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Regular Elements of Semigroups of Continuous Functions and Differentiable Functions

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Abstract : In 1974, Magill and Subbiah gave a characterization of the regular elements of $C(X)$, the semigroup of all continuous selfmaps of a topological space X. In this paper, their result is applied to determine the regular elements of $C(I)$ where I is an interval in $\mathbb{R}$, as follows : An element $f \in C(I)$ is regular if and only if $\operatorname{ran} f$ is a closed interval in I and there is a closed interval J in I such that $f_{|J}$ is a strictly monotone function from J onto $\operatorname{ran} f$. In addition, their proof is helpful to characterize the regular elements of $D(I)$ where $|I| > 1$ and $D(I)$ is the semigroup of all differentiable selfmaps of I. We show that for a nonconstant function $f \in D(I)$, f is regular if and only if f is a strictly monotone function from I onto itself and $f'(x) \neq 0$ for all $x \in I$.

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1 Introduction

For a set A, let $|A|$ and 1_A denote the cardinality of A and the identity map on A, respectively. If f is a function and A is a subset of the domain of f, we let $f_{|A}$ denote the restriction of f to A.

An element x of a semigroup S is called an *idempotent* of S if $x^2 = x$. A *regular element* of S is an element $x \in S$ such that $x = xyx$ for some $y \in S$. Following [3], let $E(S)$ and $\operatorname{Reg}(S)$ denote respectively the set of all idempotents and the set of all regular elements of S. If $\operatorname{Reg}(S) = S$, then S is called a *regular semigroup*. Note that $E(S) \subseteq \operatorname{Reg}(S)$ and if $x = xyx$, then $xy, yx \in E(S)$.

For a set X, let $T(X)$ denote the full transformation semigroup on X, that is, $T(X)$ is the semigroup, under composition, of all selfmaps of X. It is known that $T(X)$ is a regular semigroup ([3], page 4) and it is clearly seen that for $f \in T(X), f \in E(T(X))$ if and only if $f(x) = x$ for all $x \in \operatorname{ran} f$ where $\operatorname{ran} f$ is the range (image) of f. For $f, g \in T(X)$, if $f = fgf$, then $fg, gf \in E(T(X))$. Also, $\operatorname{ran} f = \operatorname{ran}(fg)$ since $\operatorname{ran} f = \operatorname{ran}(fgf) \subseteq \operatorname{ran}(fg) \subseteq \operatorname{ran} f$. Moreover,

$(fg)(f(x)) = f(x)$ and $(gf)(gf)(x) = (gf)(x)$ for all $x \in X$ which imply that $(fg)_{|\text{ran } f} = 1_{\text{ran } f}$ and $(gf)_{|\text{ran } (gf)} = 1_{\text{ran } (gf)}$, respectively.

For a topological space X, let $C(X)$ be the subsemigroup of $T(X)$ consisting of all continuous functions $f : X \to X$. A subset A of X is called a *retract* of X if $A = \text{ran } f$ for some $f \in E(C(X))$.

In 1974, Magill and Subbiah [6] characterized the regular elements of $C(X)$ as follows :

Theorem 1.1 ([6]) *Let X be a topological space and $f \in C(X)$. Then $f \in Reg\,(C(X))$ if and only if*

(i) ran f is a retract of X and

(ii) there is a retract A of X such that $f_{|A}$ is a homeomorphism from A onto ran f.

Recall that a *homeomorphism* from a topological space X onto a topological space Y is a bijection $f : X \to Y$ such that f and f^{-1} are continuous.

Next, let I be an interval in $\mathbb{R}$, the set of real numbers. By a *nontrivial interval* in $\mathbb{R}$ we mean an interval I in $\mathbb{R}$ with $|I| > 1$. Consider I as a metric space with the usual metric on $\mathbb{R}$. Then

$$C(I) = \Big\{ f : I \to I \mid f \text{ is continuous on } I \Big\}$$

and we have

$$D(I) = \Big\{ f : I \to I \mid f \text{ is differentiable on } I \Big\}$$

with $|I| > 1$ is a subsemigroup of $C(I)$. By an *interval* in I we mean a nonempty subset J of I having the property that for $x \in I$, $a \le x \le b$ for some $a, b \in J$ implies $x \in J$. Hence all intervals of I are precisely all intervals in $\mathbb{R}$ of the form $K \cap I$ where K is an interval of $\mathbb{R}$ with $K \cap I \neq \emptyset$. Also, by a *closed interval* in I we mean an interval in I which is a closed set in I. It follows as consequences of the main results in [2] that neither $C(I)$ nor $D(I)$ is a regular semigroup for every nontrivial interval I in $\mathbb{R}$.

In 1967, Magill [5] proved that every automorphism φ of $D(\mathbb{R})$ is inner, that is, there is a unit (invertible element) $g \in D(\mathbb{R})$ such that $\varphi(f) = gfg^{-1}$ for all $f \in D(\mathbb{R})$. The second author and Changphas [4] investigated the regularity of the subsemigroup $OT(I)$ of $T(I)$ consisting of all order-preserving functions $f : I \to I$. It was proved that $OT(I)$ is a regular semigroup if and only if I is closed and bounded.

Our first purpose is to determine the regular elements of $C(I)$ by Theorem 1.1. Also, the following basic results are recalled to be referred.

Proposition 1.2 ([1], page 177) *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$. If f is strictly increasing [decreasing] and continuous on I, then f^{-1} is strictly increasing [decreasing] and continuous on ran f.*

Proposition 1.3 ([1], page 179) *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$. If f is one-to-one and continuous on I, then f is strictly monotone on I.*

Our second purpose is to give necessary and sufficient conditions for elements of $D(I)$ to be regular. The proof of Theorem 1.1 given in [6] is useful for this result. Beside Proposition 1.2 and 1.3, the following basic results are also needed.

Proposition 1.4 ([1], page 198) *Let I be a nontrivial interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ strictly monotone on I. If f is differentiable on I and $f'(x) \neq 0$ for all $x \in I$, then f^{-1} is differentiable on* ran f *and*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

for all $x \in \operatorname{ran} f$.

Proposition 1.5 ([1], page 205) *Let I be a nontrivial interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ differentiable on I.*

(i) f is increasing [decreasing] on I if and only if $f'(x) \geq 0$ [$f'(x) \leq 0$] for all $x \in I$.

(ii) If $f'(x) > 0$ [$f'(x) < 0$] for all $x \in I$, then f is strictly increasing [strictly decreasing] on I.

Proposition 1.6 ([1], page 209-210) *Let I be a nontrivial interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$, $c \in I$ and assume that $f'(c)$ exists.*

(i) If $f'(c) > 0$, then there is a $\delta > 0$ such that $f(x) > f(c)$ for all $x \in I \cap (c, c+\delta)$ and $f(x) < f(c)$ for all $x \in I \cap (c-\delta, c)$.

(ii) If $f'(c) < 0$, then there is a $\delta > 0$ such that $f(x) < f(c)$ for all $x \in I \cap (c, c+\delta)$ and $f(x) > f(c)$ for all $x \in I \cap (c-\delta, c)$.

In the remainder, let I be an interval in $\mathbb{R}$.

2 The Semigroup $C(I)$

First, we provide the following two lemmas which will be used to obtain the main results.

Lemma 2.1 *If $f \in$ Reg$(C(I))$, then* ran f *is a closed interval in I.*

Proof. Let $g \in C(I)$ be such that $f = fgf$. Then $\operatorname{ran}(fg) = \operatorname{ran} f$ and $(fg)(x) = x$ for all $x \in \operatorname{ran} f$. Since f is continuous on I and $\operatorname{ran} f \subseteq I$, $\operatorname{ran} f$ is an interval in I. Let $x \in \overline{\operatorname{ran} f}$ where $\overline{\operatorname{ran} f}$ is the closure of $\operatorname{ran} f$ in I. Then there is a sequence (x_n) in $\operatorname{ran} f$ such that $\lim_{n\to\infty} x_n = x$. By the continuity of fg at x in I, $\lim_{n\to\infty} (fg)(x_n) = (fg)(x)$. But $(fg)(x_n) = x_n$ for every n, so $x = (fg)(x) \in \operatorname{ran}(fg) = \operatorname{ran} f$. □

Lemma 2.2 *For $A \subseteq I, A$ is a retract of I if and only if A is a closed interval in I.*

Proof. Since $E(C(I)) \subseteq$ Reg $(C(I))$, by Lemma 2.1, every retract of I is a closed interval in I.

Conversely, assume that A is a closed interval in I.

Case 1 : $A = (a, b)$ for some $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then $A = I = \operatorname{ran}(1_I)$ and $1_I \in E(C(I))$.

Case 2 : $A = [a, b)$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$. Then $I = [c, b)$ for some $c \in \mathbb{R}$ or $I = (c, b)$ for some $c \in \mathbb{R} \cup \{-\infty\}$. Then $f : I \to I$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ a & \text{if } x < a, \end{cases}$$

belongs to $E(C(I))$ whose range is A.

Case 3 : $A = (a, b]$ for some $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R}$. It can be shown similarly to Case 2 that $A = \operatorname{ran} f$ for some $f \in E(C(I))$.

Case 4 : $A = [a, b]$ for some $a, b \in \mathbb{R}$. Then $f : I \to I$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ a & \text{if } x < a, \\ b & \text{if } x > b, \end{cases}$$

is an element of $E(C(I))$ whose range is A. □

Now, we are ready to provide the first main result.

Theorem 2.3 *For $f \in C(I)$, $f \in$ Reg $(C(I))$ if and only if*

(i) $\operatorname{ran} f$ is a closed interval in I and

(ii) there is a closed interval J in I such that $f_{|_J}$ is a strictly monotone function from J onto $\operatorname{ran} f$.

Proof. Assume that $f \in$ Reg $(C(I))$. Then (i) holds by Lemma 2.1. By Theorem 1.1 and Lemma 2.2, there is a closed interval J in I such that $f_{|_J}$ is a homeomorphism from J onto $\operatorname{ran} f$. Then $f_{|_J}$ is one-to-one and continuous on J, so we have by Proposition 1.3 that $f_{|_J}$ is a strictly monotone function from J onto $\operatorname{ran} f$.

Conversely, assume that (i) and (ii) hold. By Lemma 2.2, $\operatorname{ran} f$ and J are retracts of I. From Proposition 1.2, $(f_{|_J})^{-1} :\ \operatorname{ran} f \to J$ is continuous. Hence $f_{|_J}$ is a homeomorphism from J onto $\operatorname{ran} f$. We therefore deduce from Theorem 1.1 that $f \in$ Reg $(C(I))$, as desired. □

As a consequence of Theorem 2.3, we have

Corollary 2.4 *If $|I| > 1$, then $C(I)$ is not a regular semigroup.*

Proof. Let $a, b, c, d \in I$ be such that $a < b < c < d$. Define $f : I \to I$ by

$$f(x) = \begin{cases} a & \text{if } x < a, \\ x & \text{if } x \in [a, b], \\ b & \text{if } x \in (b, c], \\ \frac{d-b}{d-c}(x - c) + b & \text{if } x \in (c, d], \\ d & \text{if } x > d. \end{cases}$$

Then $f \in C(I)$, $\operatorname{ran} f = [a, d]$ and f is increasing on I. It is clearly seen that there is no interval J in I such that $f_{|_J}$ is a strictly increasing function from J onto $\operatorname{ran} f$. Hence from Theorem 2.3, f is not regular in $C(I)$. $\square$

Example 2.5 Consider the following functions in $C(\mathbb{R})$:

$$f(x) = \sin x,\ g(x) = x^2,\ h(x) = e^x,\ k(x) = x \sin x.$$

Since $\operatorname{ran} f = [-1, 1]$ and $f_{|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}}$ is a strictly increasing function from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ onto $[-1, 1]$, by Theorem 2.3, $f \in \operatorname{Reg}(C(\mathbb{R}))$. Also, $g \in \operatorname{Reg}(C(\mathbb{R}))$ since $\operatorname{ran} g = [0, \infty)$ and $g_{|_{[0,\infty)}}$ is a strictly increasing function from $[0, \infty)$ onto $[0, \infty)$. Since $\operatorname{ran} h = (0, \infty)$ which is not closed in $\mathbb{R}$, by Theorem 2.3, $h \notin \operatorname{Reg}(C(\mathbb{R}))$.

By the definition of k, for every $n \in \mathbb{Z}$, $k(2n\pi) = 0$ and $k(2n\pi + \frac{\pi}{2}) = 2n\pi + \frac{\pi}{2}$ where $\mathbb{Z}$ is the set of integers. This implies that $\operatorname{ran} k = \mathbb{R}$ and for every $a \in \mathbb{R}$, $k_{|_{[a,\infty)}}$ and $k_{|_{(-\infty,a]}}$ are not one-to-one. Since $|k(x)| = |x \sin x| \leq |x|$ for every $x \in \mathbb{R}$, it follows that for any $a, b \in \mathbb{R}$ with $a < b$, $k([a, b]) \subseteq [-c, c] \neq \mathbb{R}$ where $c = \max\{|a|, |b|\}$. Consequently, there is no closed interval J in $\mathbb{R}$ such that $k_{|_J}$ is a strictly monotone function from J onto $\mathbb{R}$. Hence we have $k \notin \operatorname{Reg}(C(\mathbb{R}))$ by Theorem 2.3.

3 The Semigroup $D(I)$

Our purpose of this section is to prove the following theorem.

Theorem 3.1 *Assume that I is a nontrivial interval in $\mathbb{R}$. Then for a nonconstant function $f \in D(I)$, $f \in Reg(D(I))$ if and only if f is a strictly monotone function from I onto itself and $f'(x) \neq 0$ for all $x \in I$.*

Proof. Let $f \in D(I)$ be a nonconstant function and assume that $f \in \operatorname{Reg}(D(I))$. Then $|\operatorname{ran} f| > 1$. Let $g \in D(I)$ be such that $f = fgf$. Then $fg, gf \in E(D(I))$, $\operatorname{ran} f = \operatorname{ran}(fg)$, $(fg)_{|_{\operatorname{ran} f}} = 1_{\operatorname{ran} f}$ and $(gf)_{|_{\operatorname{ran}(gf)}} = 1_{\operatorname{ran}(gf)}$. Let $J = \operatorname{ran}(gf)$. From Lemma 2.1, J and $\operatorname{ran} f$ are closed intervals in I. Hence

$$\begin{aligned} (f_{|_J})(g_{|_{\operatorname{ran} f}}) &= (f_{|_{\operatorname{ran}(gf)}})(g_{|_{\operatorname{ran} f}}) = (fg)_{|_{\operatorname{ran} f}} = 1_{\operatorname{ran} f}, \\ (g_{|_{\operatorname{ran} f}})(f_{|_J}) &= (g_{|_{\operatorname{ran} f}})(f_{|_{\operatorname{ran}(gf)}}) = (gf)_{|_{\operatorname{ran}(gf)}} = 1_{|_J}. \end{aligned} \tag{1}$$

This implies that $f_{|_J}$ is a bijection from J onto $\operatorname{ran} f$ and $g_{|_{\operatorname{ran} f}} = (f_{|_J})^{-1}$. Since $|\operatorname{ran} f| > 1$, both J and $\operatorname{ran} f$ are nontrivial closed intervals in I. Then we deduce that $(g_{|_{\operatorname{ran} f}})'(x) = g'(x)$ for all $x \in \operatorname{ran} f$ and $(f_{|_J})'(x) = f'(x)$ for all $x \in J$. Therefore from (1), we have

$$g'(f(x))f'(x) = 1 \text{ for all } x \in J \text{ and } f'(g(x))g'(x) = 1 \text{ for all } x \in \operatorname{ran} f.$$

Hence

$$f'(x) \neq 0 \text{ for all } x \in J \text{ and } g'(x) \neq 0 \text{ for all } x \in \operatorname{ran} f. \tag{2}$$

From Proposition 1.3, $f_{|_J}$ is strictly monotone on J.

First, assume that $f_{|_J}$ is strictly increasing on J. But $g_{|_{\operatorname{ran} f}} = (f_{|_J})^{-1}$, so by Proposition 1.2, $g_{|_{\operatorname{ran} f}}$ is strictly increasing on $\operatorname{ran} f$. Proposition 1.5(i) and (2) imply that $f'(x) > 0$ for all $x \in J$ and $g'(x) > 0$ for all $x \in \operatorname{ran} f$. It remains to show that $J = I = \operatorname{ran} f$. First, suppose that $J \subsetneq I$. Then there is an element $c \in I$ such that $c > x$ for all $x \in J$ or $c < x$ for all $x \in J$.

Case 1 : $c > x$ for all $x \in J$. Since J is a closed interval in I, $\max(J)$ exists, say b. This implies that $f(b) = \max(\operatorname{ran} f)$ because $f_{|_J}$ is a strictly increasing function from J onto $\operatorname{ran} f$. But $c > b$ and $f'(b) > 0$, thus by Proposition 1.6(i), there is an element $x \in (b, c)$ such that $f(x) > f(b)$. This is a contradiction since $f(b) = \max(\operatorname{ran} f)$.

Case 2 : $c < x$ for all $x \in J$. We obtain a contradiction dually to Case 1.

Therefore we have that $J = I$. Thus $f_{|_J} = f$ and $g_{|_{\operatorname{ran} f}} = f^{-1}$ which is a strictly increasing function from $\operatorname{ran} f$ onto I. If we consider $\operatorname{ran} f$ and $g_{|_{\operatorname{ran} f}}$ replacing J and $f_{|_J}$, respectively, then we can obtain analogously that $\operatorname{ran} f = I$.

If $f_{|_J}$ is strictly decreasing on J, it can be proved similarly by Proposition 1.2, Proposition 1.5(i), (2) and Proposition 1.6(ii) that $J = I = \operatorname{ran} f$.

The converse follows directly from Proposition 1.4. □

Theorem 3.1, Proposition 1.4 and Proposition 1.5 yield the following result.

Corollary 3.2 *If f is a nonconstant function in $D(I)$, then the following statements are equivalent.*

(i) $f \in Reg\,(D(I))$.

(ii) f *is a unit of* $D(I)$.

(iii) $\operatorname{ran} f = I$ *and either* $f'(x) > 0$ *for all* $x \in I$ *or* $f'(x) < 0$ *for all* $x \in I$.

Corollary 3.3 *The semigroup $D(I)$ is not regular for every nontrivial interval I in $\mathbb{R}$.*

Proof. Let I be a nontrivial interval in $\mathbb{R}$ and let $a, b \in \mathbb{R}$ satisfy the following conditions : $a < b$, if I is bounded below, let $a = \inf(I)$ and if I is bounded above, let $b = \sup(I)$. Then the interval (a, b) is always contained in I. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{b-a}\big(x - \frac{a+b}{2}\big)^2 + \frac{a+b}{2} & \text{if } I \text{ is bounded below,} \\ \frac{1}{a-b}\big(x - \frac{a+b}{2}\big)^2 + \frac{a+b}{2} & \text{if } I \text{ is not bounded below.} \end{cases}$$

Then f is a parabola whose vertex is the point $\big(\dfrac{a+b}{2}, \dfrac{a+b}{2}\big)$. If I is bounded below, then $f(a) = f(b) = \dfrac{a+3b}{4} \in \big(\dfrac{a+b}{2}, b\big) \subseteq I$. Also, if I is not bounded below, then $f(a) = f(b) = \dfrac{3a+b}{4} \in \big(a, \dfrac{a+b}{2}\big) \subseteq I$. Consequently, $f_{|_I} \in D(I)$. Since $f'\big(\dfrac{a+b}{2}\big) = 0$, by Theorem 3.1, $f_{|_I}$ is not regular in $D(I)$. $\square$

Example 3.4 All the functions in Example 2.5 belong to $D(I)$ and it is clearly seen from Theorem 3.1 that none of them is regular in $D(I)$. Define

$$p(x) = \frac{1}{x} \quad \text{for all } x \in (0, \infty),$$
$$q(x) = x^3 \quad \text{and} \quad r(x) = x^3 + x \quad \text{for all } x \in \mathbb{R}.$$

Then $p'(x) = -\dfrac{1}{x^2} < 0$ for all $x \in (0, \infty)$, $q'(0) = 0$ and $r'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, $\operatorname{ran} p = (0, \infty)$ and $\operatorname{ran} r = \mathbb{R}$. Hence, by Corollary 3.2, $p \in \operatorname{Reg}(D((0, \infty)))$, $q \notin \operatorname{Reg}(D(\mathbb{R}))$ and $r \in \operatorname{Reg}(D(\mathbb{R}))$.

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