



Lorentzian Helicoids in Three Dimensional Heisenberg Group

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Abstract : In this paper we study the minimal surface in three dimensional Heisenberg group Heis_3 . We use Levi-Civita connections and obtain mean curvature of Lorentzian Helicoid. We characterize the Lorentzian Helicoid and obtain the condition of being minimal surface for Lorentzian Helicoid.

Keywords : Heisenberg group, Lorentzian Helicoid, Minimal surface.

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1 Introduction

The helicoid is generated by spiraling a horizontal straight line along a vertical axis, and so, it is a ruled surface which is also foliated by helices.

In Euclidean Geometry there are two equivalent approaches from which the notion of mean curvature of a submanifold arises. One starts with the definition of the second fundamental form as the orthogonal component of the directional derivative of a tangent vector field to the submanifold, and the mean curvature appears as the trace of the second fundamental form. The other one considers the volume functional defined on the submanifolds of the same dimension and the mean curvature appears as the gradient of this functional. In this paper, we use the trace of the second fundamental form when computing the mean curvature of the surface. It is known that if $M \subset \mathbb{R}^3$ is a minimal surface the Gaussian curvature $K \leq 0$ and defined on the whole plane is either a plane or has an image under the Gauss map that omits at most two points.

In [1] it is studied that helicoids are axially symmetric minimal surfaces and it is shown that helicoid is a minimal surface in Heisenberg space which is given

by the metric

$$ds^2 = dx^2 + dy^2 + \left(\frac{\lambda}{2} (x_2 dx - x_1 dy) + dz \right)^2, \quad \lambda \neq 0. \quad (1.1)$$

In [2] it is studied that Gauss map in the Heisenberg group which is endowed with the metric (2.1) for $\lambda = 1$.

In [6] minimal surfaces in the three dimensional Heisenberg group are studied and the authors obtain Weierstrass representation of $Heis_3$, which is endowed with the metric in 2.1 for $\lambda = 1$.

In [8], it is classified that space-like ruled minimal surfaces in a three dimensional Minkowski space \mathbb{R}_1^3 . Kobayashi derives two kinds of Weierstrass- Enneper representations for space-like surfaces.

In [11] minimal surfaces and one-parameter subgroups in the three dimensional Heisenberg groups are studied. The authors obtain a characterization of the one-parameter subgroups. Then Frenet formulas for one-parameter subgroups of $Heis_3$ are calculated.

In [15], it is given equation of minimal surfaces in three dimensional Minkowski space \mathbb{R}_1^3 . Woestijne obtained the plane, the helicoid, the catenoid are minimal in Minkowski space \mathbb{R}_1^3 .

Let M be a 2-manifold and $\Omega : M \longrightarrow (\tilde{M}^3, \tilde{g})$ an immersion into a Lorentzian 3-manifold. We denote by g the pull-backed tensor field of \tilde{g} by Ω :

$$g = \tilde{g}(d\Omega, d\Omega).$$

Then

i. (M, g) is said to be non-degenerate if g is non-degenerate, i.e., $\det(g) \neq 0$ on M .

ii. (M, g) is said to be a spacelike surface if g is a Riemannian metric, i.e., $\det(g) > 0$.

iii. (M, g) is said to be a timelike surface if $\det(g) < 0$.

Let M be a spacelike surface or timelike surface in \tilde{M} . Then we can take a local unit normal vector field N such that

$$\tilde{g}(N, N) = \varepsilon, \quad \varepsilon = \begin{cases} 1 & M \text{ is spacelike} \\ -1 & M \text{ is timelike} \end{cases}$$

The constant ε is called the sign of M .

Definition 1.1. The second fundamental form h derived from N is defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) N$$

where $X, Y \in \chi(M)$, $\tilde{\nabla}$ and ∇ are Levi-Civita connections of \tilde{M} and M , respectively.

Definition 1.2. A spacelike surface is said to be a maximal surface if $H = 0$.

Definition 1.3. A timelike surface is said to be a extremal surface (or minimal surface) if $H = 0$.

In [12] and [13] it is shown that three dimensional Heisenberg group has the following left-invariant Lorentz metrics

$$\begin{aligned} g_1 &= -dx^2 + dy^2 + (xdy + dz)^2 \\ g_2 &= dx^2 + dy^2 - (xdy + dz)^2 \\ g_3 &= dx^2 + (xdy + dz)^2 - ((1-x)dy - dz)^2. \end{aligned}$$

and some geometric properties of the Heisenberg group $Heis_3$ endowed with a Lorentz metric are studied.

Let $\Omega(u)$ be a curve parametrized by u . Then we have these possibilities;

- i. $g(\Omega', \Omega') > 0$ and $g(\Omega'', \Omega'') > 0$. Spacelike curve with spacelike normal.
- ii. $g(\Omega', \Omega') > 0$ and $g(\Omega'', \Omega'') < 0$. Spacelike curve with timelike normal.
- iii. $g(\Omega', \Omega') > 0$ and $g(\Omega'', \Omega'') = 0$. Spacelike curve with null normal.
- iv. $g(\Omega', \Omega') < 0$ and $g(\Omega'', \Omega'') < 0$. Timelike curve.
- v. $g(\Omega', \Omega') = 0$ and $g(\Omega'', \Omega'') > 0$. Null curve

Recall that when $\Omega(u)$ is a non-null curve in $Heis_3$ with spacelike or timelike rectifying plane, then the Frenet equations are

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= -\epsilon_0 \epsilon_1 \kappa T + \tau B \\ \nabla_T B &= -\epsilon_1 \epsilon_2 \tau N \end{aligned}$$

where $\epsilon_0 = g(T, T) = \pm 1$, $\epsilon_1 = g(N, N) = \pm 1$, $\epsilon_2 = g(B, B) = \pm 1$ and $\epsilon_0 \epsilon_1 \epsilon_2 = -1$, [5]. If $\Omega(u)$ is a nulllike curve the Frenet equations are

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= -\kappa B + \tau T \\ \nabla_T B &= \tau N \end{aligned}$$

where $g(T, T) = 1$, $g(N, N) = 0$, $g(B, B) = 0$, $g(T, N) = 0$, $g(T, B) = 0$, $g(N, B) = 1$, [6].

Lemma 1.1. Each left invariant Lorentz metric on the Heisenberg group $Heis_3$ is isometric to one of the metrics g_1, g_2, g_3 , [12].

Proposition 1.1. The left invariant Lorentz metric g_3 is flat, [12].

In this paper we obtain a characterization of the Lorentzian Helicoid in three dimensional Heisenberg group which is given by the Lorentz left invariant metrics

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2.$$

and

$$g_3 = dx^2 + (xdy + dz)^2 - ((1-x)dy - dz)^2$$

We find that the Lorentzian Helicoid has different properties in $(Heis_3, g_2)$ and $(Heis_3, g_3)$.

Heisenberg group is a matrix group which is given by:

$$Heis_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, x, y, z \in \mathbb{R} \right\}$$

In Heisenberg group the group multiplication is:

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}x_1y_2 - \frac{1}{2}y_1x_2 \right).$$

The Lie algebra of the $Heis_3$ is

$$\mathfrak{heis}_3 = \left\{ \begin{bmatrix} 0 & u_1 & u_3 \\ 0 & 0 & u_2 \\ 0 & 0 & 0 \end{bmatrix}, u_1, u_2, u_3 \in \mathbb{R} \right\}.$$

The orthonormal bases of the \mathfrak{heis}_3 are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of the tangent space at the identity.

In this paper we will use the left invariant Lorentz metrics g_1 and g_2 . For the metric

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2$$

we have the orthonormal frames

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = x \frac{\partial}{\partial z} - \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

For the metric

$$g_3 = dx^2 + (xdy + dz)^2 - ((1-x)dy - dz)^2$$

we have the orthonormal frames

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + (1-x) \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$$

The element zero $0 = (0, 0, 0)$ is the unit of this group structure and the inverse element for (z, t) is $(z, t)^{-1} = (-z, -t)$.

Let $a = (z, t)$ and $b = (w, s)$. The commutator of the elements $a, b \in H_3$ is equal to

$$\begin{aligned} [a, b] &= aba^{-1}b^{-1} \\ &= (z, t)(w, s)(-z, -t)(-w, -s) \\ &= (z + w - z + w, t + s - t - s + \alpha) = (0, \alpha) \end{aligned}$$

where $\alpha \neq 0$ in general. This shows that H_3 is not abelian. On the other hand for any $a, b, c \in H_3$, their double commutator is:

$$[[a, b], c] = (0, 0)$$

This implies that H_3 is a nilpotent Lie group with nilpotency 2.

We know that in E^3 a Helicoid is a minimal regle surface. In this paper we want to make a characterization of Helicoid in $(Heis_3, g)$.

2 Lorentzian Helicoids in Three Dimensional Heisenberg Group

2.1 Minimal Surfaces in Lorentzian Heisenberg Group $(Heis_3, g_2)$

Lorentzian Helicoid has coordinates

$$P(u, v) = \begin{cases} x = \cosh v \cos u \\ y = \cosh v \sin u \\ z = u \end{cases} \quad (2.1)$$

Let three dimensional Heisenberg group $Heis_3$ is given by the left-invariant Lorentz metric

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2.$$

The Lie algebra of $Heis_3$ has the orthonormal bases

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = x \frac{\partial}{\partial z} - \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}. \quad (2.2)$$

Lie brackets of these orthonormal basis are

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

Then we have the Levi-Civita connections

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \\ \nabla_{e_1} e_2 &= -\nabla_{e_2} e_1 = \frac{1}{2} e_3, \\ \nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = \frac{1}{2} e_2, \\ \nabla_{e_2} e_3 &= \nabla_{e_3} e_2 = -\frac{1}{2} e_1, \end{aligned}$$

Theorem 2.1: *Lorentzian helicoid in three dimensional Lorentzian Heisenberg group which has left-invariant Lorentz metric g_2 , is a minimal (or maximal) surface if*

$$v = 0.$$

□

Proof. By differentiating (2.1) with respect to u and to v we deduce the following:

$$\begin{aligned} P_u &= -\cosh v \sin u \frac{\partial}{\partial x} + \cosh v \cos u \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \\ &= -\cosh v \sin u e_1 - \cosh v \cos u e_2 + (\cosh^2 v \cos^2 u + 1) e_3 \end{aligned}$$

$$\begin{aligned} P_v &= \sinh v \cos u \frac{\partial}{\partial x} + \sinh v \sin u \frac{\partial}{\partial y} \\ &= \sinh v \cos u e_1 - \sinh v \sin u e_2 + \cosh v \sinh v \cos u \sin u e_3 \end{aligned}$$

Now we will calculate $\|P_u\|^2$, $g(P_u, P_v)$ and $\|P_v\|^2$:

$$\|P_u\|^2 = g_1(P_u, P_u) = \cosh^2 v - (\cosh^2 v \cos^2 u + 1)^2 \quad (2.3)$$

$$g_2(P_u, P_v) = -(\cosh^2 v \cos^2 u + 1)(\cosh v \sinh v \cos u \sin u) \quad (2.4)$$

$$\|P_v\|^2 = g(P_v, P_v) = \sinh^2 v - (\cosh v \sinh v \cos u \sin u)^2. \quad (2.5)$$

The normal vector field of the Lorentzian helicoid is given by the following formula:

$$N = P_u \times P_v;$$

Then we obtain the following:

$$N = \sinh v \sin u e_1 - \sinh v \cos u (1 + \cosh^2 v) e_2 + \cosh v \sinh v e_3.$$

Now we can calculate the components of the second fundamental form of Lorentzian helicoid:

$$h_{11} = g\left(\nabla_{P_u} P_u, \frac{N}{\|N\|}\right) = \frac{-\cosh^3 v \sinh v \cos u \sin u (\cosh^2 v \cos^2 u + 1)}{\|N\|}$$

$$\begin{aligned} h_{12} = g\left(\nabla_{P_u} P_v, \frac{N}{\|N\|}\right) &= \frac{\sinh^2 v}{2\|N\|} (\sin^2 u (2 \cosh^2 v \cos^2 u + 1) \\ &\quad + \cos^3 u (\cosh^2 v \cos^2 u - \cosh^2 v \sin^2 u + 1) \\ &\quad \times (\cosh^2 v + 1) - \cosh v \cos^2 u (-\cosh^2 v \\ &\quad + \cosh v (\cosh^2 v + \cos^2 u + 1)) \end{aligned}$$

$$h_{13} = g\left(\nabla_{P_v} P_v, \frac{N}{\|N\|}\right) = \frac{\sinh^3 v \cosh v \sin u \cos u (\cosh^2 v + \cos^2 u + 1)}{\|N\|}$$

The mean curvature of this surface

$$H = \frac{1}{2} g_{ij}^{-1} h_{ij}$$

is zero if and only if

$$\sinh v = 0 \implies v = 0$$

The sectional curvature associated with e_i, e_j $1 \leq i, j \leq 3$ is

$$K(e_i, e_j) = g(R_{e_i e_j}(e_i), e_j).$$

So, we can obtain the following:

$$\begin{aligned} K(e_1, e_2) &= -\frac{3}{4} \\ K(e_1, e_3) &= \frac{1}{4} \\ K(e_2, e_3) &= \frac{1}{4}. \end{aligned}$$

Principle Ricci curvatures are:

$$r(e_1) = -\frac{1}{2}, \quad r(e_2) = -\frac{1}{2}, \quad r(e_3) = \frac{1}{2}$$

and the scalar curvature is:

$$\begin{aligned} \rho &= r(e_1) + r(e_2) + r(e_3) \\ &= -\frac{1}{2}. \end{aligned}$$

2.2. Frenet Formulas for Lorentzian Helicoid in $(Heis_3, g_2)$

Definition 2.1 A curve in $Heis_3$ is a regular mapping of an open interval $I \subset \mathbb{R}$ into $Heis_3$.

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. $T(s), N(s)$ and $B(s)$ are the tangent, the principal normal and the binormal vector of the curve $\alpha(s)$, respectively.

In this section we derive some results for the curve α whose coordinates are:

$$P(u) = \begin{cases} x = \cos u \\ y = \sin u \\ z = u \end{cases} \quad (2.6)$$

Differentiating (2.6) with respect to u we deduce that:

$$P'(u) = -\sin u e_1 - \cos u e_2 + (1 + \cos u) e_3 \quad (2.7)$$

and

$$P''(u) = -\cos u e_1 + \sin u e_2 - 2 \cos u \sin u \quad (2.8)$$

where $\{e_1, e_2, e_3\}$ is the left-invariant frame associated with the basis $\{E_1, E_2, E_3\}$ of the tangent space of $Heis_3$ at the identity. From (2.7) and (2.8) we have that:

$$g_2(P'(u), P'(u)) = 1 - (1 + \cos^2 u)^2 \quad (2.9)$$

and

$$g_2(P''(u), P''(u)) = 1 - 4 \cos^2 u \sin^2 u. \quad (2.10)$$

So, from (2.9), if

$$\begin{aligned} \cos u &= 0, \\ u &= 2k\pi + \frac{\pi}{2}, \end{aligned}$$

the curve is timelike, otherwise the curve is null. From (2.10), we have the following cases:

i) If

$$u = \pi + 2k\pi, 2k\pi, \frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi,$$

the normal is spacelike,

ii) If

$$u = \frac{\pi}{4} + 2k\pi,$$

the normal is null.

iii) For all cases except (i) and (ii), the normal is timelike.

Case 1: Let the curve be timelike, and the normal be spacelike. In this case we will use the Frenet equation

$$\nabla_T T = \kappa N \quad (2.11)$$

$$\nabla_T N = \kappa T + \tau B$$

$$\nabla_T B = -\tau N.$$

The unit vector field T which is tangent to α is:

$$T = \frac{-\sin u e_1 - \cos u e_2 + (1 + \cos u) e_3}{\sqrt{1 - (\cos^2 u + 1)^2}}$$

and the covariant derivative of T is:

$$\nabla_T T = \frac{1 + \cos^2 u}{1 - (\cos^2 u + 1)^2} (\cos u e_1 - \sin u e_2).$$

Now we calculate the curvature of α :

$$\begin{aligned}\kappa &= \|\nabla_T T\|^2 = g(\nabla_T T, \nabla_T T) \\ &= \frac{1 + \cos^2 u}{1 - (\cos^2 u + 1)^2}.\end{aligned}$$

We know from (2.11) that the first Frenet formula is:

$$\nabla_T T = \kappa N \quad (2.12)$$

where N is the normal vector of α . So, from (2.12) the normal vector can be obtained as:

$$N = (\cos u e_1 - \sin u e_2).$$

From (2.11) we have

$$\nabla_T N = \kappa T + \tau B$$

Thus, we can calculate the following:

$$\begin{aligned}\nabla_T N - \kappa T &= \frac{1}{2(1 - (1 + \cos^2 u))^{\frac{3}{2}}} (1 + \cos^2 u) (\sin u) \\ &\quad \times (3 - (1 + \cos^2 u)) e_1 + \cos u (1 + \cos^2 u) (3 - (1 + \cos^2 u)) e_2 \\ &\quad + (2 - 2(1 + \cos^2 u)^2) e_3.\end{aligned} \quad (2.13)$$

From (2.13) we deduce that

$$\begin{aligned}B &= \frac{1}{(1 - (1 + \cos^2 u))^{\frac{1}{2}}} ((1 + \cos^2 u) (\sin u) (3 - (1 + \cos^2 u))) e_1 \\ &\quad + \cos u (1 + \cos^2 u) (3 - (1 + \cos^2 u)) e_2 + (2 - 2(1 + \cos^2 u)^2) e_3\end{aligned}$$

and we can obtain

$$\tau = \frac{1}{2(1 - (1 + \cos^2 u))}.$$

Case 2: Let both the curve and the normal be timelike. In this case we will use the Frenet equation

$$\begin{aligned}\nabla_T T &= \kappa N \\ \nabla_T N &= -\kappa T + \tau B \\ \nabla_T B &= \tau N.\end{aligned} \quad (2.14)$$

The unit vector field T which is tangent to α is

$$T = \frac{-\sin u e_1 - \cos u e_2 + (1 + \cos u) e_3}{\sqrt{1 - (\cos^2 u + 1)^2}}$$

and the covariant derivative of T is

$$\nabla_T T = \frac{1 + \cos^2 u}{1 - (\cos^2 u + 1)^2} (\cos u e_1 - \sin u e_2).$$

Now we us calculate the curvature of α :

$$\begin{aligned} \kappa &= \|\nabla_T T\| = g(\nabla_T T, \nabla_T T) \\ &= \frac{1 + \cos^2 u}{1 - (\cos^2 u + 1)^2}. \end{aligned}$$

We know from (2.14) that the first Frenet formula is:

$$\nabla_T T = \kappa N \tag{2.15}$$

where N is the normal vector of α . So, from (2.15) the normal vector can be obtained as:

$$N = (\cos u e_1 - \sin u e_2).$$

From (2.14) we have

$$\nabla_T N = -\kappa T + \tau B$$

Thus, we can obtain the following:

$$\nabla_T N + \kappa T = \frac{1 + (1 + \cos^2 u)^2}{2(1 - (1 + \cos^2 u))^{\frac{3}{2}}} (-(1 + \cos^2 u) \sin u) e_1 - (1 + \cos^2 u) \cos u e_2 + 2e_3$$

Then we deduce that

$$B = \frac{1 + (1 + \cos^2 u)^2}{(1 - (1 + \cos^2 u))^{\frac{1}{2}}} (-(1 + \cos^2 u) \sin u) e_1 - (1 + \cos^2 u) \cos u e_2 + 2e_3$$

and we can obtain

$$\tau = \frac{1}{2(1 - (1 + \cos^2 u))}.$$

Case 3: Let the curve be null. In this case we will use the Frenet equation

$$\begin{aligned} \nabla_T T &= \kappa N \\ \nabla_T N &= -\kappa B + \tau T \\ \nabla_T B &= \tau N. \end{aligned} \tag{2.16}$$

The unit vector field T which is tangent to α is:

$$T = \frac{-\sin u e_1 - \cos u e_2 + (1 + \cos u) e_3}{\sqrt{1 - (\cos^2 u + 1)^2}}$$

and the covariant derivative of T is:

$$\nabla_T T = \frac{1 + \cos^2 u}{1 - (\cos^2 u + 1)^2} (\cos u e_1 - \sin u e_2).$$

Now let us calculate the curvature of α :

$$\begin{aligned} \kappa &= \|\nabla_T T\| = g(\nabla_T T, \nabla_T T) \\ &= \frac{1 + \cos^2 u}{1 - (\cos^2 u + 1)^2}. \end{aligned}$$

We know from (2.16) that the first Frenet formula is:

$$\nabla_T T = \kappa N$$

where N is the normal vector of α . Then the normal vector can be obtained as

$$N = (\cos u e_1 - \sin u e_2).$$

If we denote

$$B = B_1 e_1 + B_2 e_2 + B_3 e_3,$$

from (2.16) we can deduce that:

$$B_1 = B_2 = 0 \quad \text{and} \quad B_3 = (1 + \cos^2 u) \sqrt{1 - (\cos^2 u + 1)^2}.$$

Then we have that

$$B = (1 + \cos^2 u) \sqrt{1 - (\cos^2 u + 1)^2} e_3$$

and

$$\tau = -\frac{1 + \cos^2 u}{2}.$$

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