M$.

Lemma 1.1. *(cf.[5]) Let M be a graded module over a G-graded ring R and I a graded ideal of R. Then the following hold:*

(i) If N is a graded submodule of M, $a \in h(R)$ and $m \in h(M)$, then Rm, IN and aN are graded submodule of M and Ra is a graded ideal of R.

(ii) If $\{N_i\}_{i \in J}$ is a collection of graded submodules of M, then $\sum_{i \in J} N_i$ and $\bigcap_{i \in J} N_i$ are graded submodules of M.

(iii) If P is a graded prime ideal of R and M a faithful graded multiplication R-module with $PM \neq M$, then PM is a graded prime submodule of M.

2 *Gr*-radical formula

Definition 2.1. *Let R be a G-graded ring, M be a graded R-module and N be a graded R- submodule of M.*

(i) The graded radical of N in M denoted by $Gr_M(N)$ and is defined to be the intersection of all graded prime submodules of M containing N. Should there be no graded prime submodule of M containing N, then we put $Gr_M(N) = M$. By Lemma 1.1, It is easy to see that $Gr_M(N)$ is a graded submodule of M containing N. On the other hand, $Gr(R)$ denotes the intersection of all graded prime ideals of R.

(ii) The graded envelop submodule $RGE_M(N)$ of N in M is a graded submodule of M generated by the set $GE_M(N) = \{rm : r \in h(R), m \in h(M)$ such that $r^n m \in N$ for some $n \in \boldsymbol{N}\}$.

(iii) We say that the graded submodule N of M satisfies Gr-radical formula (graded radical formula), if $Gr_M(N) = RGE_M(N)$.

(iv) A graded R-module M will be called a Gr-McCasland module if every graded submodule of M satisfies Gr-radical formula.

Lemma 2.2. *Let M be a graded module over a G-graded ring R. Then $N \subseteq RGE_M(N) \subseteq Gr_M(N)$ for every graded R-submodule N of M.*

Proof. Obvious. □

3 *Gr*-Multiplication Modules

In this section we list some basic properties of graded multiplicative module and we will show that every Gr- multiplication module is McCasland.

Lemma 3.1. *Let I be a graded ideal of a G-graded ring R and M be a graded R-module. Then there exists a proper graded submodule N of M satisfies $I = (N : M)$ if and only if $IM \neq M$, $I = (IM : M)$.*

Proof. Let N be a proper graded submodule of M and $I = (N : M)$. Then $IM \subseteq N \subsetneq M$, so $IM \neq M$. It is clear that $I \subseteq (IM : M)$. Let $r \in (IM : M)$ then $rM \subseteq IM \subseteq N$, so $r \in I$. Therefore $I = (IM : M)$. The convers is clear since IM is a proper graded submodule from Lemma 1.1. □

Theorem 3.2. *Let M be a Gr-multiplicative R-module, N a graded submodule of M and $A = (N : M)$. Then $Gr_M(N) = \sqrt{A}M = \sqrt{(N : M)}M$.*

Proof. Without loss of generality we can assume that M is faithful by [7, p. 155]. Let $\boldsymbol{P}$ denote the collection of all graded prime ideals P of R such that $A \subseteq P$. If $B = \sqrt{A}$, then $B = \bigcap_{P\in\boldsymbol{P}} P$. Choose $P \in \boldsymbol{P}$. If $PM = M$, then $Gr_M(N) \subseteq PM$. If $PM \neq M$, then since N is a graded submodule of M and M is Gr-multiplicative then $N = AM \subseteq PM$. Therefore by Lemma 1.1, since PM is a prime submodule of M, then $Gr_M(N) \subseteq PM$. Thus $BM = \bigcap_{P\in\boldsymbol{P}} PM$, by [7, Corollary 4.2.8]. So $Gr_M(N) \subseteq BM$.

Now let K be a graded prime submodule of M containing N. Then there exists a graded prime ideal $Q = (K : M)$ of R such that $K = QM$. We show that $A \subseteq Q$. By Lemma 3.1, $Q = (QM : M)$. Let $r \in A = (N : M)$. So $rM \subseteq N \subseteq K = QM$, then $r \in (QM : M) = Q$. Thus $A \subseteq Q$. As Q is a graded prime ideal containing A, so $B = \sqrt{A} \subseteq Q$. Therefore $BM \subseteq QM = K$. Hence, since K is an arbitary graded prime submodule of M containing N, then $BM \subseteq Gr_M(N)$. □

Theorem 3.3. *Let M be a Gr-multiplicative R-module. Then M is a Gr-McCasland module.*

Proof. Let N be a graded submodule of M. Then $RGE_M(N) \subseteq Gr_M(N)$ by lemma 2.2, so it suffices to show that $Gr_M(N) \subseteq RGE_M(N)$. Let $x \in Gr_M(N)$. Since $Gr_M(N) = \sqrt{(N : M)}M$, then $x = \sum_{j=1}^{k} r_j x_j$ such that $r_j \in \sqrt{(N : M)}$, $x_j \in M$. As $\sqrt{(N : M)}$, M are graded, so without loss of generality we can assume that $x = \sum_{i=1}^{n} r_{g_i} x_{g_i}$ such that $r_{g_i} \in h(R) \cap \sqrt{(N : M)}$ and $x_{g_i} \in h(M)$ for each $i = 1, 2, ..., n$. Since $r_{g_i} \in \sqrt{(N : M)}$, so there exists $n_i \in \mathbf{N}$ such that $r_{g_i}^{n_i} M \subseteq N$ for each $i = 1, 2, ..., n$. Therefore $r_{g_i}^{n_i} x_{g_i} \in N$ and $r_{g_i} x_{g_i} \in GE_M(N) \subseteq RGE_M(N)$ for each $i = 1, 2, ..., n$. Thus $x \in RGE_M(N)$. □

4 *Gr*-Semisimple Modules

In this section we list some basic properties of graded Semisimple module and we will show that every Gr- Semisimple Module is McCasland.

Lemma 4.1. *Let N_1, N_2 be graded submodules of a graded R-module M and $N_1 \subseteq N_2$. Then*

(i) $RGE_{M/N_1}(N_2/N_1) = RGE_M(N_2)/N_1$

(ii) $Gr_{M/N_1}(N_2/N_1) = Gr_M(N_2)/N_1$

Proof. (i) Let $y \in RGE_{M/N_1}(N_2/N_1)$. So $y = \sum_{i=1}^{k} r_{g_i}(m_{g'_i} + N_1)$ such that $r_{g_i} \in h(R)$, $m_{g'_i} \in h(M)$ and there exists $n_i \in \mathbf{N}$ such that $r_{g_i}^{n_i}(m_{g'_i} + N_1) \in N_2/N_1$ for each $i = 1, 2, ..., k$. Thus $r_{g_i}^{n_i} m_{g'_i} \in N_2$ and $r_{g_i} m_{g'_i} \in RGE_M(N_2)$. So $y = \sum_{i=1}^{k} r_{g_i} m_{g'_i} + N_1 \in RGE_M(N_2)/N_1$.
Now let $x \in RGE_M(N_2)/N_1$. So $x = \sum_{i=1}^{t} s_{g_i} m_{g'_i} + N_1$ such that $s_{g_i} \in h(R)$, $m_{g'_i} \in h(M)$ and there exists $n_i \in \mathbf{N}$ such that $s_{g_i}^{n_i} m_{g'_i} \in N_2$ for each $i = 1, 2, ..., t$. So $s_{g_i}^{n_i}(m_{g'_i} + N_1) \in N_2/N_1$ and $s_{g_i}(m_{g'_i} + N_1) \in RGE_{M/N_1}(N_2/N_1)$. Therefore $x = \sum_{i=1}^{t} s_{g_i} m_{g'_i} + N_1 = \sum_{i=1}^{t} s_{g_i}(m_{g'_i} + N_1) \in RGE_{M/N_1}(N_2/N_1)$.
(ii) It is clear by [2, lemma 2.8]. □

Corollary 4.2. *Let N, N' be graded submodules of graded R-modules M, M' such that $M/N \cong M'/N'$. Then $Gr_M(N) = RGE_M(N)$ if and only if $Gr_{M'}(N') = RGE_{M'}(N')$.*

Proof. By lemma 4.1, we have the following implications:
$Gr_M(N) = RGE_M(N) \Leftrightarrow Gr_M(N)/N = RGE_M(N)/N$
$\Leftrightarrow Gr_{M/N}(0) = RGE_{M/N}(0) \Leftrightarrow Gr_{M'/N'}(0) = RGE_{M'/N'}(0)$
$\Leftrightarrow Gr_{M'}(N')/N' = RGE_{M'}(N')/N' \Leftrightarrow Gr_{M'}(N') = RGE_{M'}(N')$. □

Corollary 4.3. *Let N, L be graded submodule of graded R-module M such that $M = N + L$ and $Gr_L(N \cap L) = RGE_L(N \cap L)$. Then $Gr_M(N) = RGE_M(N)$.*

Proof. Note that $M/N = (N + L)/N \cong L/N \cap L$. Apply Corollary 4.2. □

A graded R-module M is said to be a Gr-semisimple module if every graded R-submodule of M is a direct summand of M. It is clear that every graded submodule of a Gr-semisimple module is Gr-semisimple.
A graded submodule K of a graded R-module M is said to be Gr-small submodule in M, written $K \ll_{Gr} M$ if for every graded submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$.
The intersection of all graded maximal submodules of a graded R-module M is denoted by $GRad(M)$. If M has no graded maximal submodule we set $GRad(M) = M$.

Lemma 4.4. *For a graded R-module M, we have*
$GRad(M) = \bigcap\{K \subset M \mid K \text{ is a graded maximal in } M\}$
$= \sum\{L \subset M \mid L \text{ is a } Gr - \text{small submodule in } M\}$.

Proof. The first row is just the definition. If $L \ll_{Gr} M$ and K is a graded maximal submodule of M not containing L, then $K \subsetneq L + K \subseteq M$ so $L + K = M$. As K is a graded maximal submodule, then $K = M$, since $L \ll_{Gr} M$. Hence every Gr-small submodule of M is contained in $GRad(M)$.
Now assume that $m \in GRad(M) \cap h(R)$, $U \subseteq M$ with $Rm + U = M$ and U is a graded submodule of M. If $U \neq M$, set $\mathbf{A} = \{K \mid K \text{ is a graded submodule of } M \text{ with } U \subseteq K \text{ and } m \notin K\}$. Then $\mathbf{A} \neq \emptyset$. By Zorn's lemma there is a graded submodule L of M maximal with respect to $U \subseteq L$ and $m \notin L$. So $M = Rm + L$, now we show that L is a graded maximal submodule of M. Let L' be a graded submodule of M and $L \subseteq L' \subseteq M$. We divide the proof into two cases:
Case 1 If $m \notin L'$, then $L' \in \mathbf{A}$ and $L = L'$.
Case 2 If $m \in L'$, then $M = Rm + L'$ and so $L' = M$.
So L is a graded maximal submodule of M. But $m \in GRad(M) \subseteq L$ is a contradiction. So $U = M$ and $Rm \ll_{Gr} M$. Since $GRad(M)$ is a graded submodule of M and every element of $GRad(M)$ is a finite sum of homogenous elements, therefore the result holds. □

Lemma 4.5. *Let M be a Gr-semisimple R-module. Then $GRad(M) = 0$.*

Proof. Since M is a Gr-semisimple R-module so M has no proper Gr-small submodule, then by lemma 4.4, $GRad(M) = 0$. □

Lemma 4.6. *Let M be a graded R-module and M' be a graded submodule of M. If P is a graded prime submodule of M, then $P \cap M'$ is a graded prime submodule of M'.*

Proof. Set $L = P \cap M'$. Since P and M' are graded submodules of M so L is a graded submodule of M. Let $rm' \in L$ for some $r \in h(R)$ and $m \in h(M') \subseteq h(M)$. Then $rm' \in P$. So $m' \in P$ or $rM \subseteq P$ since P is a graded prime submodule of M. Thus $m' \in L$ or $rM' \subseteq L$. Therefore $L = P \cap M'$ is a graded prime submodule of M'. □

Lemma 4.7. *Let $M = M' \oplus M''$ be a graded R-module, M' and M'' be graded submodules of M such that M'' is a Gr-semisimple module. Then $Gr_M(N) = Gr_{M'}(N)$ for any graded submodule N of M'.*

Proof. Let N be a graded submodule of M'. Since M'' is Gr-semisimple and $GRad(M'') = 0$, so there exists a collection of graded maximal submodules $P_i (i \in I)$ of M'' such that $\bigcap_{i \in I} P_i = 0$ and there exists a collection of graded prime submodules $Q_j (j \in J)$ of M' such that $Gr_{M'}(N) = \bigcap_{j \in J} Q_j$. We show that for all $i \in I$ and $j \in J$, $M' \oplus P_i$ and $Q_j \oplus M''$ are graded prime submodules of M containing N. First we show that for each $i \in I$, $L' = M' \oplus P_i$ is a graded prime submodule of M containing N, the proof for $Q_j \oplus M''$ is the same.

Let $rm \in L' = M' \oplus P_i$ for some $r \in h(R)$ and $m \in h(M)$. So $m = m' + m''$ for some $m' \in M'$ and $m'' \in M''$. Thus $r(m - m') = rm'' \in P_i$. If $m - m' \in P_i$, then $m = m' + (m - m') \in M' \oplus P_i$. If $rM'' \subseteq P_i$, then $rM = rM' \oplus rM'' \subseteq M' \oplus P_i$. Hence
$Gr_M(N) \subseteq \{\bigcap_{i\in I}(M' \oplus P_i)\} \bigcap \{\bigcap_{j\in J}(Q_j \oplus M'')\} = \bigcap_{j\in J} Q_j = Gr_{M'}(N)$. By lemma 4.6, $Gr_{M'}(N) \subseteq Gr_M(N)$. □

Let M and M' be two graded R-modules. A morphism of graded R-modules $f : M \to M'$ is a morphism of R-modules verifying $f(M_g) \subseteq M'_g$ for every $g \in G$.

Lemma 4.8. *Let M and M' be two graded R-modules and N be a graded R-submodule of M. If $f : M \to M'$ is a morphism of graded R-modules, then $f(N)$ is a graded submodule of M'.*

Proof. Let $N = \bigoplus_{g\in G} N_g$ such that $N_g = N \cap M_g$ for all $g \in G$. Since f is a morphism of graded R-modules so $f(N_g) = f(N) \cap M'_g$. We show that $f(N) = \bigoplus_{g\in G} f(N_g)$. Let $y \in f(N)$. Then $y = \sum_{g\in G} m'_g$ such that $m'_g \in M'_g$ for all $g \in G$ and $y = f(n)$ for some $n \in N$. Thus $n = \sum_{g\in G} n_g$, since N is graded. Without loss of generality we can assume that $n = \sum_{i=1}^{n} n_{g_i}$ and $n_g = 0$ for each $g \notin \{g_1, ..., g_n\}$. Therefore $f(n) = \sum_{i=1}^{n} f(n_{g_i}) = y$. But every element of M' has unique representation, so $m'_{h_i} = f(n_{g_i}) \in f(N_{g_i}) \subseteq f(N)$ for some $h_i \in G$ and $m'_{h_i} = 0 \in f(N)$ for all $h \notin \{h_1, ..., h_n\}$. Therefore $m'_g \in f(N)$ for all $g \in G$ and $f(N)$ is a graded submodule of M'. □

Theorem 4.9. *Let $M = M' \oplus M''$ be a graded R-module, M' a Gr-McCasland R-submodule and M'' a Gr-semisimple R-submodule of M. Then M is a Gr-McCasland module.*

Proof. Let N be a graded R-submodule of M. It suffices to show that $Gr_M(N) \subseteq RGE_M(N)$. Let $\pi : M \to M''$ be the natural epimorphism. It is clear that π is a morphism of graded modules. Thus $\pi(N) \subseteq M''$ is a graded submodule of M'' by lemma 4.8. So there exists a graded submodule N'' of M'' such that $M'' = \pi(N) \oplus N''$. Then $M = M' \oplus M'' = M' \oplus \pi(N) \oplus N''$. We show that $M = N + (M' \oplus N'')$. Let $m \in M$. So there exists $m' \in M'$, $x \in \pi(N)$ and $n'' \in N''$ such that $m = m' + x + n''$. Since $x \in \pi(N)$, so $\pi(n) = x$ for some $n \in N$. Thus $n = n_1 + n_2$ for some $n_1 \in M'$, $n_2 \in M''$ and $n_2 = \pi(n) = x$. So $m = m' + x + n'' = m' + n - n_1 + n'' = n + (m' - n_1) + n''$, then $m \in N + (M' \oplus N'')$. Now consider submodule $L = N \cap (M' \oplus N'')$ of graded R-module $H = M' \oplus N''$. L is a graded submodule because N, M' and N'' are graded. Let $\pi' : H \to N''$ be the natural epimorphism. Then $\pi'(L) \subseteq \pi(N) \cap N'' = 0$, so $L \subseteq M'$. As N'' is Gr-semisimple and M' is a Gr-McCasland R-module, then by lemma 4.7, $Gr_H(L) = Gr_{M'}(L) = RGE_{M'}(L) \subseteq RGE_H(L)$. So $Gr_H(L) = RGE_H(L)$. Then by Corollary 4.3, $Gr_M(N) = RGE_M(N)$ since $M = N + H$ and $L = N \cap H$. □

Corollary 4.10. *Let R be a graded R-module. Then every Gr-semisimple R-module is a Gr-McCasland module.*

5 *Gr*-Divisible Modules

In this section we list some basic properties of graded divisible Module and we will show that every Gr- divisible Module is McCasland.

Let R be a graded ring. We say that R is a Gr-integral domain whenever $a, b \in h(R)$ with $ab = 0$ implies that either $a = 0$ or $b = 0$.

Let R be a Gr-integral domain. A graded R-module M is called Gr-divisible if $aM = M$ for all $0 \neq a \in h(R)$.

If R is a graded ring and M is a graded R-module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0 \; for\ some\ 0 \neq r \in h(R)\}$.

Clearly, if R is a Gr-integral domain, then $T(M)$ is a graded submodule of M. $T(M)$ is called the Gr-torsion submodule of M. A graded R-module M is called a Gr-torsion module if $M = T(M)$ and is called Gr-torsion free module if $T(M) = 0$.

Lemma 5.1. *Let R be a Gr-integral domain, M be a graded R-module and N a proper graded submodule of M. Then N is a graded prime submodule of M or if $T(M/N) = L/N$ is the Gr-torsion submodule of M/N, then $L = M$ or L is a graded prime submodule of M containing N.*

Proof. By the definition $T(M/N) = \{m + N \in M/N : rm \in N for\ some\ 0 \neq r \in h(R)\}$. We divide the proof into two cases:

Case 1 Let the graded R-module M/N be Gr-torsion free. Then $T(M/N) = 0$. If $I = (N : M) \neq 0$, then there exists $0 \neq r \in I$ such that $rM \subseteq N$. So for every $m \in M$, $r(m + N) = rm + N = N$, then $m + N \in T(M/N) = 0$ and $m \in N$. Therefore $N = M$ is a contradiction. Thus $I = (N : M) = 0$. So M/N is a Gr-torsion free $R \cong R/0$-module and $I = (N : M) = 0$ is a graded prime ideal of R. So by [2, Theorem 2.11], N is a graded prime submodule of M.

Case 2 If M/N isn't Gr-torsion free R-module, then $T(M/N) = L/N \neq 0$. If M/N is Gr-torsion, then $T(M/N) = M/N$ and $L = M$. If M/N isn't Gr-torsion, then by [6, Proposition 2.6], $T(M/N)$ is a graded prime submodule of M. Then by [2, Lemma 2.8], L is a graded prime submodule of M containing N. □

Lemma 5.2. *Let R be a graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an epimorphism of graded modules. Let N be a graded submodule of M such that $Ker\varphi \subseteq N$. Then*

(i) If P is a graded prime submodule of M containing N, then $\varphi(P)$ is a graded prime submodule of M' containing $\varphi(N)$.

(ii) If P' is a graded prime submodule of M' containing $\varphi(N)$, then $\varphi^{-1}(P')$ is a graded prime submodule of M containing N.

Proof. The proof is a direct consequence of the definition. □

Lemma 5.3. *Let M, M' be graded R-modules and N' be a graded submodule of M'. Let $\varphi : M \to M'$ be an epimorphism of graded R-modules. Then*

(i) $\varphi^{-1}(Gr_{M'}(N')) = Gr_M(\varphi^{-1}(N'))$

(ii) $\varphi^{-1}(RGE_{M'}(N')) = RGE_M(\varphi^{-1}(N'))$

Proof. (i) Let $x \in Gr_M(\varphi^{-1}(N'))$ and L be a graded prime submodule of M' containing N'. Then by lemma 5.2, $\varphi^{-1}(L)$ is a graded prime submodule of M containing $\varphi^{-1}(N')$. So $Gr_M(\varphi^{-1}(N')) \subseteq \varphi^{-1}(L)$, then $x \in \varphi^{-1}(L)$ so $\varphi(x) \in L$. Therefore $\varphi(x) \in Gr_{M'}(N')$, then $x \in \varphi^{-1}(Gr_{M'}(N'))$.
Now suppose that $y \in \varphi^{-1}(Gr_{M'}(N'))$ and K be a graded prime submodule of M containing $\varphi^{-1}(N')$. It is clear that $Ker\varphi \subseteq \varphi^{-1}(N')$, so by lemma 5.2, $\varphi(K)$ is a graded prime submodule of M' containing N'. Thus $Gr_{M'}(N') \subseteq \varphi(K)$, then $\varphi(y) \in Gr_{M'}(N') \subseteq \varphi(K)$. So there exists $m \in K$ such that $\varphi(y) = \varphi(m)$, then $y - m \in Ker\varphi \subseteq \varphi^{-1}(N') \subseteq K$. Therefore $y \in K$, so $y \in Gr_M(\varphi^{-1}(N'))$.
(ii) Let $rm \in GE_M(\varphi^{-1}(N'))$ for some $m \in h(M)$ and $r \in h(R)$. So there exists $n \in \mathbf{N}$ such that $r^n m \in \varphi^{-1}(N')$, then $r^n\varphi(m) = \varphi(r^n m) \in N'$, so $\varphi(rm) = r\varphi(m) \in RGE_{M'}(N')$ since $r \in h(R)$ and $\varphi(m) \in h(M')$. Therefore $rm \in \varphi^{-1}(RGE_{M'}(N'))$.
Now let $x \in \varphi^{-1}(RGE_{M'}(N'))$. So $\varphi(x) \in RGE_{M'}(N')$. Without loss of generality we can consider $\varphi(x) = \sum_{i=1}^{k} r_{g_i} x'_{g'_i}$ such that $r_{g_i} \in h(R)$ and $x'_{g'_i} \in h(M')$ for each $i = 1, 2, ..., k$. So there exists $n_i \in \mathbf{N}$ such that $r_{g_i}^{n_i} x'_{g'_i} \in N'$ for each $i = 1, 2, ..., k$. Since φ is an epimorphism of graded R-modules and $x'_{g'_i} \in h(M')$, then there exists $x_{g'_i} \in h(M)$ such that $\varphi(x_{g'_i}) = x'_{g'_i}$. So $\varphi(r_{g_i}^{n_i} x_{g'_i}) = r_{g_i}^{n_i} x'_{g'_i} \in N'$, then $r_{g_i}^{n_i} x_{g'_i} \in \varphi^{-1}(N')$. Thus $r_{g_i} x_{g'_i} \in GE_M(\varphi^{-1}(N'))$ for each $i = 1, 2, ..., k$. On the other hand, $\varphi(x - \sum_{i=1}^{k} r_{g_i} x_{g'_i}) = 0$. So $x - \sum_{i=1}^{k} r_{g_i} x_{g'_i} \in Ker\varphi \subseteq \varphi^{-1}(N') \subseteq RGE_M(\varphi^{-1}(N'))$. Therefore $x \in RGE_M(\varphi^{-1}(N'))$. □

Let M be a graded R-module. Then a graded homomorphic image of M is a graded R-module M' with an epimorphism of graded modules $\varphi : M \to M'$.

Theorem 5.4. *Let M be a Gr-McCasland R-module. Then every graded homomorphic image of M is Gr-McCasland.*

Proof. Let M' be a graded homomorphic image of M. Then there exists an epimorphism of graded modules $\varphi : M \to M'$. We show that M' is an Gr-McCasland module. Let N' be a graded submodule of M'. Then $\varphi^{-1}(N')$ is a graded submodule of M and since M is Gr-McCasland R-module, then $Gr_M(\varphi^{-1}(N')) = RGE_M(\varphi^{-1}(N'))$. So by lemma 5.3, $\varphi^{-1}(Gr_{M'}(N')) = \varphi^{-1}(RGE_{M'}(N'))$. Then $Gr_{M'}(N') = RGE_{M'}(N')$ since φ is an epimorphism. □

Theorem 5.5. *Let G be a group and R be a Gr-integral domain. Let $M = M_1 + M_2$ be a graded R-module, M_1 a Gr-McCasland submodule of M and M_2 be a Gr-divisible submodule of M. Then M is a Gr-McCasland R-module.*

Proof. Define $\alpha : M_1 \to M/M_2$ with $\alpha(s_1) = s_1 + M_2$ for every element $s_1 \in M_1$. It is easy to see that α is an epimorphism of graded modules. Since M_1 is Gr-McCasland, so by Theorem 5.4, M/M_2 is Gr-McCasland. Let N be a graded submodule of M. It suffices to show that $Gr_M(N) \subseteq RGE_M(N)$. Let $m \in Gr_M(N)$. Then $m + M_2 \in (Gr_M(N) + M_2)/M_2 = Gr_{M/M_2}(N + M_2/M_2) = RGE_{M/M_2}(N + M_2/M_2)$, since M/M_2 is Gr-McCasland. So $m + M_2 = \sum_{i=1}^{t} r_{g_i}(k_{g'_i} + M_2)$ and

$m - \sum_{i=1}^{t} r_{g_i} k_{g'_i} \in M_2$ such that $r_{g_i} \in h(R)$, $k_{g'_i} \in h(M)$ and there exists $s_i \in \mathbf{N}$ such that $r_{g_i}^{s_i} k_{g'_i} \in (N + M_2) \cap M_{g_i^{s_i} g'_i}$ for each $i = 1, 2, ..., t$. So there exist $n_i \in N \cap M_{g_i^{s_i} g'_i}$ and $d_i \in M_2 \cap M_{g_i^{s_i} g'_i}$ such that $r_{g_i}^{s_i} k_{g'_i} = n_i + d_i$ for each $i = 1, 2, ..., t$. By hypothesis M_2 is Gr-divisible, so there exists $c_i \in M_2$ such that $d_i = r_{g_i}^{s_i} c_i$ for each $i = 1, 2, ..., t$. Thus $r_{g_i}^{s_i}(k_{g'_i} - c_i) = r_{g_i}^{s_i} k_{g'_i} - d_i = n_i \in N$. Now we want to show that $k_{g'_i} - c_i \in h(M)$ for each $i = 1, 2, ..., t$. Since $k_{g'_i} \in M_{g'_i}$, it suffices to show that $c_i \in M_{g'_i}$. If $d_i = 0$, then $c_i = 0 \in M_{g'_i}$. So suppose that $d_i \neq 0$. Let $c_i \in M_{h_i}$ for some $h_i \in G$. Then $0 \neq r_{g_i}^{s_i} c_i = d_i \in M_{g_i^{s_i} g'_i} \cap M_{g_i^{s_i} h_i}$. Thus $g_i^{s_i} g'_i = g_i^{s_i} h_i$. So $g'_i = h_i$ since G is a group. So $r_{g_i}(k_{g'_i} - c_i) \in RGE_M(N) \subseteq Gr_M(N)$ for each $i = 1, 2, ..., t$. But $m - \sum_{i=1}^{t} r_{g_i} k_{g'_i}$, $\sum_{i=1}^{t} r_{g_i} c_i \in M_2$ so there exists $x \in M_2$ such that $m - \sum_{i=1}^{t} r_{g_i}(k_{g'_i} - c_i) = x$. Then $x \in Gr_M(N) \cap M_2$. Now by lemma 5.1, we divide the proof into two cases:

Case 1 If N is a graded prime submodule of M, then $Gr_M(N) = RGE_M(N) = N$.

Case 2 If $T(M/N) = L/N$ is Gr-torsion submodule of M/N, then $L = M$ or L is a graded prime submodule of M containing N. Therefore $Gr_M(N) \subseteq L$, so $x \in L$ and $x + N \in T(M/N)$. Thus there exists $0 \neq c \in h(R)$ such that $cx \in N$. Since M_2 is Gr-divisible module, so there exists $y \in M_2$ such that $x = cy$ so $c^2 y = cx \in N$. Since $y \in M$, without loss of generality we can assume that $y = \sum_{i=1}^{l} m_{g_i}$ and $m_h = 0$ for each $h \notin \{g_1, ..., g_l\}$. Then $c^2 y = \sum_{i=1}^{l} c^2 m_{g_i} \in N$. Then $c^2 m_{g_i} \in N$ for each $i = 1, ..., l$, since N is graded. Then $cm_{g_i} \in RGE_M(N)$ for each $i = 1, ..., l$. Therefore $x = cy = \sum_{i=1}^{l} cm_{g_i} \in RGE_M(N)$, then $m = x + \sum_{i=1}^{t} r_{g_i}(k_{g'_i} - c_i) \in RGE_M(N)$ and $Gr_M(N) \subseteq RGE_M(N)$. □

Corollary 5.6. *If G is a group and R is a Gr-integral domain, then every Gr-divisible R-module is a Gr-McCasland module.*

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