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Pseudo-Differential Operators associated with Bessel Type Operators - I

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Abstract : In this paper Bessel type differential operator $\Delta_{\alpha,\beta}$, the pseudo differential type operators are defined and the symbol classes $\mathrm{H}^m$and H_0^m are introduced. It is established that Pseudo-differential type operator associated with symbols belonging to these classes are continuous linear mappings of the Zemanian space $\mathrm{H}_{\alpha,\beta}$ into itself. Integral representation for Pseudo-differential type operator $h_{\alpha,\beta,a}$ is obtained. Finally it is shown that Pseudo-differential type operators satisfy L^1-norm inequality.

Keywords : Pseudo-Differential type operator; Bessel type operator; symbol class; integral representation; Hankel type convolution; Hankel type transformation; Sobolev type space.
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1 Introduction

The Hankel type transformation,

$$(h_{\alpha,\beta}\phi)(x) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\phi(y)dy \tag{1.1}$$

is extended by Zemanian [7] to distributions belonging to $H'_{\alpha,\beta}$, the dual of the function space $H_{\alpha,\beta}$ which consists of all complex valued infinitely differentiable functions defined on $I = (0,\infty)$ satisfying,

$$\rho^{\alpha,\beta}_{m,k}(\varphi) = \sup_{x\in I}\left|x^m (x^{-1}D_x)^k \left(x^{2\beta-1}\phi(x)\right)\right| < \infty \tag{1.2}$$

for every $m, k \in \mathbb{N}_0$.

Zaidman [5], [6] has used Schwartz's theory of the Fourier transform of distributions in $\Im'(R^n)$in the study of Pseudo-differential operators. But Zemanian's theory of the Hankel type transform has not been used so far to develop a theory of Pseudo-differential operators associated with Bessel type operators as a special case. The purpose of the present paper is to change this Scenario (situation)

2 Notations and Terminology

We define the differential operators $P_{\alpha,\beta}, Q_{\alpha,\beta}$ and $S_{\alpha,\beta}$ as,

$$P_{\alpha,\beta} = P_{\alpha,\beta,x} = x^{2\alpha}\ D_x\ x^{2\beta-1} \tag{2.1}$$

$$Q_{\alpha,\beta,x} = x^{2\beta-1}\ D_x\ x^{2\alpha} \tag{2.2}$$

$$\begin{aligned} \Delta_{\alpha,\beta} = \Delta_{\alpha,\beta,x} &= Q_{\alpha,\beta}\ P_{\alpha,\beta} = x^{2\beta-1}\ D_x\ x^{4\alpha} D_x\ x^{2\beta-1} \\ &= (2\beta-1)\,(4\alpha+2\beta-2)\,x^{4(\alpha+\beta-1)} \\ &+ 2(2\alpha+2\beta-1)\ x^{4\alpha+4\beta-3}\ D_x\ + x^{2(2\alpha+2\beta-1)}\ D_x^2 \end{aligned} \tag{2.3}$$

where, $D_x = \dfrac{d}{dx}$

Following [7, p. 139] and [2, p. 948], we can establish the following relations for any $\phi \in H_{\alpha,\beta}$

$$h_{\alpha,\beta,1}(-x\phi) = P_{\alpha,\beta}\ h_{\alpha,\beta}\ \phi \tag{2.4}$$

$$h_{\alpha,\beta,1}(P_{\alpha,\beta}\ \phi) = -y\ h_{\alpha,\beta}\ \phi \tag{2.5}$$

$$h_{\alpha,\beta}(\Delta_{\alpha,\beta}\ \phi) = -y^2\ h_{\alpha,\beta}\ \phi \tag{2.6}$$

$$\left(x^{-1}D\right)^k \left(x^{2\beta-1}\theta\phi\right) = \sum_{i=0}^{k} \binom{k}{i} \left(x^{-1}D_x\right)^i \theta \left(x^{-1}D_x\right)^{k-i} \left(x^{2\beta-1}\phi\right) \tag{2.7}$$

$$\Delta^r_{\alpha,\beta,x}\phi(x) = \sum_{j=0}^{r} b_j x^{2j+2\alpha} \left(x^{-1}D_x\right)^{r+j} \left(x^{2\beta-1}\phi(x)\right)\ , \tag{2.8}$$

where b_j are constants depending only on $\alpha - \beta$.

We also need a lemma due to Haimo [1] for the Hankel type convolution transform.

Lemma 2.1. *Let $\Delta(x,y,z)$ be the area of a triangle with sides x, y, z if such a triangle exists for fixed $(a-b) \geq -\frac{1}{2}$, set,*

$$D(x,y,z) = \frac{2^{3(a-b)-1}\ \left(\Gamma(a-b+1)\right)^2}{\sqrt{\pi}\,\Gamma(a-b+\frac{1}{2})}\,(xyz)^{-2(a-b)}\ \left[\Delta(x,y,z)\right]^{2(a-b)-1} \tag{2.9}$$

if Δ exists and zero otherwise. We note that $D(x,y,z) \geq 0$ and that it is symmetric in x, y, z. Further we have the following basic formula:

$$\int_0^\infty i(zt)\, D(x,y,z)\, d\,\mu(z) = i(xt)\, i(yt) \tag{2.10}$$

$$\text{where,} \quad d\,\mu(x) = \frac{x^{2(a-b)+1}}{2^{a-b}\Gamma(a-b+1)} dx \tag{2.11}$$

$$i(x) = 2^{a-b}\, \Gamma(a-b+1)\, x^{-(a-b)}\, J_{a-b}\,(x)\, \cdot \tag{2.12}$$

Let $f \in L^1(0,\infty)$ Then its associated function $f(x,y)$ is defined by,

$$f(x,y) = \int_0^\infty f(z)\, D(x,y,z)\, d\,\mu(z) \quad 0 < x,\ y < \infty\, \cdot \tag{2.13}$$

Lemma 2.2. *Let f and g be functions of $L^1(0,\infty)$ and let,*

$$f \# g(x) = \int_0^\infty f(x,y)\, g(y)\, d\,\mu(y) \quad 0 < x < \infty\, \cdot \tag{2.14}$$

Then the integral defining f#g(x) converges for almost all x, $\quad 0 < x < \infty$, and

$$\|f \# g\|_{L^1} \leq \|f\|_{L^1}\, \|g\|_{L^1}\, \cdot \tag{2.15}$$

3 The Pseudo-Differential type Operator $h_{\alpha,\beta,a}$

Definition 3.1. *Let $a(x,y)$ be a complex valued function belonging to the space $C^\infty\,(I \times I)$, where $I = (0,\infty)$ and let its derivatives satisfy certain growth conditions such as* (3.5). *Then the Pseudo-differential type operator $h_{\alpha,\beta,a}$ associated with the symbol a(x,y) is defined by,*

$$(h_{\alpha,\beta,a}\, u)\,(x) = \int_0^\infty (xy)^{\alpha+\beta}\, J_{\alpha-\beta}\,(xy)\, a(x,y)\, U_{\alpha,\beta}\,(y)\, dy \tag{3.1}$$

where,

$$U_{\alpha,\beta}\,(y) = (h_{\alpha,\beta}\, u)\,(y) = \int_0^\infty (xy)^{\alpha+\beta}\, J_{\alpha-\beta}\,(xy)\, u(x)\, dx; \quad (\alpha-\beta) \geq -\frac{1}{2}\cdot \tag{3.2}$$

In case $a(x,y) = b(y)$, then clearly we have,

$$(h_{\alpha,\beta,a}\, u)(x) = h_{\alpha,\beta}[b(y)\, U_{\alpha,\beta}(y)]\, \cdot$$

If $a(x,y)$ possesses a power series expansion in $(-y^2)$ with variable coefficients depending on x, that is,

$$a(x,y)=\sum_{k=0}^{N} a_k\,(x)\,(-y^2)^k\ \cdot \tag{3.3}$$

Then using formula (2.6) one can show that

$$(h_{\alpha,\beta,a}\,u)\,(x)=\sum_{k=0}^{N} a_k\,(x)\,(\Delta_{\alpha,\beta})^k\,u(x) \tag{3.4}$$

so that the Pseudo-differential type operator associated with the symbol (3.3) could be a finite order differential operator involving $\Delta_{\alpha,\beta}$.

Definition 3.2. *The function $a(x,y) : C^\infty\,(I\times I)\to\mathbb{C}$ belongs to class H^m if and only if for every $q\in\mathbb{N}_0,\ i\in\mathbb{N}_0,\ p\in\mathbb{N}_0$, there exists $K_{p,i,m,q}>0$ such that,*

$$(1+x)^q\left|\left(x^{-1}\,D_x\right)^i\left(y^{-1}\,D_y\right)^p\,a\,(a,y)\right|\le\ K_{p,i,m,q}\,(1+y)^{m-p} \tag{3.5}$$

where $D_y=\frac{d}{dy}$ and m is a fixed real number.

If $a(x,y)$ satisfies (3.5) with $q=0$, then the symbol class will be denoted by H_0^m; clearly $H^m\subset H_0^m$. One can easily show that $a(x,y)=\left(1+x^2\right)^{-n}\left(1+y^2\right)^{\frac{m}{2}}\ n\ >\ 0$, $m\in\mathbb{R}$ is an element of H_0^m, but it does not belong to H^m.
Nevertheless $a(x,y)=e^{-x^2}\,(1+y^2)^{\frac{m}{2}},\quad m\in\mathbb{R}$ belongs to H^m.

Theorem 3.3. *Let the symbol $a(x,y)\in H_0^m$(or H^m). Then for $(\alpha-\beta)\ge-\frac{1}{2}$, the pseudo-differential type operator $h_{\alpha,\beta,a}$ is a continuous linear mapping of $H_{\alpha,\beta}$ into itself.*

Proof. Let $\phi(y)=(h_{\alpha,\beta,a}\,u)\,(y),\quad u\in H_{\alpha,\beta}\,(I)$. Then using formulae (2.4), (2.5) and Zemanian's technique, [6, p. 141], we have,

$$(P_{\alpha,\beta,k-1}\ldots\ldots P_{\alpha,\beta}\,\phi)\ (y)=$$
$$\sum_{r=0}^{k} C_r\int_0^\infty y^{r+\frac{1}{2}}\,x^{\frac{1}{2}}\,(y^{-1}\,D_y)^r\,a(x,y)\,(-x)^{k-r}\,u(x)\,J_{\alpha-\beta+k-r}\,(xy)\,dx,$$

where C_r are certain positive real numbers.

Set $a_r\,(x,y)=(y^{-1}\,D_y)^r\,a(x,y)$

Now using formula (2.7) and induction, we obtain,

$$\begin{aligned}&(-y)^n\,(P_{\alpha,\beta,k-1}\ldots\ldots P_{\alpha,\beta}\,\phi\,(y))\\ =\ &\sum_{r=0}^{k}(-1)^{k-r}\,C_r\int_0^\infty y^{r+\frac{1}{2}}\,x^{\alpha-\beta+k+n-r+1}\sum_{i=0}^{n}\tbinom{n}{i}\ (x^{-1}\,D_x)^i\\ \times\ \ &a_r\,(x,y)\,(x^{-1}\,D_x)^{n-i}\left(x^{2\beta-1}\,u(x)\right)\ J_{\alpha-\beta+k+n-r}\,(xy)\,dx\ .\end{aligned}$$

Setting $\alpha - \beta + k - r = \lambda$ and using formula (2.1), we can obtain,

$$(-1)^n \, y^n \, (y^{-1} \, D_y)^k \left(y^{2\beta-1} \, \phi(y)\right) = \sum_{r=0}^{k} (-1)^{k-r} \, C_r \int_0^\infty x^{2\lambda+n+1} \cdot$$
$$\sum_{i=0}^{n} \tbinom{n}{i} \, (x^{-1} \, D_x)^i \, a_r \, (x,y) \, (x^{-1} \, D_x)^{n-i} \, \left(x^{2\beta-1} \, u(x)\right) \, J_{\lambda+n} \, (xy) \, (xy)^{-\lambda} \, dx \, .$$

Now setting $n = t + s$ where $s, t \in \mathbb{N}_0$, and $s \geq m$, and $n = t$ in turn, in the above expression and using (3.5) with $q = 0$ and assumption $(\alpha - \beta) \geq -\frac{1}{2}$, we can estimate the above expression in absolute value as follows:

There exists a constant $K_{m,\nu,r}$ such that,

$$(1 + y^s) \left| y^t \, (y^{-1} \, D_y)^k \, y^{2\beta-1} \, \phi \, (y) \right|$$
$$\leq \sum_{r=0}^{k} \int_0^\infty 2^m \, (1 + y^m) \, (1 + y)^{-p} \cdot$$
$$\left[\begin{array}{l} (1+x)^{2\lambda+t+s+1} \displaystyle\sum_{i=0}^{t+s} K_{m,v,r} \tbinom{t+s}{i} (x^{-1} D_x)^{t+s-i} \left(x^{2\beta-1} u(x)\right) \\ +(1+x)^{2\lambda+t+1} \displaystyle\sum_{i=0}^{t} K_{m,v,r} \tbinom{t}{i} (x^{-1} D_x)^{t-i} \left(x^{2\beta-1} u(x)\right) \end{array} \right] dx$$

Thus, $$\left| y^t \, (y^{-1} \, D_y)^k \, y^{2\beta-1} \, \phi(y) \right| \; \leq \; K \sum_{r=0}^{k} \int_0^\infty \; 2^m \, (1 + y^m) \, (1 + y^s)^{-1} \cdot$$
$$\left[\begin{array}{l} (1+x)^{2\lambda+t+s+1} \displaystyle\sum_{i=0}^{t+s} \tbinom{t+s}{i} (x^{-1} D_x)^{t+s-i} (x^{2\beta-1} u(x) \\ +(1+x)^{2\lambda+t+1} \displaystyle\sum_{i=0}^{t} \tbinom{t}{i} (x^{-1} D_x)^{t-i} \left(x^{2\beta-1} u(x)\right) \end{array} \right] dx$$

Now, we can choose a non-negative integer N such that $N > 2(\alpha - \beta + k) + t + s + 3$. Then using the fact that $(1 + y^m)/(1 + y^s) \leq 2$ for $y \geq 0, \; s \geq m$, we obtain,

$$\left| y^t \, (y^{-1} \, D_y)^k \, y^{2\beta-1} \, \phi(y) \right| \; \leq \; K' \sum_{r=0}^{k} 2^{m+1} \int_0^\infty (1 + x)^{N-2} \cdot$$
$$\left[\sum_{i=0}^{t+s} \tbinom{t+s}{i} \left| (x^{-1} \, D_x)^{t+s+i} \, x^{2\beta-1} \, u(x) \right| + \sum_{i=0}^{t} \tbinom{t}{i} \left| (x^{-1} \, D_x)^{t-i} \, x^{2\beta-1} \, u(x) \right| \right] dx$$
$$\leq K'' \sum_{i=0}^{k} \left[\sum_{i=0}^{t+s} \tbinom{t+s}{i} \sum_{j=0}^{N} \tbinom{N}{j} \, \rho_{j,t+s-i}^{\alpha,\beta} \, (u) + \sum_{i=0}^{t} \tbinom{t}{i} \sum_{j=0}^{N} \tbinom{N}{j} \, P_{j,t-i}^{\alpha,\beta} \, (u) \right] .$$

Therefore by using (1.2), we have,

$$\rho_{t,k}^{\alpha,\beta}(\phi) \leq K'' \sum_{r=0}^{k}\sum_{j=0}^{N}\binom{N}{j}\left[\sum_{i=0}^{t+s}\binom{t+s}{i}\ \rho_{j,t+s-i}^{\alpha,\beta}(u) + \sum_{i=0}^{t}\binom{t}{i}\ \rho_{j,t-i}^{\alpha,\beta}(u)\right] \quad (3.6)$$

where K'' is a positive constant. The continuity of $h_{\alpha,\beta,a}$ follows from (3.6) and hence the proof is complete. □

4 Integral Representation for $h_{\alpha,\beta,a}$

The function $a_\eta(y)$, associated with the symbol $a(x,y)$ and defined by,

$$a_\eta(y) = \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\ \left[(x\eta)^{\alpha+\beta} J_{\alpha-\beta}(xt)\, a(x,\eta)\right] dx \quad (4.1)$$

will play a fundamental role in our investigation. An estimate for $a_\eta(y)$ is given by the following lemma:

Lemma 4.1. *Let the symbol $a(x,y) \in H^m$. Then the function $a_\eta(y)$ defined by* (4.1) *satisfies the inequity,*
$|a_\eta(y)| \leq A_{\alpha,\beta,m,r,q}(1+\eta)^{2\alpha+m+4r}(1+y)^{2\alpha}(1+y^{2r})^{-1}$ *where $A_{\alpha,\beta,m,r,q}$ is a positive constant.*

Proof. For $r \in \mathbb{N}_0$, using formula (2.6), we have,

$$\begin{aligned}(-y^2)^r a_\eta(y) &= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\,(\Delta_{\alpha,\beta})^r\ \left[(x\eta)^{\alpha+\beta} J_{\alpha-\beta}(x\eta)\, a(x,\eta)\right]\ dx \\ &= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \sum_{j=0}^{r} b_j\, x^{2j+2\alpha}\,(x^{-1}D_x)^{r+j}\, x^{2\beta-1} \\ &\times\ \left[(x\eta)^{\alpha+\beta} J_{\alpha-\beta}(x\eta)\, a(x,\eta)\right]\ dx\ .\end{aligned}$$

Using formula (2.7) we obtain,

$$\begin{aligned}(-y^2)^r\, a_\eta(y) &= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \sum_{j=0}^{r} b_j\, x^{2j+2\alpha} \sum_{i=0}^{r+j}\binom{r+j}{i} \\ &\times\ (x^{-1}D_x)^i a(x,\eta)\,(x^{-1}D_x)^{r+j-i}\left(x^{-(\alpha-\beta)} J_{\alpha-\beta}(x\eta)\right)\eta^{\alpha+\beta} dx\ .\end{aligned}$$

Now using the formula,

$$(x^{-1}D_x)^m x^{-(\alpha-\beta)} J_{\alpha-\beta}(x\eta) = (-\eta)^m x^{-(\alpha-\beta)-m} J_{\alpha-\beta+m}(x\eta)\ .$$

We can obtain,

$$\left|(-y^2)^r\, a_\eta(y)\right| \leq \int_0^\infty \left|(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \sum_{j=0}^{r} b_j\, x^{2j+2\alpha}\right.$$

$$\times \sum_{i=0}^{r+j} \binom{r+j}{i} (x^{-1}\, D_x)^i\, a(x,\eta)\, \eta^{2r+2j-2i+2\alpha} (x\eta)^{-(\alpha-\beta)-r-j-i}\, J_{\alpha-\beta+r+j-i}\, (x\eta)\Bigg|\, dx$$

$$\le y^{2\alpha} \int_0^\infty x^{2\alpha} \left|(xy)^{-(\alpha-\beta)}\, J_{\alpha-\beta}(xy)\right|$$

$$\times \left|\sum_{j=0}^{r} b_j\, x^{2j+2\alpha} \sum_{i=0}^{r+j} \binom{r+j}{i} (x^{-1}\, D_x)^i\, a(x,\eta)\, \eta^{2r+2j-2i+2\alpha}\right|$$

$$\times \left|(x\eta)^{-(\alpha-\beta)-r-j-i}\, J_{\alpha-\beta+r+j-i}\, (x\eta)\right| dx$$

$$\le B_{\alpha,\beta}\, y^{2\alpha} \sum_{i=0}^{r} \sum_{i=0}^{r+j} \binom{r+j}{i} D'\, \eta^{2r+2j-2i+2\alpha}\, (1+\eta)^m \int_0^\infty x^{2(\alpha-\beta)+1+2j}\, (1+x)^{-q} dx$$

$$\le B_{\alpha,\beta}\, y^{2\alpha} \sum_{j=0}^{r} \sum_{i=0}^{r+j} \binom{r+j}{i} D'\, \eta^{2r+2j-2i+2\alpha}\, (1+\eta)^m$$

$$\times B\, \left(2(\alpha-\beta)+2j+2, q-2(\alpha-\beta)-2j-2\right)\, ;$$

for $q > 2(\alpha-\beta+j+1)$.

Therefore there exists a constant $A_{\alpha,\beta,m,r,q}$ such that, $|a_\eta(y)| \;\le\; A_{\alpha,\beta,m,r,q}(1+\eta)^{m+4r+2\alpha}(1+y)^{2\alpha}\, (1+y^{2r})^{-1}$, for every $r > 0$.

This completes the proof. □

Now we are ready to obtain an integral representation for the Pseudo-differential type operator $h_{\alpha,\beta,a}$ as the following theorem:

Theorem 4.2. *For any symbol $a(x,y) \in H^m$, the associated operator $h_{\alpha,\beta,a}$ can be represented by*

$$\left(h_{\alpha,\beta,a}\, u\right)(x) = \int_0^\infty (xy)^{\alpha+\beta}\, J_{\alpha-\beta}\, (xy) \left[\int_0^\infty a_\eta(y)\, U_{\alpha,\beta}(\eta)\right] dy, u \in H_{\alpha,\beta}\, (I)\, , \tag{4.2}$$

where $U_{\alpha,\beta}(\eta) = (h_{\alpha,\beta}\, u)\,(\eta)$ and all involved integrals are convergent.

Proof. We have,

$$a_\eta(y) = \int_0^\infty (xy)^{\alpha+\beta}\, J_{\alpha-\beta}\, (xy) \left[(x\eta)^{\alpha+\beta}\, J_{\alpha-\beta}\, (x\eta)\, a(x,\eta)\right]\, dx$$

Now by inversion, formula, we have,

$$\int_0^\infty a_\eta(y)\, (xy)^{\alpha+\beta}\, J_{\alpha-\beta}\, (xy)\, dy = (x\eta)^{\alpha+\beta}\, J_{\alpha-\beta}\, (x\eta)\, a(x,\eta)$$

Therefore,

$$\begin{aligned}(h_{\alpha,\beta,a}\, u)(x) &= \int_0^\infty (x\eta)^{\alpha+\beta}\, J_{\alpha-\beta}\,(x\eta)\, a(x,\eta)\, U_{\alpha,\beta}(\eta)\, dx \\ &= \int_0^\infty U_{\alpha,\beta}(\eta)\, d\,\eta \int_0^\infty a_\eta(y)\,(xy)^{\alpha+\beta}\, J_{\alpha-\beta}\,(xy)\, dy \\ &= \int_0^\infty (xy)^{\alpha+\beta}\, J_{\alpha-\beta}\,(xy)\, dy \int_0^\infty U_{\alpha,\beta}(\eta)\, a_\eta(y)\, d\,\eta\ .\end{aligned}$$

Using (4.1) for $a_\eta(y)$, the above change in the order of integration can be justified and the existence of the last integral can be proved.

Note also that as $U_{\alpha,\beta}(\eta) \in H_{\alpha,\beta}(I)$, we have,

$$|U_{\alpha,\beta}(\eta)| \;\le\; C\,\eta^{2\alpha}\,(1+\eta)^{-\ell}, \qquad \text{for all}\;\; \ell > 0\ .$$

Thus,

$$\begin{aligned}|(h_{\alpha,\beta,a}\, u)(x)| \;\le\; &\int_0^\infty \int_0^\infty (xy)^{2\alpha}\, \left|(xy)^{-(\alpha-\beta)}\, J_{\alpha-\beta}\,(xy)\right| \\ &\times A_{\alpha,\beta,m,r,q}\,(1+y)^{2\alpha}\,(1+y^{2r})^{-1} \times (1+\eta)^{m+4r+2\alpha}\, C\,\eta^{2\alpha}\,(1+\eta)^{-\ell}\, d\,\eta\, dy \\ &\le D'_{\alpha,\beta,m,r,q}\, x^{2\alpha} \int_0^\infty (1+y)^{2(\alpha-\beta)+1}\,(1+y^{2r})^{-1}\, dy \\ &\quad \times \int_0^\infty (1+\eta)^{2(\alpha-\beta)+m+4r-\ell+1}\, d\,\eta\end{aligned}$$

Since $(\alpha-\beta) \ge -\frac{1}{2}$, and ℓ and r can be chosen sufficiently large, the above integrals are convergent. This completes the proof. □

5 An L^1 - Norm Inequality

In this section we shall need the following

Definition 5.1. *(Sobolev type space) The space $G^s(\mathbb{R})$, $s \in \mathbb{R}$, is defined to be the set of all those elements $u \in H_{\alpha,\beta}(I)$, which satisfy,*

$$\|u\|_{G^s} = \left\|\eta^{s+2\beta-1}\, h_{\alpha,\beta}(u)\right\|_{L^1} \;<\; \infty\ . \tag{5.1}$$

Lemma 5.2. *For $(\alpha-\beta) \ge -\frac{1}{2}$ and $r \in \mathbb{N}_0$, there exists a constant $C_{r,m,q} > 0$ such that,*

$$|A_\eta(y)| \;\le\; C_{r,m,q}\,(1+\eta)^m\, y^{2\alpha}\,(1+y^{2r})^{-1}\ , \tag{5.2}$$

$$\textit{where,}\quad A_\eta(y) = h_{\alpha,\beta}\left(x^{2\alpha}\, a(x,\eta)(y)\right)\ . \tag{5.3}$$

Proof. Proceeding as in the proof of Lemma 4.1, we can obtain,

$$
\begin{aligned}
(-y^2)^r A_\eta(y) &= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, (\Delta_{\alpha,\beta})^r \left[x^{2\alpha}\, a(x,\eta)\right]\, dx \\
&= \int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy) \sum_{j=0}^{r} b_j\, x^{2j+2\alpha}\, (x^{-1} D_x)^{r+j}\, a(x,\eta)\, dx\, .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|(-y^2)^r A_\eta(y)\right| &\le \int_0^\infty \left|(xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\right| \sum_{j=0}^{r} |b_j|\, x^{2j+2\alpha} \\
&\times\ D_{r+j,m,q}\, (1+\eta)^m\, (1+x)^{-q}\, dx \\
&\le \int_0^\infty y^{2\alpha}\, x^{2\alpha} \left|(xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\right| \sum_{j=0}^{r} |b_j|\, x^{2j+2\alpha} \\
&\times\ D_{r+j,m,q}\, (1+\eta)^m\, (1+x)^{-q}\, dx \\
&\le \sum_{j=0}^{r} y^{2\alpha} B_{j,m,q}\, (1+\eta)^m \int_0^\infty (1+x)^{2(\alpha-\beta)+2j+1-q}\, dx\, .
\end{aligned}
$$

Choosing $q > 2(\alpha-\beta+r+1)$, we can obtain,

$$
|A_\eta(y)| \le C_{r,m,q}\, (1+\eta)^m\, (1+y^{2r})^{-1}\, y^{2\alpha}\, ,
$$

where $C_{r,m,q}$ is a positive constant. Thus the proof is complete. □

Now we prove our main theorem.

Theorem 5.3. *Let $(\alpha-\beta) \ge -\frac{1}{2}$. Then for all $m \in \mathbb{N}_0$, there exists $C > 0$ such that,*

$$
\|h_{\alpha,\beta,a}(u)\|_{G^0} \le C \sum_{\ell=0}^{m} \|u\|_{G^\ell}\ \ u \in H_{\alpha,\beta}(I)\, . \tag{5.4}
$$

Proof. From Theorem 4.2, equation (4.1) and the relation (2.10) we have,

$$
\int_0^\infty (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, (h_{\alpha,\beta,a}\, u)(x)\, dx = \int_0^\infty a_\eta(y)\, U_{\alpha,\beta}(\eta)\, d\,\eta
$$

Hence

$$
\int_0^\infty y^{2\beta-1} (xy)^{\alpha+\beta} J_{\alpha-\beta}(xy)\, (h_{\alpha,\beta,a}\, u)(x)\, dx =
$$

$$
A \int_0^\infty \eta^{2\alpha}\, U_{\alpha,\beta}(\eta)\, d\eta \int_0^\infty z^{2\alpha} D(\eta,y,z)\, dz \cdot \int_0^\infty x^{2\alpha}\, a(x,\eta)\, (zx)^{\alpha+\beta} J_{\alpha-\beta}(zx)\, dx\, , \tag{5.5}
$$

where, $A = \dfrac{1}{\left[2^{\alpha-\beta}\Gamma(\alpha-\beta+1)\right]^2}$

An application of the estimate (5.2) to (5.5) yields,

$$\left|y^{2\beta-1}\, h_{\alpha,\beta}\, (h_{\alpha,\beta,a}\, u)(y)\right|$$

$$\le\ C_{r,m,q} A \int_0^\infty (1+\eta)^m \eta^{2\alpha} U_{\alpha,\beta}(\eta) d\eta \cdot \int_0^\infty z^{2(\alpha-\beta)+1}(1+z^{2r})^{-1} D(\eta,y,z) dz \quad (5.6)$$

$$\le\ D_{\alpha,\beta,m,r,q} \sum_{\ell=0}^{m} \binom{m}{\ell} \int_0^\infty \eta^{l+2\alpha} U_{\alpha,\beta}(\eta) d\eta \cdot \int_0^\infty (1+z^{2r})^{-1} D(\eta,y,z) z^{2(\alpha-\beta)+1} 2^{-(\alpha-\beta)}$$

$$\times \left(\Gamma(\alpha-\beta+1)\right)^{-1}\ dz\ .$$

Set $f(z) = (1+z^{2r})^{-1} \in L^1(0,\infty)$; for $r > 0$, and

$g(\eta) = 2^{\alpha-\beta}\Gamma(\alpha-\beta+1)\eta^{\ell+2\beta-1} U_{\alpha,\beta}(\eta) \in L^1(0,\infty)$, for all $\ell = 0, 1, 2, \ldots\ldots m$

Thus according to (2.13) and (2.14) we obtain,

$$f(\eta,\ y) \quad = \quad \int_0^\infty f(z)\, D(\eta,y,z)\, z^{2(\alpha-\beta)+1}\, 2^{-(\alpha-\beta)}\, (\Gamma(\alpha-\beta+1)^{-1}\ dz$$

and

$$(f \# g)(y) \quad = \quad \int_0^\infty f(\eta,y)\, g(\eta)\, \eta^{2(\alpha-\beta)+1}\, 2^{-(\alpha-\beta)}\, (\Gamma(\alpha-\beta+1)^{-1}\ d\,\eta$$

Now if we apply (2.15) to (5.6), we get

$$\left\|y^{2\beta-1}\, h_{\alpha,\beta}\, (h_{\alpha,\beta,a}\, u)(y)\right\|_{L^1}$$

$$\le\ D_{\alpha,\beta,r,m,q} \sum_{\ell=0}^{m} \binom{m}{\ell} \left\|\, \eta^{\ell+2\beta-1}\, U_{\alpha,\beta}(\eta)\right\|_{L^1} \cdot \left\|\, (1+z^{2r})^{-1}\right\|_{L^1}$$

$$\le\ C \sum_{\ell=0}^{m} \binom{m}{\ell}\ \left\|\eta^{\ell+2\beta-1}\, h_{\alpha,\beta}\, u(\eta)\right\|_{L^1} \qquad (5.7)$$

Now inequality (5.4) follows from the inequality (5.7).
This completes the proof. □

Conclusions

1. If we take $\alpha = \frac{1}{4} + \frac{\mu}{2}$ and $\beta = \frac{1}{4} - \frac{\mu}{2}$ in (2.1), (2.2) and (2.3), we obtain respectively,
   $P_\mu = P_{\mu,x} = x^{\mu+\frac{1}{2}}\, D_x\, x^{-\mu-\frac{1}{2}}$,
   $Q_\mu = Q_{\mu,x} = x^{-\mu-\frac{1}{2}}\, D_x\, x^{\mu+\frac{1}{2}}$
   and
   $\Delta_\mu = \Delta_{\mu,x} = Q_\mu\, P_\mu = x^{-\mu-\frac{1}{2}}\, D_x\, x^{2\mu+1}\, x^{-\mu-\frac{1}{2}} = D_x^2 + \dfrac{(1-4\mu^2)}{x^2}$
   which are operators studied in Zeamnian [7] and later on in Pathak and Pandey [3] with P_μ, Q_μ, Δ_μ respectively replaced by N_μ, M_μ, S_μ.
2. Author claims that results developed in this paper are stronger than Pathak and Pandey [3].

Remark : *It is proposed to obtain more results on Pseudo-differential operators associated with Bessel type operators in future.*

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