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On the Bessel Ultra-Hyperbolic Heat Equation

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Abstract : In this article, we study the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\Box_B^k u(x,t)$$

with the initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}_n^+$. The operator $\Box_B^k$ is named the Bessel ultra-hyperbolic operator iterated $k-$times and is defined by

$$\Box_B^k = \left(B_{x_1} + B_{x_2} + ... + B_{x_p} - B_{x_{p+1}} - ... - B_{x_{p+q}}\right)^k$$

where k is a non-negative integer, $p+q=n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$ [3,5-10], $x_i > 0$, $i = 1,2,...,n$, and n is the dimension of the $\mathbb{R}_n^+$, $u(x,t)$ is an unknown for $(x,t) = (x_1,...,x_n,t) \in \mathbb{R}_n^+ \times (0,\infty)$, $f(x)$ is a given generalized function and c is a positive constant. We obtain the solution of such equation which is related to the spectrum and the kernel which is so called the Bessel ultra-hyperbolic heat kernel. Moreover, such the Bessel ultra-hyperbolic heat kernel has interesting properties and also related to the kernel of an extension of the heat equation.

Keywords : Heat kernel, Dirac-delta distribution, Bessel ultra-hyperbolic operator, Fourier Bessel transform, B-convolution, Spectrum.
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1 Introduction

It is known that for the ultra-hyperbolic heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\Box^k u(x,t) \tag{1.1}$$

with the initial condition $u(x,0) = f(x)$ where $\Box^k$ is the ultra-hyperbolic operator iterated $k-$times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + ... + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - ... - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and k is a positive integer. In [?] Nonlaopon and Kananthai obtained the following solution

$$u(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y) \exp\left(c^2 t \left[\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{j=1}^{p} \xi_j^2\right]^k + i(\xi, x-y) \right) d\xi dy$$

or the solution in the classical convolution form

$$u(x,t) = E(x,t) * f(x) \tag{1.2}$$

where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left(c^2 t \left[\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{j=1}^{p} \xi_j^2\right]^k + i(\xi, x) \right) d\xi \tag{1.3}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$.

We can extend (1.1) to the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box_B u(x,t) \tag{1.4}$$

with the initial condition

$$u(x,0) = f(x) \tag{1.5}$$

where $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $p+q = n$ is the dimension $\mathbb{R}_n^+$, $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, ..., x_n),\ x_1 > 0, ..., x_n > 0$ and $\Box_B$ is the Bessel ultra-hyperbolic operator, defined by

$$\Box_B = B_{x_1} + B_{x_2} + ... + B_{x_p} - B_{x_{p+1}} - ... - B_{x_{p+q}}, \quad p + q = n.$$

Then, we obtain

$$u(x,t) = E(x,t) * f(x) \tag{1.6}$$

as a solution of (??) which satisfies (??) where $E(x,t)$ is the kernel of (??) or the elementary solution of (??) and is defined by

$$E(x,t) = C_v \int_{\mathbb{R}_n^+} e^{-c^2 t V(y)} \prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} dy \tag{1.7}$$

where $V(y) = \sum_{i=1}^{p} y_i^2 - \sum_{j=p+1}^{p+q} y_j^2 > 0$.

Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$ where δ is the Dirac-delta distribution, we studied the Bessel ultra-hyperbolic heat kernel which is related to spectrum.

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\Box_B^k u(x,t) \tag{1.8}$$

which the initial condition

$$u(x,0) = f(x), \text{ for } x \in \mathbb{R}_n^+, \tag{1.9}$$

where the operator $\Box_B^k$ is named the Bessel ultra-hyperbolic operator iterated $k-$times, defined by

$$\Box_B^k = \left(B_{x_1} + B_{x_2} + ... + B_{x_p} - B_{x_{p+1}} - ... - B_{x_{p+q}}\right)^k, \tag{1.10}$$

where k is a positive integer.

We obtain $u(x,t) = E(x,t) * f(x)$ a solution in the B-convolution form of (**??**) which satisfies condition (**??**) where

$$E(x,t) = C_v \int_\Omega e^{(-1)^k c^2 t V^k(y)} \prod_{i=1}^n J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} dy \tag{1.11}$$

and $\Omega \subset \mathbb{R}_n^+$ is the spectrum of $E(x,t)$ for any fixed $t > 0$. The function $E(x,t)$ is called the Bessel ultra-hyperbolic heat kernel iterated $k-$times or the elementary solution of (**??**). And all properties of $E(x,t)$ will be studied in details.

2 Preliminaries

The generalized shift operator T^y has the following form [**?**, **?**, **?**]:

$$\begin{aligned} T^y\varphi(x) \quad = \quad & C_v^* \int_0^\pi ... \int_0^\pi \varphi(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, ..., \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n}) \\ & \times (\prod_{i=1}^n \sin^{2v_i - 1}\theta_i) d\theta_1 ... d\theta_n \end{aligned}$$

where $x, y \in \mathbb{R}_n^+$,

$$C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i + 1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}.$$

We remark that this shift operator is closely connected with the Bessel differential operator $B = (B_{x_1}, ..., B_{x_n})$ [**?**].

The convolution operator determined by the T^y is as follows.

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T^y \varphi(x) (\prod_{i=1}^n y_i^{2v_i}) dy. \tag{2.1}$$

Convolution (**??**) known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

a. $T^y.1 = 1$
b. $T^0.f(x) = f(x)$
c. If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x \in \mathbb{R}_n^+$ and

$$\int\limits_{\mathbb{R}_n^+} |f(x)| \, (\prod_{i=1}^{n} x_i^{2v_i}) dx < \infty$$

then

$$\int\limits_{\mathbb{R}_n^+} T^y f(x) g(y) (\prod_{i=1}^{n} y_i^{2v_i}) dy = \int\limits_{\mathbb{R}_n^+} f(y) T^y g(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

d. From **c.,** we have the following equality for $g(x) = 1$.

$$\int\limits_{\mathbb{R}_n^+} T^y f(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy = \int\limits_{\mathbb{R}_n^+} f(y) (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

e. $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows (see, [**?**]-[**?**])

$$(F_B f)(x) = C_v \int\limits_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i y_i) \, y_i^{2v_i} \right) dy \tag{2.2}$$

$$\left(F_B^{-1} f\right)(x) = (F_B f)(-x), \; C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma \left(v_i + \frac{1}{2} \right) \right)^{-1} \tag{2.3}$$

where $J_{v_i - \frac{1}{2}} (x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following equalities for Fourier-Bessel transformation is true (see, [**?**, **?**, **?**]),

$$F_B \delta(x) = 1$$

$$F_B (f * g)(x) = F_B f(x) . F_B g(x) \tag{2.4}$$

Definition 1. *The spectrum of the kernel* $E(x,t)$ (**??**) *is the bounded support of the Fourier Bessel transform* $F_B E(x,t)$ *for any fixed* $t > 0$.

Definition 2. *Let* $x = (x_1, x_2, ..., x_n) \in \mathbb{R}_n^+$ *and denote by*

$$\Gamma_+ = \left\{ x \in \mathbb{R}_n^+ : x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2 > 0 \right\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ *denotes the clousure of* Γ_+.

*Let Ω be spectrum of $E(x,t)$ defined by definition (**??**) and $\Omega \subset \overline{\Gamma}_+$. Let $F_B E(x,t)$ be the Fourier Bessel transform of $E(x,t)$ and define*

$$F_B E(x,t) = \begin{cases} e^{(-1)^k c^2 t (x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2)^k} & \text{for } x_i \in \Omega \\ 0 & \text{for } x_i \notin \Omega. \end{cases} \tag{2.5}$$

Lemma 1. ***(Fourier Bessel Transform of $\Box_B^k$ operator)***

$$F_B \Box_B^k u(x) = (-1)^k c^2 V^k(x) F_B u(x).$$

Proof. We can use the mathematical induction method, for $k=1$, we have

$$\begin{aligned} F_B\left(\Box_B u\right)(x) &= C_{v\mathbb{R}_n^+}\left(\Box_B u(y)\right)\left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy \\ &= C_{v\mathbb{R}_n^+}\left(\sum_{i=1}^{p} \frac{\partial^2 u(y)}{\partial y_i^2}\right)\left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy \\ &+ C_{v\mathbb{R}_n^+}\left(\sum_{i=1}^{p} \frac{2v_i}{y_i}\frac{\partial u(y)}{\partial y_i}\right)\left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy \\ &- C_{v\mathbb{R}_n^+}\left(\sum_{i=p+1}^{p+q} \frac{\partial^2 u(y)}{\partial y_i^2}\right)\left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy \\ &- C_{v\mathbb{R}_n^+}\left(\sum_{i=p+1}^{p+q} \frac{2v_i}{y_i}\frac{\partial u(y)}{\partial y_i}\right)\left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

If we apply partial integration to twice in the I_1 and I_2 integrals and once in the I_3 and I_4 integrals, then we have

$$\begin{aligned} F_B\left(\Box_B u\right)(x) &= C_{v\mathbb{R}_n^+} u(y)\left(\left(\sum_{i=1}^{p} B_{y_i}\right)\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy. \\ &- C_{v\mathbb{R}_n^+} u(y)\left(\left(\sum_{i=p+1}^{p+q} B_{y_i}\right)\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy. \end{aligned}$$

Here, if we use the following equality [**?**],

$$\int_0^\infty u(y) B_{y_i} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i} dy_i = -x_i^2 \int_0^\infty u(y) J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i} dy_i$$

then, we get

$$\begin{aligned} F_B\left(\Box_B u\right)(x) &= -V(x) C_{v\mathbb{R}_n^+} u(y)\left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}\left(x_i y_i\right) y_i^{2v_i}\right) dy \\ &= -V(x) F_B u(x). \end{aligned}$$

Then, from inverse Fourier transform we finally obtain

$$\square_B u(x) = -F_B^{-1}V(x)F_B u(x).$$

Assume the statement is true for $k-1$, i.e,

$$\square_B^{k-1}u(x) \quad = \quad (-1)^{k-1}F_B^{-1}V^{k-1}(x)F_B u(x).$$

Then, we must prove that it is also true for $k \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} \square_B^k u(x) &= \square_B\left(\square_B^{k-1}u(x)\right) \\ &= (-1)F_B^{-1}V(x)F_B(-1)^{k-1}F_B^{-1}V^{k-1}(x)F_B u(x) \\ &= (-1)^k F_B^{-1}V^k(x)F_B u(x). \end{aligned}$$

This completes the proof. □

The following result can be found in [**?**, **?**, **?**], which will be applied in the sequel.

Lemma 2. *For all $t > 0$, c is a positive constant and all $x \in \mathbb{R}_n^+$ we have*

$$\overset{\infty}{\underset{0}{}}e^{-c^2x^2t}x^{2\upsilon}dx = \frac{\Gamma(\upsilon)}{2c^{2\upsilon+1}t^{\upsilon+\frac{1}{2}}} \tag{2.6}$$

and

$$\overset{\infty}{\underset{0}{}}e^{-c^2x^2t}J_{\upsilon-\frac{1}{2}}(xy)x^{2\upsilon}dx = \frac{\Gamma(\upsilon+\frac{1}{2})}{2(c^2t)^{\upsilon+\frac{1}{2}}}e^{-\frac{y^2}{4c^2t}}. \tag{2.7}$$

3 Main Results

In this section, we will state our results and given their proofs.

Lemma 3. *Let L be the operator defined by*

$$L = \frac{\partial}{\partial t} - c^2\square_B^k \tag{3.1}$$

*where $\square_B^k$ is defined by (**??**), k is a positive integer, $(x_1, ..., x_n) \in \mathbb{R}_n^+$, and c is a positive constant. Then we obtain*

$$E(x,t) = C_\upsilon \underset{\Omega}{}e^{(-1)^kc^2tV^k(y)}\prod_{i=1}^{n}J_{\upsilon_i-\frac{1}{2}}(x_iy_i)y_i^{2\upsilon_i}dy \tag{3.2}$$

*as elementary solution of (**??**) in the spectrum $\Omega \subset \mathbb{R}_n^+$ for $t > 0$.*

Proof. Let $E(x,t)$ is the kernel or the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - (-1)^k c^2 \Box_B^k E(x,t) = \delta(x)\delta(t).$$

Take the Fourier Bessel transform defined by (**??**) to both sides of the equation, using Lemma **??** and $F_B\delta(x) = 1$, we obtain

$$\frac{\partial}{\partial t}F_B E(x,t) - (-1)^k c^2 V^k(x) F_B E(x,t) = \delta(t).$$

Thus

$$F_B E(x,t) = H(t)e^{(-1)^k c^2 t V^k(x)}$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$.Therefore,

$$F_B E(x,t) = e^{(-1)^k c^2 t V^k(x)}$$

which has been already by (**??**). Thus, from (**??**), we have

$$E(x,t) = C_{\upsilon} \underset{\Omega}{} e^{(-1)^k c^2 t V^k(y)} \prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} dy \quad \text{for } t > 0,$$

where Ω is the spectrum of $E(x,t)$. $\square$

Theorem 1. *Let us consider the equation*

$$\frac{\partial}{\partial t}u(x,t) - c^2 \Box_B^k u(x,t) = 0 \tag{3.3}$$

with the initial condition

$$u(x,0) = f(x) \tag{3.4}$$

where $\Box_B^k$ *is defined by* (**??**)*,* k *is a positive integer,* $u(x,t)$ *is an unknown function for* $(x,t) = (x_1, ..., x_n, t) \in \mathbb{R}_n^+ \times (0,\infty)$ *,* $f(x)$ *is the given generalized function, and c is a positive constant. Then we obtain*

$$u(x,t) = E(x,t) * f(x) \tag{3.5}$$

as a solution of (**??**) *which satisfies* (**??**) *where* $E(x,t)$ *is given by* (**??**)*.*

Proof. Taking the Fourier Bessel transform defined by (**??**) to both sides of (**??**) for $x \in \mathbb{R}_n^+$ and using Lemma **??**, we obtain

$$\frac{\partial}{\partial t}F_B u(x,t) = (-1)^k c^2 V^k(x) F_B u(x,t). \tag{3.6}$$

Thus, we condider the initial condition (**??**) then we have following equality for the (3.6)

$$u(x,t) \quad = \quad f(x) * F_B^{-1} e^{(-1)^k c^2 t V^k(x)} \tag{3.7}$$

Here, if we use the (**??**), (**??**), then we have

$$\begin{aligned}
u(x,t) &= f(x) * F_B^{-1} e^{(-1)^k c^2 t V^k(x)} \\
&= {}_{\mathbb{R}_n^+} F_B^{-1} e^{(-1)^k c^2 t V^k(y)} T^y f(x) \left(\prod_{i=1}^{n} y_i^{2\upsilon_i} \right) dy \\
&= {}_{\mathbb{R}_n^+} \left[C_\upsilon {}_{\mathbb{R}_n^+} e^{(-1)^k c^2 t V^k(z)} \left(\prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(y_i z_i) z_i^{2\upsilon_i} \right) dz \right] T^y f(x) \left(\prod_{i=1}^{n} y_i^{2\upsilon_i} \right) dy.
\end{aligned} \tag{3.8}$$

Set

$$E(x,t) = C_\upsilon {}_{\mathbb{R}_n^+} e^{(-1)^k c^2 t V^k(y)} \left(\prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \right) dy. \tag{3.9}$$

Since the integral of (**??**) is divergent, therefore we choose $\Omega \subset \mathbb{R}_n^+$ be the spectrum of $E(x,t)$ and by (**??**), we have

$$\begin{aligned}
E(x,t) &= C_\upsilon {}_{\mathbb{R}_n^+} e^{(-1)^k c^2 t V^k(y)} \left(\prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \right) dy \\
&= C_\upsilon {}_{\Omega} e^{(-1)^k c^2 t V^k(y)} \left(\prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \right) dy.
\end{aligned} \tag{3.10}$$

Thus (**??**) can be written in the B-convolution form

$$u(x,t) = E(x,t) * f(x).$$

Moreover, since $E(x,t)$ exist, then

$$\begin{aligned}
\lim_{t \to 0} E(x,t) &= C_\upsilon {}_{\Omega} \prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} dy \\
&= C_\upsilon {}_{\mathbb{R}_n^+} \prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} dy \\
&= \delta(x), \text{ for } x \in \mathbb{R}_n^+,
\end{aligned} \tag{3.11}$$

see [**?**].

Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (**??**), then we have

$$\begin{aligned}
u(x,0) &= \lim_{t \to 0} u(x,t) \\
&= \lim_{t \to 0} E(x,t) * f(x) \\
&= \delta * f(x) \\
&= f(x)
\end{aligned}$$

which satisfies (**??**).

In particular, if we put $k = 1$ and $q = 0$ in (**??**), then from Lemma **??** we obtain

$$E(x,t) = C_{\upsilon} \int_{\mathbb{R}_n^+} e^{-c^2 tV(y)} \prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} dy$$

$$= 2^{-(\frac{n+2|\upsilon|}{2})}(c^2 t)^{-(\frac{n+2|\upsilon|}{2})} e^{-\frac{1}{4c^2 t}|x|^2}$$

where $|x|^2 = \sum_{i=1}^{n} x_i^2$. This completes the proof. □

Theorem 2. *The kernel* $E(x,t)$ *defined by* (**??**) *has the following properties:*

i. $E(x,t) \in C^{\infty}(\mathbb{R}_n^+ \times (0,\infty))$- the space of continuous function with infinitely differetiable.

ii. $(\frac{\partial}{\partial t} - c^2 \Box_B^k)E(x,t) = 0$ for all $x \in \mathbb{R}_n^+,\ t > 0$.

iii. $\lim_{t \to 0} E(x,t) = \delta(x)$ for all $x \in \mathbb{R}_n^+$.

Proof. **i.** From (**??**), and

$$\frac{\partial^n}{\partial t^n} E(x,t) = C_{\upsilon} \int_{\Omega} \frac{\partial^n}{\partial t^n} e^{(-1)^k c^2 t \left(y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2\right)^k} \left(\prod_{i=1}^{n} J_{\upsilon_i - \frac{1}{2}}(x_i y_i) y_i^{2\upsilon_i} \right) dy,$$

$E(x,t) \in C^{\infty}$ for $x \in \mathbb{R}_n^+$, $t > 0$.

ii. From $u(x,t) = E(x,t) * f(x)$, we have following equality for $f(x) = \delta(x)$ by Fourier Bessel Transformation

$$u(x,t) = E(x,t).$$

Then by direct computation, we obtain,

$$(\frac{\partial}{\partial t} - c^2 \Box_B^k)E(x,t) = 0.$$

iii. This case is obvious by (**??**). □

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