



On the Graph Relabeling Problem

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Abstract : We study the GRAPH RELABELING PROBLEM—given an undirected, connected, simple graph $G = (V, E)$, two labelings L and L' of the vertices of G , and a label *flip* operation that interchanges the labels on adjacent vertices, determine the complexity in terms of the number of flips of transforming L into L' . First we review the well-known classic case when G is a simple path. We then study the case when G is the star and define a parameter that explicitly measures the complexity of transforming one labeling into another. This value corresponds to computing the exact distance between two vertices in the corresponding *Cayley graph*. Lastly we explore relabelings with *privileged labels*, and provide a precise characterizations of when these problems are solvable. This work has applications in areas such as bioinformatics, networks, and VLSI.

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1 Introduction

Graph labeling is a well-studied subject in computer science and mathematics, and a problem that has widespread applications, including in many other disciplines. Here we explore a variant of graph labeling called the GRAPH RELABELING PROBLEM of transforming one given labeling into another by a sequence of label flips, where each flip interchanges the labels of two adjacent vertices. Some instances of this problem were explored by Kantabutra [22] and later by the authors of this paper in [2]. A shorter preliminary version of this paper appeared in [3]. Here we first review the classic case where the graph G is a simple path, and then extend the known results for the path for the case when G is the star (a tree with one nonleaf vertex). We define a parameter that measures how far one labeling of

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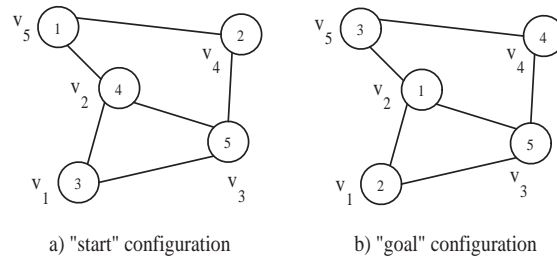


Figure 1: A label relocation problem instance.

the star is from the desired final labeling, and how this parameter alters with each flip. Finally, we provide precise characterizations of when instances of relabeling with privileged labels are solvable.

The problem of graph labeling has a rich and long history, and we recommend Gallian's extensive survey for an introduction to this topic and for a cataloging of the many different variants of labeling that have been studied [12]. Puzzles have always intrigued computer scientists and mathematicians alike, and a number of puzzles can be viewed as relabeled graphs (for example, see [47]). One of the most famous of these puzzles is the so-called 15-PUZZLE [40]. The 15-PUZZLE consists of 15 tiles numbered from 1 to 15 that are placed on a 4×4 board leaving one position empty. The goal is to reposition the tiles of an arbitrary arrangement into increasing order from left-to-right and from top-to-bottom by shifting tiles around while making use of the open hole. In [22] a generalized version of this puzzle called the $(n \times n)$ -PUZZLE was used to show a variant of the VERTEX RELABELING PROBLEM WITH PRIVILEGED LABELS is *NP*-complete. Other well-known problems, for example, the PANCAKE FLIPPING PROBLEM, can also be viewed as a special case of the graph relabeling problem [14].

Graph labeling has been studied in the context of cartography [21, 29]. And, of course, there are many special types of labelings which are of great interest—codings [32], colorings of planar graphs [5], and rankings [25] to name but three. In these cases we are typically interested in placing labels on the vertices or edges of a graph in some constrained manner so that certain properties are met. The GRAPH RELABELING PROBLEM is not only interesting in its own right but also has applications in several areas such as bioinformatics, networks, and VLSI. New applications for such work are constantly emerging, and sometimes in unexpected contexts. For instance, the GRAPH RELABELING PROBLEM can be used to model a *wormhole routing* in processor networks in which one-byte messages called *flits* [46] are sent among processors. In this example each processor has a limited buffer, one byte, and the only way to send a message is by exchanging it with another processor.

This paper is organized as follows: §2 contains definitions and some preliminary general results. We provide a recap of the case when G is a simple path. In §3

we define our main object, a parameter that measures how far a given labeling is from a final desired one, and explore how this parameter changes with each flip. In §4 we study the VERTEX RELABELING PROBLEM WITH PRIVILEGED LABELS and characterize exactly when this problem is solvable if all but two vertices are privileged. In the last section we present conclusions and open problems.

For background material on algorithms we refer the reader to [7], for graph theory to [1], and for basic notations of complexity theory including reducibility to [18].

2 Definitions, Preliminaries, and General Results

Throughout the paper $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of the natural numbers; $G = (V, E)$ a simple, undirected, and connected graph, with $V = V(G)$ as its set of vertices and $E = E(G)$ as its set of edges. We let $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$, where $|V| = n$ is the *order* and $|E| = m$ is the *size* of G . A *vertex labeling* L_V of V is a map $L_V : V \mapsto \{1, \dots, n\}$ and an *edge labeling* L_E of E a mapping $L_E : E \mapsto \{1, \dots, m\}$. A *vertex label flip* or just a *flip* (when no danger of ambiguity) is the interchange of labels on two adjacent vertices from $V(G)$. Similarly, an *edge label flip* is an interchange of labels of two adjacent edges (with a common endvertex).

Definition 2.1. (VERTEX RELABELING PROBLEM)

INSTANCE: A graph G , vertex labelings L_V and L'_V , and $t \in \mathbb{N}$.

QUESTION: Can labeling L_V be transformed into L'_V in t or fewer flips?

The EDGE RELABELING PROBLEM is defined analogously.

We now show that for an arbitrary graph and two arbitrary labelings at most $n(n-1)/2$ flips are required to transform one vertex labeling into another. We then discuss the case when G is the simple path which shows that this bound is the best possible among all simple graphs of order n .

Proposition 2.2. (VERTEX RELABELING UPPER BOUND)

For a graph $G = (V, E)$ of order n , we can always transform one vertex labeling L_V into another L'_V by $n(n-1)/2$ flips.

Proof. For a graph $G = (V, E)$, we need to consider the number of flips required to change an arbitrary labeling L_V into an arbitrary labeling L'_V .

We first construct a spanning tree T of G . Let v_1, v_2, \dots, v_n be the fixed indexing of the vertices (not labels) that denotes the *Prüfer code* order where the leaves of T are deleted during the process of constructing a Prüfer code; note, $v_j \in \{v_i \mid 1 \leq i \leq n\}$ for $1 \leq j \leq n$. The Prüfer code iteratively requires the lowest numbered vertex of degree one to be removed.¹ The idea is to transform

¹Any such ordering suffices, but we use a well-known fixed ordering (see [17] for more on the Prüfer code).

labels from L_V into their positions in L'_V in the order specified by the v_i 's and along the path in the spanning tree from their starting position in L_V to their final position in L'_V .

Let π be a permutation of $\{1, \dots, n\}$ (presented as π_1, \dots, π_n) such that $L_V(v_{\pi_i}) = L'_V(v_i)$ for each $i \in \{1, \dots, n\}$. To move $L_V(v_{\pi_1}) = L'_V(v_1)$ from the initial labeling to its final position can take at most $n - 1$ flips, since v_1 is an initial leaf in T , and T contains exactly $n - 1$ edges. To move $L_V(v_{\pi_2}) = L'_V(v_2)$ from the initial labeling to its final position, we need at most $n - 2$ flips, since $L'_V(v_1)$ is already in its rightful place. In general, after i iterations, where all of the labels $L'_V(v_1)$ through and including $L'_V(v_i)$ are in their correct places, then, to move $L_V(v_{\pi_{i+1}}) = L'_V(v_{i+1})$ to its correct place, we need at most $n - i - 1$ flips, since the remaining spanning tree induced by the vertex set, $V(T) - \{v_\ell \mid 1 \leq \ell \leq i\}$, has exactly $n - i - 1$ edges. Note, we do not perform any flips in locations of the tree that have already been completed.

All in all, we use at most $(n - 1) + (n - 2) + \dots + 1 = n(n - 1)/2$ flips to obtain L'_V from L_V . \square

Note that the proof of Proposition 2.2 is constructive and provides the sequence of flips to transform one labeling into another. The complexity of the algorithm in Proposition 2.2 is the complexity of computing a spanning tree, $\theta(n + m)$, plus the complexity of computing the Prüfer code elimination order, $\theta(n)$, plus the complexity of the flips, $\theta(n(n - 1)/2)$, which overall is therefore $\theta(n^2)$. It is interesting to consider that in the parallel setting we might be able to compute the sequence of flips required for the evolution much more quickly than we could actually execute them sequentially. We leave this as an open problem.

We now discuss the matching lower bound given in Proposition 2.2 and, for the sake of making the paper self-contained, review some well-known folklorish but relevant results for the case when G is the simple path P_n on n vertices. This work is a well-known and classic case, and most of the statements can be found in the original work by Thomas Muir [33], and the expanded and edited version [34]. But since the proofs are scattered throughout the literature, we will for the remainder of this section provide self-contained proofs of them in our notation. These methods for the path will then also be referred to in the case when G is the n -star in §3.

For convenience we represent a vertex labeling of P_n by a permutation π of $\{1, 2, \dots, n\}$ which we can view as a string $s = \pi_1\pi_2 \dots \pi_n$. For each such string s let $p(s)$ be the number of *inversions* (also known as *inversion pairs*) of s , that is, $p(s) = |\{\{i, j\} : 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|$. Note that each flip of P_n reduces or increases the value of $p(\cdot)$ by exactly one, so if s' is the string obtained from s by some flip, then $|p(s') - p(s)| = 1$. This well-known observation is stated as a lemma in the original treatise [33, p. 27] on determinants. From this we see that $p(s)$ is the number of flips necessary to obtain $\pi_1\pi_2 \dots \pi_n$ from $1\ 2 \dots n$ [34]. In particular, $p(1\ 2 \dots n) = 0$ and $p(n\ (n - 1) \dots 1) = \binom{n}{2} = n(n - 1)/2$, which shows that the bound in Proposition 2.2 is tight.

REMARKS: (i) When we view a labeling of the path P_n on n vertices as a string $s = \pi_1\pi_2 \dots \pi_n$, we note that the transformation of s to $1\ 2 \dots n$ strongly resembles

standard *bubble sort*—the simplest of the sorting algorithms on n elements (see [24, p. 108] for discussion and analysis). (ii) The parameter $p(s)$ can be viewed as a measure on how far a permutation as a string s is from $1\,2\,\dots\,n$. This idea is what we will mimic for the star in §3.

Consider the transformation of one labeling of the path P_n into another. It is clear that the minimum number of flips needed to transform $s = \pi_1 \pi_2 \dots \pi_n$ into $s' = 1\,2\,\dots\,n$ is the same as the minimum number of flips required to transform s' into s . Hence, for the sake of simplicity, we will assume that we are to transform s into s' . A *flip sequence* $(s_i)_{i=0}^m$ is a sequence of strings with $s_0 = s$, $s_m = s'$, and where s_{i+1} is obtained from s_i by a single flip, $0 \leq i \leq m-1$. In this case we see that for an arbitrary labeling $s = \pi_1 \pi_2 \dots \pi_n$, we have

$$p(s) = |p(s_0) - p(s_m)| = \left| \sum_{i=0}^{m-1} (p(s_i) - p(s_{i+1})) \right| \leq \sum_{i=0}^{m-1} |p(s_i) - p(s_{i+1})| = m, \quad (2.1)$$

reestablishing what we know that at least $p(s)$ flips are needed to transform s into s' .

By induction on n , it is easy to see that $p(s)$ flips suffice to transform s to s' : this claim is clearly true for $n = 2$.

Assume that this assertion is true for length $(n-1)$ -strings, and let $s = \pi_1 \pi_2 \dots \pi_n$ be such that $n = \pi_i$, for a fixed i , $1 \leq i \leq n$. In this case we have $p(s) = n - i + p(\hat{s})$, where $\hat{s} = \pi_1 \dots \pi_{i-1} \pi_{i+1} \dots \pi_n$. Clearly, in s we can move $n = \pi_i$ to the rightmost position by precisely $n - i$ flips. By induction, we can obtain $1\,2\,\dots\,(n-1)$ from \hat{s} by $p(\hat{s})$ flips. Hence, we are able to transform s into s' using $p(s)$ flips.

Finally, we note that if we have two vertex labelings L_V and L'_V of the vertices of the path P_n , we can define the corresponding *relative parameter* $p(L_V, L'_V)$ as $p(s)$, where s is the unique permutation obtained from L_V by renaming the labels in L'_V from left-to-right as $1, 2, \dots, n$ and reflecting these new names in L_V . By our previous comment, we have the symmetry $p(L_V, L'_V) = p(L'_V, L_V)$. This well-known result can now be stated in our notation as follows.

Observation 2.3. (TIGHT BOUND ON PATH RELABELING COMPLEXITY)

For that path P_n and vertex labelings L_V and L'_V , then L_V can be transformed into L'_V using t or fewer flips, if and only if $t \geq p(L_V, L'_V)$.

Finally, for the exact value of t in the above observation can be obtained as follows. By Observation 2.3 we can always transform L_V into L'_V using the minimum of $p(L_V, L'_V)$ flips, and repeating the last flip (or any fixed flip for that matter!) $2k$ times is not going to alter L'_V , since repeating a fixed flip an even number of times corresponds to the identity (or neutral) relabeling. Hence, for any nonnegative integer k one can always transform L_V into L'_V using $t = p(L_V, L'_V) + 2k$ flips.

We will now verify that if L_V can be transformed into L'_V in t flips, then they must have the same parity, that is $t - p(L_V, L'_V)$ must be even. – By renaming

the labels, we may assume L_V is given by the string $s = \pi_1 \pi_2 \dots \pi_n$ and L'_V by the string $s' = 1 2 \dots n$. Now let $(s_i)_{i=0}^m$ and $(s'_i)_{i=0}^{m'}$ be two flip sequences with $s_0 = s'_0 = s$ and $s_m = s'_{m'} = s'$. Since $p(s_0) = p(s'_0) = p(s)$ and $p(s_m) = p(s'_{m'}) = 0$, we have

$$p(s) = p(s_0) - p(s_m) = \sum_{i=0}^{m-1} (p(s_i) - p(s_{i+1})) = P_+ - P_-,$$

and

$$p(s) = p(s'_0) - p(s'_{m'}) = \sum_{i=0}^{m'-1} (p(s'_i) - p(s'_{i+1})) = P'_+ - P'_-,$$

where

$$\begin{aligned} P_+ &= |\{i \in \{0, \dots, m-1\} : p(s_i) - p(s_{i+1}) = 1\}|, \\ P_- &= |\{i \in \{0, \dots, m-1\} : p(s_i) - p(s_{i+1}) = -1\}|, \\ P'_+ &= |\{i \in \{0, \dots, m'-1\} : p(s'_i) - p(s'_{i+1}) = 1\}|, \text{ and} \\ P'_- &= |\{i \in \{0, \dots, m'-1\} : p(s'_i) - p(s'_{i+1}) = -1\}|. \end{aligned}$$

In particular, we have $P'_+ - P'_- = P_+ - P_-$. Since $m = P_+ + P_-$ and $m' = P'_+ + P'_-$, we obtain

$$m' - m = (P'_+ + P'_-) - (P_+ + P_-) = (P'_+ - P_+) + (P'_- - P_-) = 2(P'_+ - P_+), \quad (2.2)$$

and thus m and m' must have the same parity. This shows that if L_V is transformed into L'_V in exactly t flips, then $t - p(s)$ must be even. This proves the following well-known fact about permutations, which in our setting reads as follows.

Theorem 2.4 (Muir). *Let P_n be the path on n vertices and L_V and L'_V vertex labelings. Then we can transform the labeling L_V into L'_V using t flips, if and only if $t = p(L_V, L'_V) + 2k$ for some nonnegative integer k .*

REMARK: In many places in the literature (especially in books on abstract algebra), a permutation of $\{1, 2, \dots, n\}$ that swaps two elements $i \leftrightarrow j$ is called a *transposition* or a *2-cycle* and is denoted by (i, j) . If $i < j$, then a flip of P_n in our context is a transposition where $j = i + 1$. In general, by first moving j to the place of i and then moving i up to the place of j , we see that (i, j) can be obtained by exactly $2(j - i) - 1$ flips. Since every permutation π of $\{1, 2, \dots, n\}$ is a composition of transpositions, say t of them, then π can be obtained from $1 2 \dots n$ by N flips, where N is a sum of t odd numbers. By Theorem 2.4, we therefore have that $p(\pi) \equiv N \equiv t \pmod{2}$. This gives an alternative and more quantitative proof of the classic group-theoretic fact that the parity of the number of transpositions in a composition that yields a given permutation is unique and only depends on the permutation itself (see [20, p. 48] for the classic proof).

3 Exact Computations for the Star

In this section we discuss the case $G = K_{1,n-1}$, the star on n vertices, and we show that we can define an explicit parameter q that measures the distance between two vertex labelings of the star, analogous to the parameter p for the path P_n , the number of inversions of the corresponding permutation (see Definition 3.3).

This case of the star has also been investigated before in the literature, in particular, in [4] and from an algorithmic point of view in [30] and [31]; these references are all interesting papers on how this work applies to connectivity in computer networks. Here we will generalize these results and show how some of their results follow from ours as special cases.

Our main contribution in this section is Definition 3.3, the key definition of our parameter q , a function from the set of vertex labelings of the star $K_{1,n-1}$ to non-negative integers and Theorem 3.9 that shows how this parameter q alters with each flip.

For our general setup in this section, let $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(G) = \{\{v_0, v_i\} : i = 1, 2, \dots, n-1\}$, so we assume that v_0 is the center vertex of our star G . If L_V and L'_V are two vertex labelings of G , we may (by renaming the vertices) assume $L'_V(v_i) = i$ for each $i \in \{0, 1, \dots, n-1\}$. In this case the initial labeling is given by $L_V(v_i) = \pi(i)$, where π is a permutation of $\{0, 1, \dots, n-1\}$, and so $\pi \in S_n$, the *symmetric group* on n symbols $\{0, 1, \dots, n-1\}$ in our case here. Call the set of the elements moved by π the *support* of π , denote this set by $\text{Sp}(\pi)$, and let $|\text{Sp}(\pi)| = |\pi|$ be its cardinality. If π has the set S as its support, then we say that π is a permutation *on* S (as supposed to a permutation of S). Recall that a *cycle* $\sigma \in S_n$ is a permutation such that $\sigma(i_\ell) = i_{\ell+1}$ for all $\ell = 1, \dots, c-1$, and $\sigma(i_c) = i_1$, where $\text{Sp}(\sigma) = \{i_1, \dots, i_c\} \subseteq \{0, 1, \dots, n-1\}$ is the support of the cycle, so $|\sigma| = c$ here. Such a cycle σ is denoted by (i_1, \dots, i_c) . Each permutation $\pi \in S_n$ is a product of disjoint cycles $\pi = \sigma_1 \sigma_2 \cdots \sigma_k$ (see [20, p. 47]), and this product/composition is unique. (Note that every two disjoint cycles commute as compositions of maps $\{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, n-1\}$). For each permutation π , denote its number of disjoint cycles by $\zeta(\pi)$. Note that for the star G every flip has the form f_i , where f_i swaps the labels on v_0 and v_i for $i \in \{1, 2, \dots, n-1\}$. Hence, we have $f_i = (0, i)$, the 2-cycle transposing 0 and i .

Lemma 3.1. *Let $G = K_{1,n-1}$ be the star on n vertices. Let L_V and L'_V be vertex labelings such that $L_V(v_i) = \sigma(i)$ and $L'_V(v_i) = i$, where σ is a cycle with $\text{Sp}(\sigma) \subseteq \{1, 2, \dots, n-1\}$. In this case the labeling L_V can be transformed into L'_V in $|\sigma| + 1$ flips.*

Proof. If $\sigma = (i_1, \dots, i_c)$, where $\{i_1, \dots, i_c\} \subseteq \{1, 2, \dots, n-1\}$, then apply the composition $f_\sigma := f_{i_1} f_{i_2} \cdots f_{i_c} f_{i_1}$ to the labeling L_V and obtain L'_V since

$$f_{i_1} f_{i_2} \cdots f_{i_c} f_{i_1} \sigma = (0, i_1)(0, i_2) \cdots (0, i_c)(0, i_1)(i_1, \dots, i_c)$$

is the identity permutation. Since f_σ consists of $c + 1$ flips altogether, we have the lemma. \square

For a cycle σ with $\text{Sp}(\sigma) \subseteq \{1, 2, \dots, n-1\}$, let f_σ denote the composition of the $|\sigma| + 1$ label flip functions as in the previous proof. By Lemma 3.1 we have the following corollary.

Corollary 3.2. *Let $G = K_{1, n-1}$ be the star on n vertices. Let L_V and L'_V be vertex labelings such that $L_V(v_i) = \pi(i)$ and $L'_V(v_i) = i$ for $i \in \{0, 1, \dots, n-1\}$ where $\pi(0) = 0$. In this case the labeling L_V can be transformed into L'_V in $|\pi| + \zeta(\pi)$ flips.*

Proof. If $\pi = \sigma_1 \cdots \sigma_k$, a product of k disjoint cycles each having its support in $\{1, 2, \dots, n-1\}$, then apply the composition $f_{\sigma_k} f_{\sigma_{k-1}} \cdots f_{\sigma_1}$ to the labeling L_V and obtain L'_V . This composition consists of $\sum_{i=1}^k (|\sigma_i| + 1) = |\pi| + k = |\pi| + \zeta(\pi)$ flips altogether. \square

Corollary 3.2 establishes an upper bound on how many flips are needed to transform one labeling into another. This upper bound is the easier part and coincides with [4, Lemma 1, p. 561].

To analyze and obtain the tight lower bound, we will define a parameter $q(\cdot)$, a function from the set of all possible labelings of G into the set of nonnegative integers, such that each flip either reduces or increases the parameter by exactly one, just like the number $p(\cdot)$ of inversions of a permutation on the path. Before we present the formal definition of the parameter q , we need some notation. For each permutation π on $\{0, 1, \dots, n-1\}$, we define a corresponding permutation π^0 on the same set in the following way:

1. If $\pi(0) = 0$, then $\pi^0 := \pi$.
2. If $\pi(0) = i \neq 0$, then let $j \in \{1, 2, \dots, n-1\}$ be the unique element with $\pi(j) = 0$. In this case we let $\pi^0 := \pi(0, j)$.

Note that for any permutation π on $\{0, 1, \dots, n-1\}$ we always have $\pi^0(0) = 0$. If L_V is a vertex labeling of the star G such that $L_V(v_i) = \pi(i)$ for each $i \in \{0, 1, \dots, n-1\}$, then let L_V^0 be the vertex labeling corresponding to the permutation π^0 , so $L_V^0(v_i) = \pi^0(i)$ for each $i \in \{0, 1, \dots, n-1\}$. With this notation we can now define our parameter — the main object of this section.

Definition 3.3. *Let $L_V : V(G) \rightarrow \{0, 1, \dots, n-1\}$ be a vertex labeling of the star $G = K_{1, n-1}$ given by $L_V(v_i) = \pi(i)$, where π is some permutation of $\{0, 1, \dots, n-1\}$.*

1. If $\pi(0) = 0$, then let $q(L_V) = |\pi| + \zeta(\pi)$.
2. Otherwise, if $\pi(0) = i \neq 0$ and hence $\pi(j) = 0$ for some j , then let

$$q(L_V) = \begin{cases} q(L_V^0) + 1 & \text{if } i = j, \\ q(L_V^0) - 1 & \text{if } i \neq j. \end{cases}$$

Note that $L_V(v_i) = i$ for each $i \in \{0, 1, \dots, n-1\}$ if and only if $q(L_V) = 0$.

We now want to show that if L_V is a vertex labeling of the star G , and L'_V is obtained from L_V by a single flip, then $|q(L_V) - q(L'_V)| = 1$. First we note that if one of the labels swapped by the single flip is zero, then we either have $L'_V = L_V^0$ or vice versa $L_V = L_V^0$. Hence, in this case we have directly by Definition 3.3 that $|q(L_V) - q(L'_V)| = 1$.

Assume now that neither labels i nor j swapped by the flip is zero. In this case we have $L_V(v_0) = i$ and $L'_V(v_0) = j$, and hence $L_V(v_\ell) = L'_V(v_\ell) = 0$ for some $\ell \in \{1, 2, \dots, n-1\}$. Let the labelings L_V and L'_V on $\{0, 1, \dots, n-1\}$ be given by the permutations π and π' , respectively. Since $\pi(k) = j$ and $\pi'(k) = i$ for some $k \neq \ell$ and $\pi'(\ell) = \pi(\ell) = 0$, we have $\pi' = \pi(0, k)$. Using the notation introduced earlier, we have $\pi^0 = \pi(0, \ell)$ and $\pi'^0 = \pi'(0, \ell)$. Since $\pi = \pi(0, \ell)(0, \ell) = \pi^0(0, \ell)$ and $(0, \ell)(0, k)(0, \ell) = (k, \ell)$, we have

$$\pi'^0 = \pi'(0, \ell) = \pi(0, k)(0, \ell) = \pi^0(0, \ell)(0, k)(0, \ell) = \pi^0(k, \ell). \quad (3.1)$$

Note that (3.1) also implies that $\pi^0(k, \ell) = \pi^0$, and so this observation yields a symmetry $\pi^0 \leftrightarrow \pi'^0$ that we will use later. Also, since $\pi^0(k) = \pi(k) = j$, $\pi^0(\ell) = \pi(0) = i$, $\pi'^0(k) = \pi'(k) = i$, and $\pi'^0(\ell) = \pi'(0) = j$, we see that the labeling L_V^0 is obtained from L_V by swapping the labels i on v_k and j on v_ℓ .

By Definition 3.3 we have $q(L_V^0) = |\pi^0| + \varsigma(\pi^0)$, and further by (3.1) we get the following:

$$q(L_V^0) = |\pi'^0| + \varsigma(\pi'^0) = |\pi^0(k, \ell)| + \varsigma(\pi^0(k, \ell)). \quad (3.2)$$

Note that what happens with the parameter q depends on whether $\ell \in \{i, j\}$ or not. Before we consider these cases, we dispatch with some basic but relevant observations on permutations.

Claim 3.4. *Let σ_1 and σ_2 be two disjoint cycles. If $i_1 \in \text{Sp}(\sigma_1)$ and $i_2 \in \text{Sp}(\sigma_2)$, then $\sigma_1\sigma_2(i_1, i_2)$ is a cycle on $\text{Sp}(\sigma_1) \cup \text{Sp}(\sigma_2)$.*

Proof. Let $\sigma_1 = (a_1, \dots, a_h)$ and $\sigma_2 = (b_1, \dots, b_k)$, where $h, k \geq 2$. We may assume that $i_1 = a_1$ and $i_2 = b_1$. In this case we have

$$\sigma_1\sigma_2(i_1, i_2) = (a_1, \dots, a_h)(b_1, \dots, b_k)(a_1, b_1) = (a_1, b_2, \dots, b_k, b_1, a_2, \dots, a_h).$$

□

Claim 3.5. *Let σ be a cycle and $i_1, i_2 \in \text{Sp}(\sigma)$ be distinct. Then one of the following holds for $\sigma(i_1, i_2)$:*

1. $\text{Sp}(\sigma(i_1, i_2)) = \text{Sp}(\sigma)$ and $\sigma(i_1, i_2) = \sigma_1\sigma_2$ —a product of disjoint cycles with $\text{Sp}(\sigma_1) \cup \text{Sp}(\sigma_2) = \text{Sp}(\sigma)$.
2. $\text{Sp}(\sigma(i_1, i_2)) = \text{Sp}(\sigma) \setminus \{i^*\}$, where $i^* \in \{i_1, i_2\}$ and $\sigma(i_1, i_2)$ is a cycle on $\text{Sp}(\sigma) \setminus \{i^*\}$.

3. $\text{Sp}(\sigma(i_1, i_2)) = \emptyset$ and $\sigma = (i_1, i_2)$.

Proof. Let $\sigma = (a_1, \dots, a_h)$, where $h \geq 2$. We may assume $(i_1, i_2) = (a_1, a_i)$ for some $i \in \{2, \dots, h\}$. We now consider the following cases for h and i :

If $h = 2$, then $i = 2$ and $\sigma = (a_1, a_2) = (i_1, i_2)$, and we have part 3.

If $h \geq 3$ and $i = 2$, then $\sigma(i_1, i_2) = (a_1, \dots, a_h)(a_1, a_2) = (a_1, a_3, \dots, a_h)$, and we have part 2.

If $h \geq 3$ and $i = h$, then $\sigma(i_1, i_2) = (a_1, \dots, a_h)(a_1, a_h) = (a_2, a_3, \dots, a_h)$, and again we have part 2.

Finally, if $h \geq 3$ and $i \notin \{2, h\}$, then $i \in \{3, \dots, h-1\}$ (and hence $h \geq 4$), and $\sigma(i_1, i_2) = (a_1, \dots, a_h)(a_1, a_i) = (a_1, a_{i+1}, \dots, a_h)(a_2, \dots, a_i)$, and we have part 1. \square

We are now ready to consider the cases of whether $\ell \in \{i, j\}$ or not.

FIRST CASE: $\ell \notin \{i, j\}$. Directly by definition we have here that $q(L_V) = q(L_V^0) - 1$ and $q(L'_V) = q(L_V^0) - 1$, and hence $q(L_V) - q(L'_V) = q(L_V^0) - q(L_V^0)$.

Proposition 3.6. *If $\ell \notin \{i, j\}$, then $q(L_V) - q(L'_V) = q(L_V^0) - q(L_V^0) = \pm 1$.*

Proof. Assuming $\ell \notin \{i, j\}$, we have $\pi^0(\ell) = i$ and $\pi'^0(\ell) = j$, and hence ℓ is in the support of both π^0 and π'^0 .

If $k \notin \{i, j\}$, then $\{k, \ell\}$ is contained in both $\text{Sp}(\pi^0)$ and $\text{Sp}(\pi'^0)$, and hence by definition we have that $|\pi^0(k, \ell)| = |\pi^0|$. Since π^0 is a product of disjoint cycles, then, either (i) there are two cycles σ_1 and σ_2 of π^0 such that $k \in \text{Sp}(\sigma_1)$ and $\ell \in \text{Sp}(\sigma_2)$, or (ii) there is one cycle σ of π^0 such that $\{k, \ell\} \subseteq \text{Sp}(\sigma)$. Since the cycles of π commute, we have by Claim 3.4 in case (i) that $\zeta(\pi^0(k, \ell)) = \zeta(\pi^0) - 1$, and by Claim 3.5 in case (ii) part 1 that $\zeta(\pi^0(k, \ell)) = \zeta(\pi^0) + 1$. By (3.2) this completes the argument when $k \notin \{i, j\}$.

If $k \in \{i, j\}$, we may by symmetry ($\pi^0 \leftrightarrow \pi'^0$) assume that $k = i \neq j$. In this case we have $\pi^0(k) = j$ so $k \in \text{Sp}(\pi^0)$, and $\pi'^0(k) = i$ so $k \notin \text{Sp}(\pi'^0)$. Hence, we have $|\pi^0(k, \ell)| = |\pi^0| - 1$. Since $\pi^0(\ell) = i = k$, we see that both k and ℓ are contained in the same cycle σ of π^0 in its disjoint cycle decomposition, and they are consecutive. Moreover, since $\pi^0(k) = j \neq i$, we see that $|\sigma| \geq 3$. Again, since disjoint cycles commute, we have by Claim 3.5 part 2 that $\zeta(\pi^0(k, \ell)) = \zeta(\pi^0)$. By (3.2) this fact completes the argument when $k = i$, and hence the proof of the proposition. \square

SECOND CASE: $\ell \in \{i, j\}$. By symmetry we may assume $\ell = i$. In this case we have directly by definition that $q(L_V) = q(L_V^0) + 1$ and $q(L'_V) = q(L_V^0) - 1$. Before continuing we need one more basic observation about permutations.

Claim 3.7. *Let σ be a cycle. If $i_1 \in \text{Sp}(\sigma)$ and $i_2 \notin \text{Sp}(\sigma)$, then $\sigma(i_1, i_2)$ is a cycle on $\text{Sp}(\sigma) \cup \{i_2\}$.*

Proof. Let $\sigma = (a_1, \dots, a_h)$. We may assume $(i_1, i_2) = (a_1, b)$, where $b \notin \{a_1, \dots, a_h\}$, and so we get $\sigma(i_1, i_2) = (a_1, \dots, a_h)(a_1, b) = (a_1, b, a_2, \dots, a_h)$. \square

Proposition 3.8. *If $\ell = i$, then $q(L_V) - q(L'_V) = q(L_V^0) - q(L'^0_V) + 2 = \pm 1$.*

Proof. Assuming $\ell = i$, we have $\pi^0(\ell) = i$ and $\pi'^0(\ell) = j$, and hence $\ell \in \text{Sp}(\pi^0) \setminus \text{Sp}(\pi'^0)$.

If $k \in \{i, j\}$, then since $k \neq \ell$, we have $k = j$. Also, since $\pi^0(k) = j$ and $\pi'^0(k) = i$, we have $k \in \text{Sp}(\pi'^0) \setminus \text{Sp}(\pi^0)$. Since π^0 and π'^0 only differ on k and ℓ , we have $\text{Sp}(\pi'^0) = \text{Sp}(\pi^0) \cup \{k, \ell\}$, this union being disjoint. From this fact it is immediate that $|\pi'^0| = |\pi^0(k, \ell)| = |\pi^0| + 2$ and $\zeta(\pi'^0(k, \ell)) = \zeta(\pi^0) + 1$, and hence by (3.2), we have the following:

$$q(L'^0_V) = |\pi^0(k, \ell)| + \zeta(\pi^0(k, \ell)) = |\pi^0| + \zeta(\pi^0) + 3 = q(L_V^0) + 3,$$

and hence $q(L_V^0) - q(L'^0_V) + 2 = q(L_V^0) - (q(L_V^0) + 3) + 2 = -1$, which completes the argument when $k \in \{i, j\}$.

If $k \notin \{i, j\}$, then since $\pi^0(k) = j$ and $\pi'^0(k) = i$, we have that k is contained in both $\text{Sp}(\pi^0)$ and $\text{Sp}(\pi'^0)$, and therefore $|\pi^0(k, \ell)| = |\pi^0| + 1$. Since π^0 is a product of disjoint cycles, there is a unique cycle σ of π^0 whose support contains k . By Claim 3.7, $\sigma(k, \ell)$ is a cycle on $\text{Sp}(\sigma) \cup \{\ell\}$, and hence $\zeta(\pi^0(k, \ell)) = \zeta(\pi^0)$. By (3.2) we therefore have

$$q(L'^0_V) = |\pi^0(k, \ell)| + \zeta(\pi^0(k, \ell)) = |\pi^0| + \zeta(\pi^0) + 1 = q(L_V^0) + 1,$$

and hence $q(L_V^0) - q(L'^0_V) + 2 = q(L_V^0) - (q(L_V^0) + 1) + 2 = 1$, which completes the argument when $k \notin \{i, j\}$. This result completes the proof. \square

We now have our main theorem of this section.

Theorem 3.9. *Let $G = K_{1, n-1}$ be the star on n vertices. If L_V is a vertex labeling of G and L'_V is a vertex labeling obtained from L_V by a single flip, then $|q(L_V) - q(L'_V)| = 1$.*

Theorem 3.9 shows in particular that the upper bound given in Corollary 3.2 is also a lower bound. We summarize these results in the following.

Proposition 3.10. *Let $G = K_{1, n-1}$ be the star on n vertices. Let L_V and L'_V be vertex labelings such that $L'_V(v_i) = i$ and $L_V(v_i) = \pi(i)$ for $i \in \{0, 1, \dots, n-1\}$, where $\pi(0) = 0$. In this case the labeling L_V can be transformed into L'_V in t flips if and only if $t \geq |\pi| + \zeta(\pi)$.*

In Proposition 3.10 we restricted to labelings L_V and L'_V with $L_V(v_0) = L'_V(v_0) = 0$, which by Definition 3.3 is the fundamental case for defining the parameter $q(L_V)$. Just as we summarized for the case of the path $G = P_n$ in the beginning of this section, we can likewise define the *relative star parameter* $q(L_V, L'_V)$ for any two vertex labelings L_V and L'_V of the n -star $G = K_{1, n-1}$ to be $q(L''_V)$, where L''_V is the unique vertex labeling obtained from L_V by renaming the labels of L'_V so that $L''_V(v_i) = i$ for all i . (Strictly speaking, if L_V and L'_V

are given by permutations π and π' of $\{0, 1, \dots, n-1\}$, then L_V'' is given by the permutation $\pi'' = \pi(\pi'^{-1})$. Clearly, this relative parameter q is symmetric, $q(L_V, L_V') = q(L_V', L_V)$, as was the case for the path.

As with the path P_n , where the parameter $p(\cdot)$ increased or decreased by exactly one with each flip, by Theorem 3.9, so does $q(\cdot)$ for the star $G = K_{1, n-1}$. Hence, exactly the same arguments used for (2.1) and (2.2) can be used to obtain the following corollary.

Corollary 3.11. *Let $G = K_{1, n-1}$ be the star on n vertices, L_V and L_V' vertex labelings, and $t \in \mathbb{N}$. Then we can transform the labeling L_V into L_V' using t flips if and only if $t = q(L_V, L_V') + 2k$ for some nonnegative integer k , where q is the relative parameter corresponding to the one in Definition 3.3.*

Corollary 3.11 generalizes the results both from [4] and [31].

Consider the graph C where its vertex set $V(C)$ consists of all the $n!$ vertex labelings of the star $G = K_{1, n-1}$, so each vertex v_π of C corresponds to a permutation $\pi \in S_n$, and where two vertices v_π and $v_{\pi'}$ are connected in C if and only if $\pi' = \pi(0, i)$ for some $i \in \{1, 2, \dots, n-1\}$. Here $C = (V(C), E(C))$ is an example of a *Cayley graph*, and this particular one is sometimes ambiguously also referred to as the *star graph* in the literature [4, p. 561], [30], and [31, p. 374]. In terms of Cayley graphs, we can interpret Corollary 3.11 as follows:

Corollary 3.12. *Let C be the Cayley graph of the n -star $G = K_{1, n-1}$. For any $\pi, \pi' \in S_n$, let $v_\pi, v_{\pi'} \in V(C)$ be the corresponding vertices of C , and L_V and L_V' the corresponding vertex labelings of G . Then the following holds:*

1. *The distance between v_π and $v_{\pi'}$ in C is precisely $q(L_V, L_V')$.*
2. *There is a walk between v_π and $v_{\pi'}$ in C of length d if and only if $d = q(L_V, L_V') + 2k$ for some nonnegative integer k .*

Other related results regarding the Cayley graph of the star can be found in [45] where the distance distribution among the vertices of the star graph is computed, and in [35] where the cycle structure of the Cayley graph of the star is investigated.

Let $n \in \mathbb{N}$ be given. Among all permutations π on $\{0, 1, \dots, n-1\}$ with $\pi(0) = 0$, clearly a maximum value of $|\pi|$ is $n-1$, obtained when $\text{Sp}(\pi) = \{1, 2, \dots, n-1\}$. Also, the maximum value of $\zeta(\pi)$ is $\lfloor (n-1)/2 \rfloor$, obtained when every cycle of π has support of two when $n-1$ is even, or when every cycle except one (with support of three) has support of two when $n-1$ is odd. Hence, among all permutations π on $\{0, 1, \dots, n-1\}$, the maximum value of $|\pi^0| + \zeta(\pi^0)$ is always $n-1 + \lfloor (n-1)/2 \rfloor = \lfloor 3(n-1)/2 \rfloor$.

Consider the star $G = K_{1, n-1}$ and a vertex labeling L_V of G with $q(L_V)$ at maximum. Let π be the permutation on $\{0, 1, \dots, n-1\}$ corresponding to L_V , so $L_V(v_i) = \pi(i)$. If $\pi(0) = 0$, then $\pi = \pi^0$ and by Definition 3.3, the value $q(L_V)$ is at most $\lfloor 3(n-1)/2 \rfloor$. Assume now that $\pi(0) = i \neq 0$, and hence $\pi(j) = 0$ for some j . If $i \neq j$, then by Definition 3.3 and previous remarks

$q(L_V) = q(L_V^0) - 1 \leq \lfloor 3(n-1)/2 \rfloor - 1$. Finally if $i = j$, then $q(L_V) = q(L_V^0) + 1$. Since $\pi^0 = \pi(0, j)$, we obtain in this case that $\pi^0(i) = [\pi(0, j)](i) = [\pi(0, i)](i) = i$, and hence $i \notin \text{Sp}(\pi^0)$. Therefore, $|\pi^0| \leq n - 2$ and $\zeta(\pi^0) \leq \lfloor 3/2(n-2) \rfloor$, and so

$$\begin{aligned} q(L_V) &= q(L_V^0) + 1 \\ &= |\pi^0| + \zeta(\pi^0) + 1 \\ &\leq n - 2 + \lfloor (n-2)/2 \rfloor + 1 \\ &= \lfloor (3n-4)/2 \rfloor \\ &\leq \lfloor 3(n-1)/2 \rfloor. \end{aligned}$$

From this inequality we see, in particular, that for a given $n \in \mathbb{N}$, the maximum value of $q(L_V)$ among all vertex labelings of the star on n vertices is $\lfloor 3(n-1)/2 \rfloor$. Hence, we obtain the next observation as a special case. This special case was also observed both in [4, p. 561] and in [31, p. 378]. In our setting we can state the following.

Observation 3.13. *For $G = K_{1,n-1}$ the star on n vertices, any vertex labeling L_V can be transformed into another L'_V in at most $\lfloor 3(n-1)/2 \rfloor$ flips. Moreover, this value of n is the smallest possibly with this property.*

We conclude this section by some observations that generalize further what we have done for the path and the star, but first we need some additional notation and basic results.

For $n \in \mathbb{N}$ let K_n be the complete graph on n vertices, and let $V(K_n) = \{v_1, \dots, v_n\}$ be a fixed indexing of the vertices. Clearly, for each edge $e = \{v_i, v_j\}$ of K_n there is a corresponding transposition $\tau_e = (i, j)$ in the symmetric group S_n on $\{1, 2, \dots, n\}$, and vice versa, for each transposition $\tau = (i, j) \in S_n$ yields an edge $e_\tau = \{v_i, v_j\}$ of K_n . This correspondence is 1-1 in the sense that $e_{\tau_e} = e$ and $\tau_{e_\tau} = \tau$ for every e and every τ . For edges e_1, \dots, e_m of K_n let $G[e_1, \dots, e_m]$ be the simple graph induced (or formed) by these edges. In light of Proposition 2.2 the following observation is clear.

Observation 3.14. *The transpositions $\tau_1, \dots, \tau_m \in S_n$ generate the symmetric group S_n if and only if the graph $G[e_{\tau_1}, \dots, e_{\tau_m}]$ contains a spanning tree of K_n . In particular, $m \geq n - 1$ must hold.*

Consider now a connected simple graph $G = (V(G), E(G))$ on n vertices, where $V(G) = \{v_1, \dots, v_n\}$ is a fixed indexing. As before, a vertex labeling $L_V : V(G) \rightarrow \{1, 2, \dots, n\}$ corresponds to a permutation π of $\{1, 2, \dots, n\}$ in S_n . Since G is connected, it contains a spanning tree; and hence, each vertex labeling L_V of G can be transformed to any other labeling L'_V of G by a sequence of edge flips. As for the path and star, we have in general the following.

Theorem 3.15. *Let G be a connected simple graph with $V(G) = \{v_1, \dots, v_n\}$. For vertex labelings $L_V, L'_V : V(G) \rightarrow \{1, 2, \dots, n\}$ there exists a symmetric nonnegative parameter $p_G(L_V, L'_V)$ and a function $p_G(n)$ such that we have the following:*

1. The labeling L_V can be transformed into L'_V in exactly t edge flips if and only if $t = p_G(L_V, L'_V) + 2k$ for some nonnegative integer k .
2. Every labeling L_V can be transformed into another labeling L'_V in at most t edge flips if and only if $t \geq p_G(n)$.

Proof. For a given graph G and given vertex labelings L_V and L'_V of G , we define the parameter $p_G(L_V, L'_V)$ as the minimum number of edge flips needed to transform L_V into L'_V . This existence is guaranteed since every nonempty subset of $\mathbb{N} \cup \{0\}$ contains a least element. If L_V can be transformed into L'_V in t edge flips, then by reversing the process L'_V can be transformed into L_V in t edge flips as well, so $p_G(L_V, L'_V)$ is symmetric. By repeating the last edge flip an even number of times, it is clear that L_V can be transformed into L'_V in $t + 2k$ edge flips. Assume that L_V can be transformed into L'_V in t' edge flips. By viewing the edge flips of t and t' as permutations of S_n , they must have the same parity, so $t - t'$ must be even. This completes the proof of the first part.

By Theorem 2.2 we have that $p_G(L_V, L'_V) \leq n(n-1)/2$ for all vertex labelings L_V and L'_V of G . Hence, the maximum of $p_G(L_V, L'_V)$ among all pairs of vertex labelings L_V and L'_V is also at most $n(n-1)/2$. Letting $p_G(n)$ be this very maximum, the second part clearly follows. \square

REMARK: Using the notation of Theorem 3.15, what we have in particular is (i) $p_G(n) \leq n(n-1)/2$ for every connected graph G on n vertices, (ii) $p_{P_n}(n) = n(n-1)/2$, the classical result on the number of inversions by Muir [34], and (iii) $p_{K_{1,n-1}}(n) = \lfloor 3(n-1)/2 \rfloor$ for the star.

4 Relabeling with Privileged Labels

In this section we describe the last variants of the relabeling problem that we consider in this paper. We impose an additional restriction on the flip operation. Some labels are designated as *privileged*. Our restricted flips can only take place if *at least* one label of the pair to be flipped is a privileged label. The problem can be defined for vertices and for edges, the following is in terms of vertices. The version for the edges is similar.

Definition 4.1. (VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM)
 INSTANCE: A graph G , labelings L_V and L'_V , a nonempty set $S \subseteq \{1, 2, \dots, n\}$ of privileged labels, and $t \in \mathbb{N}$.
 QUESTION: Can labeling L_V transform into L'_V in t or fewer restricted vertex flips?

The problem in Definition 4.1 is increasingly restricted as the number of privileged labels decreases. Of course, one question is whether the problems are solvable at all. If $|S| = 1$, the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM can be reduced to the $(n \times n)$ -PUZZLE PROBLEM, in which half of the starting

configurations are not solvable [43]. This result proved in [22] shows that the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM is *NP*-complete.

We do not yet know if the EDGE RELABELING WITH PRIVILEGED LABELS PROBLEM with $|S| = 1$ is *NP*-complete. It is interesting to note that many other similar games and puzzles such as the GENERALIZED HEX PROBLEM [9], $(n \times n)$ -CHECKERS PROBLEM [11], $(n \times n)$ -GO PROBLEM [28], and the GENERALIZED GEOGRAPHY PROBLEM [38] are also *NP*-complete. Other unsolvable instances of the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEMS do exist and we will now provide some simple examples of unsolvable instances and then provide characterizations of both solvable and unsolvable instances of these problems. We begin with an example.

EXAMPLE A: Let $n \geq 2$ and consider two vertex labelings L_V and L'_V of the path P_n , where we have precisely k privileged labels p_1, \dots, p_k , where $k \in \{0, 1, \dots, n-2\}$. For a fixed horizontal embedding of P_n in the plane, assume the labelings are given in the following left-to-right order:

$$\begin{aligned} L_V & : (p_1, \dots, p_k, 1, 2, 3, \dots, n-k), \text{ and} \\ L'_V & : (p_1, \dots, p_k, 2, 1, 3, \dots, n-k). \end{aligned}$$

Note that by any restricted flip, where one of the labels are among $\{p_1, \dots, p_k\}$, the relative left/right order of the non-privileged labels will remain unchanged. Since the order of the two non-privileged labels 1 and 2 in L'_V is different from the one of L_V , we see that it is impossible to transform L_V to L'_V by restricted flips only. This example yields the following observation.

Observation 4.2. (GENERAL INSOLUBILITY, PRIVILEGED LABELS)

Among all connected vertex labeled graphs of order n with $k \in \{0, 1, \dots, n-2\}$ privileged labels, the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM is, in general, unsolvable.

Note that it is clear that for any connected graph G with all labels but one being privileged, any flip is a legitimate transformation, since for any edge $e = \{u, v\}$ either the label on u or v is privileged. Hence, among all connected graphs on n vertices with $n-1$ privileged labels, the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM is solvable and in *P*.

Restricting now to the class of 2-connected simple graphs, we consider a slight variation of Example A.

EXAMPLE B: Let $n \geq 3$ and consider two vertex labelings L_V and L'_V of the cycle C_n , where we have precisely k privileged labels p_1, \dots, p_k , where $k \in \{0, 1, \dots, n-3\}$. For a fixed planar embedding of C_n , assume the labelings are given cyclically in clockwise order as follows:

$$\begin{aligned} L_V & : (p_1, \dots, p_k, 1, 2, 3, \dots, n-k), \text{ and} \\ L'_V & : (p_1, \dots, p_k, 2, 1, 3, \dots, n-k). \end{aligned}$$

Note that by any restricted flip, where one of the labels are among $\{p_1, \dots, p_k\}$, the relative orientation (clockwise or counterclockwise) of the non-privileged labels

1, 2, and 3 will remain unchanged. Since the orientation of 1, 2, and 3 in L'_V is counterclockwise, and the opposite of the clockwise order of 1, 2, and 3 in L_V , we see again that it is impossible to transform L_V to L'_V by restricted flips. We summarize the implication of Example B in the following observation.

Observation 4.3. (2-CONNECTED INSOLUBILITY, PRIVILEGED LABELS)

Among all 2-connected vertex labeled graphs of order n with $k \in \{0, 1, \dots, n - 3\}$ privileged labels where the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM is, in general, unsolvable.

Note that in Observations 4.2 and 4.3 we can push the labels onto the edges and thereby obtain similar observations for the EDGE RELABELING WITH PRIVILEGED LABELS PROBLEM.

We will now fully analyze the case where G is connected and all but two of the labels are privileged.

Claim 4.4. *If a simple graph is neither a path nor a cycle, then it has a spanning tree that is not a path (and hence contains a vertex of degree at least three).*

Proof. Let G be a graph that is neither a path nor a cycle. Then G contains a vertex u of degree three or greater. Assigning the weight of one to each edge, we start by choosing three edges with u as an end-vertex and complete the construction of our spanning tree using Kruskal's algorithm. \square

Claim 4.5. *Among vertex labeled trees, which are not paths, with exactly two non-privileged labels, any two labels can be swapped using restricted flips.*

Proof. Let $G = (V, E)$ be a tree that is not a path, and L_V a labeling of the vertices. For any two distinct vertices x and y denote the unique path between them by $P(x, y)$.

Assume that we want to swap the labels $L_V(u)$ and $L_V(v)$ on vertices u and v . We first consider the case where all labels, except possibly one, on $P(u, v)$, are privileged. Restricting to $P(u, v)$, there are $2\partial(u, v) - 1$ legitimate flips that swap the labels on u and v . (Here $\partial(u, v)$ denotes the distance between u and v in the tree, or the length of $P(u, v)$. This fact was noted in the remark right after the proof of Theorem 2.4.) Let us denote such a privileged swap by $SW(u, v)$.

Consider next the case where the labels of u and v are both non-privileged. Let u' and v' be vertices such that the (u', v') -path P^* is of maximum length in the tree and such that it contains $P(u, v)$ as a sub-path. Hence, the three paths $P(u', u)$, $P(u, v)$, and $P(v, v')$ make up this maximum length path P^* . By the maximality of P^* and our assumption on the tree, there is an internal vertex w on P^* (note $w \notin \{u', v'\}$) of degree three or more, and hence that has a neighbor w' not on P^* . We now perform the following procedure of legitimate swaps:

1. $SW(u, u')$ and $SW(v, v')$,
2. $SW(u', w')$,

3. $SW(u', v')$,
4. $SW(v', w')$, and
5. $SW(u, u')$ and $SW(v, v')$.

This procedure has legitimately swapped the labels on u and v .

If at least one of the labels of u and v is privileged, but both of the non-privileged labels do lie on $P(u, v)$, say x and y , then we can perform at least one of the swaps $SW(u, x)$ or $SW(y, v)$, say $SW(u, x)$, after which we perform the swaps $SW(x, v)$ and $SW(u, x)$ to complete the legitimate swap. The case where $SW(y, v)$ was performed first is handled similarly. This completes the proof. \square

We can now state the following lemma.

Lemma 4.6. *Among vertex labeled trees, which are not paths, with exactly two non-privileged labels, the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM is solvable and in P .*

Proof. Since any transformation from one labeling L_V to another L'_V is a composition of transpositions, this lemma follows from Claim 4.5. \square

We now have the following summarizing theorem.

Theorem 4.7. (VERTEX SOLUBILITY, TWO PRIVILEGED LABELS)

Among all connected vertex labeled graphs G on $n \geq 4$ vertices with all but two vertex labels privileged, the VERTEX RELABELING WITH PRIVILEGED LABELS PROBLEM is solvable if and only if G is not a path.

Proof. We see from Example A that for $n \geq 2$ there are labelings of the vertices of the path P_n that cannot transform into one another using restricted flips.

If G is a cycle on $n \geq 4$ vertices, we can first move the labels of the non-privileged labels to their desired places by using appropriate clockwise and/or anti-clockwise sequences of flips, and then move all the privileged labels to their places using flips as on a path.

If G is neither a path nor a cycle, then by Claim 4.4 G has a spanning tree T that is not a path. Restricting to T we can by Lemma 4.6 move all the labels to their desired places within T and hence within G . This completes our proof. \square

5 Conclusions and Open Problems

We have defined several versions of a graph relabeling problem, including variants involving vertices, edges, and privileged labels, and proved numerous results about the complexity of these problems, answering several open problems along the way. A number of interesting open problems remain as follows:

- Determine if the VERTEX RELABELING PROBLEM can be reduced to the EDGE RELABELING PROBLEM.

- Study other types of flip functions where, for example, labels along an entire path are flipped, or where labels can be reused.
- In the parallel setting, compute the sequence of flips required for the transformation of one labeling into another. The parallel time for computing the sequence could be much smaller than the sequential time to execute the flip sequence.

One result of interest in this direction is the problem of given a labeled graph, a *prescribed flipping sequence*, and two designated labels l_1 and l_2 are l_1 and l_2 flipped? A prescribed flipping sequence is an ordering of edges in which each succeeding edge's labels may be flipped if and only if neither of its labels has already been flipped. This problem is *NC*-equivalent to the Lexicographically First Maximal Matching Problem, and so *CC*-complete; see [19] for a list of *CC*-complete problems.

- For various classes of graphs determine the probability of one labelings evolving naturally into another. Such an evolution of a labeling could be used to model flip periods.
- Study the properties of the graphs of all labelings. In this graph all labelings of a given graph are vertices and two vertices are connected if they are one flip apart. Other conditions for edge placement may also be worthwhile to examine.
- Determine if there is a version of the EDGE RELABELING WITH PRIVILEGED LABELS PROBLEM that is *NP*-complete.
- Define the *cost of a flip sequence* to be the sum of the weights on all edges that are flipped. Determine flip sequences that minimize the cost of transforming one labeling into another. Explore other cost functions.

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