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Common Fixed Point Theorems in F-complete Topological Spaces

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Abstract : In this paper we partially answer three open problems, given two in Sastry and Srinivasa Rao [6] and one in Sastry and Srinivasa Rao [5].

Keywords : F-complete topological space, Common fixed point, Selfmap.
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1 Introduction

In 1994, Choudhury [1] introduced the notion of F-complete topological spaces as a generalization of d-complete topological spaces due to Hicks[4] and complete Menger spaces due to Schweizer and Sklar [7] as follows:

Definition 1.1. A function $F : R \to [0,\ 1]$ is said to be a distribution function if

(i) F is non-decreasing

(ii) F is left-continuous

(iii) $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$.

The set of all distribution functions is denoted by $\mathscr{D}$.

Definition 1.2. (Choudhury [1]) Let $(M,\ \mathcal{T})$ be a topological space and $F : M \times M \to \mathscr{D}$ be such that

(i) $F_{xy}(0) = 0$ for all $x,\ y \in M$

(ii) $F_{xy}(\epsilon) = 1$ for all $\epsilon > 0$ if and only if $x = y$.

(iii) If $\{x_i\}$ is a sequence of elements in M, $\{s_i\}$ is a sequence of non negative real numbers such that $\sum_{i=1}^{\infty} s_i < \infty$ and $\prod_{i=1}^{\infty} F_{x_i x_{i+1}}(s_i) > 0$, then $\{x_n\}$ is convergent in

$(M, \mathcal{T})$. Then the triplet $(M, \mathcal{T}, F)$ is called a F - complete topological space.

Choudhury (1994) [1] studied conditions for the existence and uniqueness of fixed points for selfmaps on F - complete topological spaces. A further study on the existence of common fixed points was done by Choudhury and Sarkar (1996) [3] and Choudhury (1997) [2]. Continuing the same vein, Sastry and Srinivasa Rao (2000) [6] proved fixed point theorems one is for single selfmap and the other for two selfmaps.

Theorem 1.3. (Sastry and Srinivasa Rao [6]) *Let $(M, \mathcal{T}, F)$ be a F- complete topological space. Let T be a selfmap on M such that the following are satisfied: there exists $0 < q < 1$ such that for $x, y \in M$, $x \neq y$*

$$F_{TxTy}(tq) \geq F_{xy}(t) \text{ for } t \geq 0 \tag{1.3.1}$$

there exists $x \in M$ such that

$$x \neq Tx \text{ and } F_{xTx}(t) = 1 \text{ for some } t > 0 \tag{1.3.2}$$

$$F_{xy} = F_{yx} \text{ for all } x, y \in M \tag{1.3.3}$$

$$F_{x_m y_n} \to F_{xy} \text{ whenever } x_m \to x \text{ and } y_n \to y \tag{1.3.4}$$

Then T has a unique fixed point.

Theorem 1.4. (Sastry and Srinivasa Rao [6]) *Let $(M, \mathcal{T}, F)$ be a F - complete topological space. Let S and T be two selfmaps on M such that the following are satisfied:*
there exists $0 < q < 1$ such that for $x, y \in M$, $x \neq y$

$$F_{SxTy}(tq) \geq F_{xy}(t) \text{ for } t \geq 0 \tag{1.4.1}$$

there exists $x_0 \in M$ such that $x_0 \neq Sx_0$, $F_{x_0 S_{x_0}}(t) = 1$ for some $t > 0$ and the sequence $\{x_n\}$ defined by $x_{2n+1} = Sx_{2n}$ for $= 0, 1, 2, \cdots$ and $x_{2n} = Tx_{2n-1}$ for $n = 1, 2, \cdots$ has the property that

$$x_n \neq x_{n+1} \text{ for } n = 0, 1, 2, \cdots \tag{1.4.2}$$

$$F_{xy} = F_{yx}, \text{ for all } x, y \in M \tag{1.4.3}$$

$$F_{x_m y_n} \to F_{xy}, \text{ whenever } x_m \to x \text{ and } y_n \to y. \tag{1.4.4}$$

Then S and T have a unique common fixed point. In fact, the sequence $\{x_n\}$ of (1.4.2) *is Cauchy and hence converges to a point, say, z and z is the unique common fixed point of S and T.*

Sastry and Srinivasa Rao [6] raised the following two open problems:

Open Problem 1. Is Theorem 1.3 true, if we replace (1.3.2) by

(1.3.2)′ there exists $x \in M$ such that $x \neq Tx$ and $\sup_{t>0} F_{xTx}(t) = 1$?

Open Problem 2. Is theorem 1.4 valid, if we replace in (1.4.2), '$F_{x_0 S_{x_0}}(t) = 1$' for some $t > 0$ by '$\sup_{t>0} F_{x_0 Sx_0}(t) = 1$'?

We answer these two open problems in Theorem 2.1 and Theorem 2.3 partially.

In 2000, Sastry and Srinivasa Rao [5] proved the following result.

Theorem 1.5. (Sastry and Srinivasa Rao [5], Corollary 2.4) *Let $(M,\ \mathcal{T},\ F)$ be a F - complete topological space satisfying:*

there exists $0 < a < 1$ such that for any $x,\ y \in M$ and $s > 0$,

$$(F_{SxTy}(s) + F_{xSx}(s))(F_{SxTy}(s) + F_{yTy}(s)) \geq 4max\{F_{xTx}(\tfrac{s}{a})F_{yTy}(s),\ F_{xTx}(s)F_{yTy}(\tfrac{s}{a})\} \tag{1.5.1}$$

there exists $x_0 \in M$, a sequence $\{\lambda_n\} \in (0,\ 1)$ and a sequence $\{t_n\}$ of positive numbers such that

$$\sum \lambda_n < \infty, \quad \sum_{n=0}^{\infty} a^n t_n < \infty \text{ and } F_{x_0Sx_0}(t_n) > 1 - \lambda_n \text{ for } n = 0,\ 1,\ 2,\ \cdots \tag{1.5.2}$$

$$F_{xy} = F_{yx} \text{ for all } x,\ y \in M \tag{1.5.3}$$

$$F_{x_my_n} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \text{ in } (M,\ \mathcal{T}) \tag{1.5.4}$$

$$F_{xy}(s) > 0 \text{ for all } s > 0 \text{ and for all } x,\ y \in M. \tag{1.5.5}$$

Then S and T have the same fixed point set which is a singleton.

Sastry and Srinivasa Rao [5] raised the following open problem:

Open Problem 3. Is the conclusion of Theorem 1.5 valid, if the 'max' on the right hand side of (1.5.1) is replaced by 'min'?

We answer this problem partially in Theorem 2.4.

2 Main Results

In this section, Theorem 2.1 and Theorem 2.3, partially answer the two open problems due to Sastry and Srinivasa Rao [6] by replacing the condition (1.3.2) by (2.1.2) and (1.4.2) by (2.3.2) in Theorem 2.1 and Theorem 2.3 respectively .

Theorem 2.1. *Let $(M,\ \mathcal{T},\ F)$ be a F - complete Hausdorff topological space. Let T be a selfmap on M such that the following are satisfied:*
there exists $0 < q < 1$ such that for any $x,\ y \in M,\quad x \neq y$

$$F_{TxTy}(tq) \geq F_{xy}(t), \text{ for } t \geq 0 \tag{2.1.1}$$

there exists $x_0 \in M$ and $k \in R$ such that $1 < k < \frac{1}{q}$ and

$$F_{x_0Tx_0}(k^n) > 1 - (kq)^n \tag{2.1.2}$$

$$F_{xy} = F_{yx} \text{ for all } x,\ y \in M \tag{2.1.3}$$

$$F_{x_my_n} \to F_{xy} \text{ whenever } x_m \to x \text{ and } y_n \to y. \tag{2.1.4}$$

Then T has a unique fixed point.

Proof. Let x_0 be as in (2.1.2). Define $\{x_n\}$ by $x_n = Tx_{n-1}$, $n = 1, 2, \cdots$.

$$F_{x_{n+1}x_n}(k^nq^n) = F_{Tx_nTx_{n-1}}(k^nq^n) \geq F_{x_nx_{n-1}}(k^nq^{n-1}) \geq \cdots \geq F_{x_0x_1}(k^n) > 1-(kq)^n.$$

Hence $\prod_{n=1}^{\infty} F_{x_{n+1}x_n}(k^nq^n) > 0$ (since $\sum(kq)^n < \infty$).

Therefore $\{x_n\}$ converges to z (say).

For any $s > 0$,

$F_{x_{n+1}Tz}(sq) = F_{Tx_nTz}(sq) \geq F_{x_nz}(s)$.

On letting $n \to \infty$, $F_{zTz}(sq) \geq F_{zz}(s) = 1$.

Therefore $Tz = z$.

Uniqueness of z follows from (2.1.1).

Theorem 2.2. *Let $(M, \mathcal{T}, F)$ be a F - complete topological space. Let S and T be two selfmaps on M satisfying:*
there exists $0 < q < 1$ such that for any $x, y \in M$ with $x \neq y$

$$F_{SxTy}(tq) \geq F_{xy}(q) \text{ for } t \geq 0 \tag{2.2.1}$$

If 'a' is a fixed point for T and 'z' is a fixed point for S, then $a = z$. Thus $F(T) \neq \phi$ and $F(S) \neq \phi$ implies $F(T) = F(S) =$ singleton.

Proof. Clearly the proof follows from (2.2.1).

Note: In view of Theorem 2.2, if $x_n = x_{n+1}$ for one odd 'n' and for one even 'n', then S and T have a unique common fixed point.

Theorem 2.3. *Let $(M, \mathcal{T}, F)$ be a F - complete Hausdorff topological space. Let S and T be two selfmaps on M such that the following are satisfying:*

there exists $0 < q < 1$ such that for any $x, y \in M$ with $x \neq y$

$$F_{SxTy}(tq) \geq F_{xy}(q) \text{ for } t \geq 0 \tag{2.3.1}$$

there exists $x_0 \in M$ and $k \in R$ such that $x_0 \neq Sx_0$ and $1 < k < \frac{1}{q}$ with $F_{x_0Sx_0}(k^n) > 1-(kq)^n$ and the sequence $\{F_{x_nx_{n+1}}(s)\}$ is increasing for $s > 0$ where $\{x_n\}$ is defined by $x_{2n+1} = Sx_{2n}$ for $n = 0, 1, 2, \cdots$ and $x_{2n} = Tx_{2n-1}$

$$\text{for } n = 1, 2, \cdots \tag{2.3.2}$$

$$F_{xy} = F_{yx} \text{ for all } x, y \in M \tag{2.3.3}$$

$$F_{x_my_n} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \tag{2.3.4}$$

then S and T have a unique common fixed point.

Proof. Let x_0 and $\{x_n\}$ be as in (2.3.2).

Case(i). $x_n \neq x_{n+1}$ for all n.
From (2.3.1)

$F_{x_{2n+1}x_{2n}}(k^{2n}q^{2n}) = F_{Sx_{2n}Tx_{2n-1}}(k^{2n}q^{2n}) \geq F_{x_{2n}x_{2n-1}}(k^{2n}q^{2n-1}) \geq \cdots \geq F_{x_0Sx_0}(k^{2n})$

Therefore it follows that, $F_{x_{n+1}x_n}(k^nq^n) \geq F_{x_0Sx_0}(k^n) > 1-(kq)^n$.

Hence $\prod_{n=1}^{\infty} F_{x_{n+1}x_n}((kq)^n) > 0$ (since $\sum_{n=1}^{\infty}(kq)^n < \infty$).

Therefore $\{x_n\}$ converges to z (say).

We now show that $x_n \neq z$ for large n.

If possible suppose that $x_n = z$ for infinitely many n.

Write $A = \{2n : x_{2n} = z\}$ and $B = \{2n+1 : x_{2n+1} = z\}$.

Suppose A is infinite. Then for $2n \in A$, we have

$F_{zx_{2n+1}}(s) \leq F_{SzTx_{2n+1}}(sq) = F_{Szx_{2n+2}}(sq)$.

On letting $n \to \infty$, $\quad F_{zz}(s) \leq F_{Szz}(sq)$ for $s > 0$.

Thus $sz = z$. Hence z is a fixed point for S.

Now, for $2n \in A$
$x_{2n} = z = Sz = Sx_{2n} = x_{2n+1}$.
Therefore $x_{2n} = x_{2n+1}$, a contradiction.

Hence, A is finite. Similarly B is finite.
Thus $x_n \neq z$ for large n.
Then, for large n,
$F_{x_{2n}z}(s) \leq F_{Sx_{2n}Tz}(sq) = F_{x_{2n+1}Tz}(sq)$.

On letting $n \to \infty$, $F_{zz}(s) \leq F_{zTz}(sq)$.

Therefore $F_{zTz}(sq) = 1$ for all $s > 0$.

Therefore $z = Tz$. Similarly we can show that $Sz = z$.

Case(ii): $x_n = x_{n+1}$ for some n.
Then $F_{x_nx_{n+1}}(s) = 1$.
Hence, $F_{x_mx_{m+1}}(s) = 1$ for all $m \geq n$ (by (2.3.2))
consequently, $x_m = x_{m+1}$ for all $m \geq n$.

Thus S and T have a unique common fixed point, which follows from the above Note.

In the following theorem we partially answer the Open Problem 3.

Theorem 2.4. *Let $(M,\ \mathcal{T},\ F)$ be a F - complete Hausdorff topological space and S and T be selfmaps on M satisfying:*

there exists $0 < a < 1$ *and for* $s > 0$

$(F_{SxTy}(s)+F_{xSx}(s))(F_{SxTy}(s)+F_{yTy}(s)) \geq 4min\ \{F_{xsx}(\frac{s}{a})F_{yTy}(s),$
$F_{xSx}(s)F_{yTy}(\frac{s}{a})\}$

(2.4.1)

$$\frac{F_{TxSTx}(\frac{s}{a})}{F_{xTx}(\frac{s}{a})} \geq \frac{F_{TxSTx}(s)}{F_{xTx}(s)} \text{ for all } s>0 \tag{2.4.2}$$

there exists $x_0 \in M$ such that sequence $\{\lambda_n\} \in (0,\ 1)$ and a sequence $\{t_n\}$ of positive numbers such that

$$\sum_{n=0}^{\infty} \lambda_n < \infty \text{ and } \sum_{n=0}^{\infty} a^n t_n < \infty \text{ and } F_{x_0Sx_0}(t_n) > 1-\lambda_n \tag{2.4.3}$$

$$F_{xy} = F_{yx} \text{ for all } x,\ y \in M \tag{2.4.4}$$

$$F_{x_m y_n} \to F_{xy} \text{ as } x_m \to x \text{ and } y_n \to y \tag{2.4.5}$$

Then S and T have the same fixed point set which is singleton.

Proof. Choose x_0 as in (2.4.3). Let $s>0$ and $x=x_{2n},\ y=x_{2n-1}$, from (2.4.1),

$$\begin{aligned}
&(F_{x_{2n+1}x_{2n}}(s)+F_{x_{2n}x_{2n+1}}(s))(F_{x_{2n+1}x_{2n}}(s)+F_{x_{2n-1}x_{2n}}(s))\\
&\qquad \geq 4min\ \{F_{x_{2n}x_{2n+1}}(\tfrac{s}{a})F_{x_{2n-1}x_{2n}}(s),\ F_{x_{2n}x_{2n+1}}(s)F_{x_{2n-1}x_{2n}}(\tfrac{s}{a})\}\\
&F_{x_{2n+1}x_{2n}}(s)(F_{x_{2n+1}x_{2n}}(s)+F_{x_{2n-1}x_{2n}}(s))\\
&\qquad \geq 2min\ \{F_{x_{2n}x_{2n+1}}(\tfrac{s}{a})F_{x_{2n-1}x_{2n}}(s),\ F_{x_{2n}x_{2n+1}}(s)F_{x_{2n-1}x_{2n}}(\tfrac{s}{a})\}\\
&\qquad = 2min\ \{F_{Tx_{2n-1}STx_{2n-1}}(\tfrac{s}{a})F_{x_{2n-1}Tx_{2n-1}}(s),\\
&\qquad\quad F_{Tx_{2n-1}STx_{2n-1}}(s)F_{x_{2n-1}Tx_{2n-1}}(\tfrac{s}{a})\}\\
&\qquad = 2F_{Tx_{2n-1}STx_{2n-1}}(s)F_{x_{2n-1}Tx_{2n-1}}(\tfrac{s}{a}) \text{ (by (2.4.2))}\\
&\qquad = F_{x_{2n}x_{2n+1}}(s)F_{x_{2n-1}x_{2n}}(\tfrac{s}{a})+F_{x_{2n}x_{2n+1}}(s)F_{x_{2n-1}x_{2n}}(\tfrac{s}{a})
\end{aligned}$$

Thus, $F_{x_{2n}x_{2n+1}}(s)F_{x_{2n}x_{2n+1}}(s)+F_{x_{2n}x_{2n+1}}(s)F_{x_{2n-1}x_{2n}}(s)$

$$\geq F_{x_{2n+1}x_{2n}}(s)F_{x_{2n-1}x_{2n}}(s)+F_{x_{2n}x_{2n+1}}(s)F_{x_{2n-1}x_{2n}}(\tfrac{s}{a}).$$

Therefore, $F_{x_{2n}x_{2n+1}}(s) \geq F_{x_{2n-1}x_{2n}}(\frac{s}{a})$.

Similarly by taking $x=x_{2n-2}$ and $y=x_{2n-1}$

$F_{x_{2n}x_{2n-1}}(\frac{s}{a}) \geq F_{x_{2n-1}x_{2n-2}}(\frac{s}{a^2})$.

Therefore $F_{x_{2n+1}x_{2n}}(sa^n) \geq F_{x_{2n}x_{2n-1}}(sa^{n-1}) \geq \cdots \geq F_{x_1x_0}(s)$.

Taking $s=t_n$,

$F_{x_{2n+1}x_{2n}}(a^nt_n) \geq F_{x_{2n}x_{2n-1}}(a^{n-1}t_n) \geq \cdots \geq F_{x_1x_0}(t_n) > 1-\lambda_n$

write $r_n = a^nt_n$ for $n=0,\ 1,\ \cdots$. Then

$$\prod_{n=0}^{\infty} F_{x_{n+1}x_n}(r_n) = F_{x_1x_0}(r_0)\prod_{n=1}^{\infty}(F_{x_{2n+1}x_{2n}}(r_{2n})F_{x_{2n}x_{2n-1}}(r_{2n+1}))$$

$$\geq (1-\lambda_0)\prod_{n=1}^{\infty}(1-\lambda_n)^2 > 0$$

$\sum\limits_{n=0}^{\infty} r_n < \infty$ (by 2.4.3). Therefore $\{x_n\}$ converges to z (say).

Taking $x = z$ and $y = x_{2n+1}$, from (2.4.1), $(F_{Szx_{2n+2}}(s) + F_{zSz}(s))(F_{Szx_{2n+2}}(s) + F_{x_{2n+1}x_{2n+2}}(s))$

$$\geq 4min\ \{F_{zSz}(\tfrac{s}{a})F_{x_{2n+1}x_{2n+2}}(s),\ F_{zSz}(s)F_{x_{2n+1}x_{2n+2}}(\tfrac{s}{a})\}.$$

On letting $n \to \infty$,

$(F_{Szz}(s) + F_{zSz}(s))(F_{Szz}(s) + F_{zz}(s)) \geq 4min\ \{F_{zSz}(\frac{s}{a})F_{zz}(s),\ F_{zSz}(s)F_{zz}(\frac{s}{a})\}$

$F_{zSz}(s)(F_{Szz}(s) + F_{zz}(s)) \geq 2F_{zSz}(s)F_{zz}(s)$.

Hence, $F_{Szz}(s) \geq F_{zz}(s) = 1$.

Therefore $Sz = z$.

Similarly z is also a fixed point for T.

The conclusion of the theorem follows from (2.4.1).

Note: If we replace (2.4.2) by

(2.4.2)$'$: $\frac{F_{SyTSy}(\frac{s}{a})}{F_{ySy}(\frac{s}{a})} \geq \frac{F_{SyTSy}(s)}{F_{ySy}(s)}$ for all $s > 0$

and $F_{x_0Sx_0}(t_n) > (1-\lambda_n)$ by $F_{y_0Ty_0}(t_n) > (1-\lambda_n)$, then also we can prove the existence of unique common fixed point by suitably modifying the definition of the sequence.

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