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Starlikeness and Subordination of Two Integral Operators

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Abstract : In this paper, we consider some sufficient conditions for two integral operators to be starlike in the open unit disk.

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1 Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function f belonging to $\mathcal{A}$ is said to be starlike of order α if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U})$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Clearly $\mathcal{S}^*(0) = \mathcal{S}^*$the class of all starlike functions with respect the origin.

Recently, Breaz and Breaz in [3] and Breaz et al. [7] introduced and studied the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.1}$$

and

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} \ldots (f_n'(t))^{\alpha_n} \, dt \tag{1.2}$$

where $f_i \in \mathcal{A}$ and for $\alpha_i > 0$, for all $i = 1, \ldots, n$ (see also [1, 4, 6]).

Breaz and Güney [5] considered the above integral operators and they obtained their properties on the classes $\mathcal{S}_\alpha^*(b)$, $\mathcal{C}_\alpha(b)$ of starlike and convex functions of complex order b and type α introduced and studied by Frasin [8] (see [2]).

Very recently, Frasin [9] obtained some sufficient conditions for the above integral operators to be in the classes $\mathcal{S}^*$, $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$, where $\mathcal{C}(\alpha)$ and $\mathcal{UCV}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, close -to-convex of order $\alpha(0 \leq \alpha < 1)$ in $\mathcal{U}$ and uniformly convex functions.

In the present paper, we obtain some sufficient conditions for starlikeness of the above integral operators F_n and $F_{\alpha_1,\ldots,\alpha_n}$.

In order to derive our main results, we have to recall here the following results:

Lemma 1.1. *([10]) If $f \in \mathcal{A}$ satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{\beta+1}{2(\beta-1)} \qquad (z \in \mathcal{U}) \tag{1.3}$$

for some $2 \leq \beta < 3$, or

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{5\beta-1}{2(\beta+1)} \qquad (z \in \mathcal{U}) \tag{1.4}$$

for some $1 < \beta \leq 2$, then $f \in \mathcal{S}^*$.

Lemma 1.2. *([10]) If $f \in \mathcal{A}$ satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{\beta+1}{2\beta(\beta-1)} \qquad (z \in \mathcal{U}) \tag{1.5}$$

for some $\beta \leq -1$, or

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3\beta+1}{2\beta(\beta+1)} \qquad (z \in \mathcal{U}) \tag{1.6}$$

for some $\beta > 1$, then $f \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$.

2 Starlikeness for the integral operator F_n

Applying Lemma 1.1, we derive

Theorem 2.1. *Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies*

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < 1 + \frac{3-\beta}{2(\beta-1)n\alpha_i} \qquad (z \in \mathcal{U}) \tag{2.1}$$

for some $2 \leq \beta < 3$, or

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) < 1 + \frac{3(\beta-1)}{2(\beta+1)n\alpha_i} \qquad (z \in \mathcal{U}) \tag{2.2}$$

for some $1 < \beta \leq 2,$ then $F_n \in \mathcal{S}^*$.

Proof. It follows from (1.1) that

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdots \left(\frac{f_n(z)}{z}\right)^{\alpha_n}. \tag{2.3}$$

Thus we have

$$F_n^{''}(z) = \left[\alpha_1\left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n\left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right)\right] F_n'(z). \tag{2.4}$$

Then from (2.4) , we obtain

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) \tag{2.5}$$

or, equivalently,

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)}\right) + 1 - \sum_{i=1}^{n} \alpha_i. \tag{2.6}$$

Taking the real part of both terms of (2.6), we have

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) &= \sum_{i=1}^{n} \alpha_i \operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right) + 1 - \sum_{i=1}^{n} \alpha_i \\ &= \alpha_1 \operatorname{Re}\left(\frac{zf_1'(z)}{f_1(z)}\right) + \alpha_2 \operatorname{Re}\left(\frac{zf_2'(z)}{f_2(z)}\right) + \cdots \\ &\quad + \alpha_n \operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) + 1 - [\alpha_1 + \alpha_2 + \cdots + \alpha_n], \end{aligned} \tag{2.7}$$

using the hypothesis (2.1) it follows from (2.7) that

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) &< \alpha_1\left(1 + \frac{3-\beta}{2(\beta-1)n\alpha_1}\right) + \alpha_2\left(1 + \frac{3-\beta}{2(\beta-1)n\alpha_2}\right) + \cdots \\ &\quad + \alpha_n\left(1 + \frac{3-\beta}{2(\beta-1)n\alpha_n}\right) + 1 - [\alpha_1 + \alpha_2 + \cdots + \alpha_n] \\ &< \frac{\beta+1}{2(\beta-1)} \qquad (z \in \mathcal{U}), \end{aligned}$$

for some $2 \le \beta < 3$. Also from the hypothesis (2.2) and (2.7), we get

$$\text{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right) < \frac{5\beta-1}{2(\beta+1)} \qquad (z\in\mathcal{U}),$$

for some $1 < \beta \le 2$. Hence by Lemma 1.1, we get $F_n \in \mathcal{S}^*$. This completes the proof. □

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.1, we have

Corollary 2.2. *Let $\alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1+\frac{3-\beta}{2(\beta-1)\alpha} \qquad (z\in\mathcal{U}),$$

for some $2 \le \beta < 3$, or

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) < 1+\frac{3(\beta-1)}{2(\beta+1)\alpha} \qquad (z\in\mathcal{U}),$$

for some $1 < \beta \le 2$, then $\int\limits_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \in \mathcal{S}^*$.

Applying Lemma 1.2, we derive

Theorem 2.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies*

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > 1+\frac{\beta-2\beta^2-1}{2\beta(\beta-1)n\alpha_i} \qquad (z\in\mathcal{U}), \tag{2.8}$$

for some $\beta \le -1$, or

$$Re\left(\frac{zf_i'(z)}{f_i(z)}\right) > 1+\frac{\beta-2\beta^2+1}{2\beta(\beta+1)n\alpha_i} \qquad (z\in\mathcal{U}), \tag{2.9}$$

for some $\beta > 1$, then $F_n \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$.

Proof. Using (2.7), (2.8), (2.9) and applying Lemma 1.2, we have $F_n \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$. □

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.3, we have

Corollary 2.4. *Let $\alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 1+\frac{\beta-2\beta^2-1}{2\beta(\beta-1)\alpha} \qquad (z\in\mathcal{U}),$$

for some $\beta \le -1$, or

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 1+\frac{\beta-2\beta^2+1}{2\beta(\beta+1)\alpha} \qquad (z\in\mathcal{U}),$$

for some $\beta > 1$, then $\int\limits_0^z \left(\frac{f(t)}{t}\right)^\alpha dt \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$.

3 Starlikeness for the integral operator $F_{\alpha_1,\ldots,\alpha_n}$

Applying Lemma 1.1 , we derive

Theorem 3.1. *Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies*

$$\mathrm{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right) < \frac{3-\beta}{2(\beta-1)n\alpha_i} \tag{3.1}$$

for some $2 \le \beta < 3$, or

$$\mathrm{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right) < \frac{3(\beta-1)}{2(\beta+1)n\alpha_i} \tag{3.2}$$

for some $1 < \beta \le 2$, then $F_{\alpha_1,\ldots,\alpha_n} \in \mathcal{S}^*$.

Proof. From (1.2) , we easily get

$$\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right). \tag{3.3}$$

It follows from (3.3) that

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) &= 1+\sum_{i=1}^{n}\alpha_i\mathrm{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right)\\ &= 1+\alpha_1\mathrm{Re}\left(\frac{zf_1''(z)}{f_1'(z)}\right)+\alpha_2\mathrm{Re}\left(\frac{zf_2''(z)}{f_2'(z)}\right)+\cdots\\ &\quad +\alpha_n\mathrm{Re}\left(\frac{zf_n''(z)}{f_n'(z)}\right),\end{aligned} \tag{3.4}$$

which, in the light of the hypothesis (3.1), yields

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) &< 1+\alpha_1\left(\frac{3-\beta}{2(\beta-1)n\alpha_1}\right)+\alpha_1\left(\frac{3-\beta}{2(\beta-1)n\alpha_2}\right)+\cdots\\ &\quad +\alpha_n\left(\frac{3-\beta}{2(\beta-1)n\alpha_n}\right)\\ &< \frac{\beta+1}{2(\beta-1)} \qquad (z\in\mathcal{U}),\end{aligned}$$

for some $2 \le \beta < 3$.On the other hand, by the hypothesis (3.2) and (3.4), we have

$$\mathrm{Re}\left(1+\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}\right) < \frac{5\beta-1}{2(\beta+1)} \qquad (z\in\mathcal{U}),$$

for some $1 < \beta \le 2$. Hence by Lemma 1.1, we get $F_{\alpha_1,\ldots,\alpha_n} \in \mathcal{S}^*$. □

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.1, we have

Corollary 3.2. *Let $\alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{3-\beta}{2(\beta-1)\alpha}$$

for some $2 \le \beta < 3$, or

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) < \frac{3(\beta-1)}{2(\beta+1)\alpha}$$

for some $1 < \beta \le 2$, then $\int_0^z (f'(t))^\alpha \, dt \in \mathcal{S}^*$.

Finally, we have

Theorem 3.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, \ldots, n$. If $f_i \in \mathcal{A}$ for all $i = 1, \ldots, n$ satisfies*

$$\mathrm{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right) > \frac{\beta-2\beta^2-1}{2\beta(\beta-1)n\alpha_i} \tag{3.5}$$

for some $\beta \le -1$, or

$$\mathrm{Re}\left(\frac{zf_i''(z)}{f_i'(z)}\right) > \frac{\beta-2\beta^2+1}{2\beta(\beta+1)n\alpha_i} \tag{3.6}$$

for some $\beta > 1$, then $F_{\alpha_1,\ldots,\alpha_n} \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$.

Proof. The theorem follows easily by using (3.4), (3.5), (3.6) and applying Lemma 1.2. □

Letting $n = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.3, we have

Corollary 3.4. *Let $\alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)}\right) > \frac{\beta-2\beta^2-1}{2\beta(\beta-1)\alpha}$$

for some $\beta \le -1$, or

$$Re\left(\frac{zf''(z)}{f'(z)}\right) > \frac{\beta-2\beta^2+1}{2\beta(\beta+1)\alpha}$$

for some $\beta > 1$, then then $\int_0^z (f'(t))^\alpha \, dt \in \mathcal{S}^*\left(\frac{\beta+1}{2\beta}\right)$.

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References

[1] D. Breaz, *Certain Integral Operators on the Classes* $\mathcal{M}(\beta_i)$ and $\mathcal{N}(\beta_i)$, J. Ineq. Appl.Volume 2008, Article ID 719354, 4 pages

[2] S. Bulut, *A Note on the paper of Breaz and Güney*, J. Math. Ineq. Vo. 2, No.4 (2008), 549-553.

[3] D. Breaz and N. Breaz, *Two integral operator,* Studia Universitatis Babes-Bolyai, Mathematica, Cluj-Napoca, 3 (2002), 13-21.

[4] D. Breaz and N. Breaz, *Some convexity properties for a general integral operator*, JIPAM, Volume 7, Issue 5, Article 177, 2006.

[5] D. Breaz and H. Güney, *The integral operator on the classes* $\mathcal{S}^*_\alpha(b)$ *and* $\mathcal{C}_\alpha(b)$, J. Math. Ineq. Vol. 2, No.1 (2008), 97-100.

[6] D. Breaz and V. Pescar, *Univalence conditions for some general integral operators,* Banach J. Math. Anal. (1) 2 (2008), 53-58.

[7] D. Breaz, S.Owa, N. Breaz, *A new integral univalent operator*, Acta Univ. Apul. 16 (2008), 11-16.

[8] B.A. Frasin, *Family of analytic functions of complex order*, Acta Math. Acad. Paed. Ny. 22 (2006), 179-191.

[9] B.A. Frasin, *Some sufficient conditions for certain integral operators*, J. Math. Ineq. Vol. 2, No.4 (2008), 527-535

[10] H. Shiraishi and S. Owa, *Starlikeness and convexity for anlytic functions concerned with Jack's lemma,* Int. J. Open Problems Compt. Math., Vol. 2, No. 1, (2009), 37-46.

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