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Generalized Difference Sequence Spaces Defined by Orlicz Functions and their Köthe-Toeplitz and Null Duals

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Abstract : In this article we define some difference sequence spaces and its subspaces using an Orlicz function. We find their isometrically isomorphic spaces and study some other properties. Moreover we compute Köthe-Toeplitz and null duals of all these spaces.

Keywords : Difference sequence spaces, Orlicz function, Köthe Toeplitz dual, Null dual.
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1 Introduction

Throughout this section w, ℓ_∞, ℓ_1, c and c_0 denote the spaces of all, bounded, absolutely summable, convergent and null sequences $x = (x_k)$ with complex terms respectively. The notion of difference sequence space was introduced by Kizmaz [4], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [3] by introducing the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$.

Let n be non-negative integers then for Z a given sequence space we have

$$Z(\Delta^m) = \{x = (x_k) \in w : (\Delta^m x_k) \in Z\},$$

where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation.

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [4].

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \longrightarrow \infty$, as $x \longrightarrow \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that

$$M(2u) < KM(u), \quad \text{where} \ \ u \geq 0$$

The Δ_2-condition is equivalent to $M(lu) \leq KlM(u)$, for all values of u and for $l > 1$.

An Orlicz function M can always be represented in the following integral form:

$$M(x) = \int\limits_0^x p(t)dt$$

where p, known as kernel of M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing, and $p(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Consider the kernel $p(t)$ associated with the Orlicz function $M(t)$, and let

$$q(s) = sup\{t : p(t) \leq s\}$$

Then q possesses the same properties as the function p. Suppose now

$$\Phi(x) = \int\limits_0^x q(s)ds$$

Then Φ is an Orlicz function. The functions M and Φ are called mutually complementary Orlicz functions.

Now we state the following results which can be found in [5].
Let M and Φ are mutually complementary Orlicz functions. Then we have (Young's inequality)

(i) For $x, y \geq 0, \quad xy \leq M(x) + \Phi(y)$.

Also we have

(ii) $M(\lambda x) < \lambda M(x)$ for all $x \geq 0$ and λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [6] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

A norm $\| \bullet \|$ on a vector space X is said to be equivalent to a norm $\| \bullet \|_0$ on X if there are positive numbers A and B such that for all $x \in X$, we have

$$A\|x\|_0 \leq \|x\| \leq B\|x\|_0.$$

This concept is motivated by the fact that equivalent norms on X define the same topology for X.

An isomorphism of a normed space X onto a normed space Y is a bijective linear operator $T : X \longrightarrow Y$ which preserves the norm, that is, for all $x \in X$,

$$\|Tx\| = \|x\|. \text{ (Hence } T \text{ is isometric)}$$

X is then called isomorphic with Y, and X and Y are called isomorphic normed spaces.

Let m be a non-negative integer. Then we can have the following sequence spaces for an Orlicz function M as:

$$c_0(M,\Delta^m)=\{x=(x_k):\lim_k M\left(\frac{|\Delta^m x_k|}{\rho}\right)=0,\ \text{for some } \rho>0\},$$

$$c(M,\Delta^m)=\{x=(x_k):\lim_k M\left(\frac{|\Delta^m x_k-L|}{\rho}\right)=0,\ \ \text{for some } L \text{ and } \rho>0\},$$

$$\ell_\infty(M,\Delta^m)=\{x=(x_k):\sup_k M\left(\frac{|\Delta^m x_k|}{\rho}\right)<\infty,\ \text{for some } \rho>0\},$$

where $\Delta^m x_k=\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}$, $\Delta_r^m x_k=\sum\limits_{i=0}^{m}(-1)^i\binom{m}{i}x_{k+i}$ for all $k\in N$.

It is obvious that $\quad c_0(M,\Delta^m)\subset c(M,\Delta^m)\subset \ell_\infty(M,\Delta^m) \qquad (1.1)$

Several authors have studied different algebraic and topological properties of such spaces. In this article our main aim to compute the Köthe-Toepliz and Null duals of such spaces.

Throughout the paper X will denote one of the sequence spaces c_0, c and ℓ_∞. The sequence spaces $X(M,\Delta^m)$ are Banach spaces normed by

$$\|x\|_{\Delta^m}=\sum_{i=1}^{m}|x_i|+\inf\left\{\rho>0:\sup_k M\left(\frac{|\Delta^m x_k|}{\rho}\right)\le 1\right\} \tag{1.2}$$

Now we take

$$\Delta^{(m)}x_k=\sum_{i=0}^{m}(-1)^i\binom{m}{i}x_{k-i}.$$

It is trivial that $(\Delta^m x_k)\in X(M)$ if and only if $(\Delta^{(m)}x_k)\in X(M)$. Now for $x\in X(M,\Delta^{(m)})$, we define

$$\|x\|_{\Delta^{(m)}}=\inf\left\{\rho>0:\sup_k M\left(\frac{|\Delta^m x_k|}{\rho}\right)\le 1\right\}.$$

It can be shown that $X(M,\Delta^{(m)})$ is a BK space under the norm $\|x\|_{\Delta^{(m)}}$ and the norms $\|x\|_{\Delta^m}$ and $\|x\|_{\Delta^{(m)}}$ are equivalent. Obviously
$\Delta^{(m)}:X(M,\Delta^{(m)})\longrightarrow X(M)$, denoted by $\Delta^{(m)}x=y=(\Delta^{(m)}x_k)$, is isometric isomorphism.

Hence $c_0(M,\Delta^m)$, $c(M,\Delta^m)$ and $\ell_\infty(M,\Delta^m)$ are isometrically isomorphic to $c_0(M)$, $c(M)$ and $\ell_\infty(M)$ respectively. From abstract point of view $X(M,\Delta_r^m)$ is identical with $X(M)$, for $X=c_0$, c and ℓ_∞.

Now we define the spaces $\overline{c}_0(M,\Delta^m)$, $\overline{c}(M,\Delta^m)$ and $\overline{\ell}_\infty(M,\Delta^m)$ as follows:
$\overline{c}_0(M,\Delta^m)$ is a subspace of $c_0(M,\Delta^m)$ consisting of those x in $c_0(M,\Delta^m)$ such that $\lim\limits_k M\left(\frac{|\Delta^m x_k|}{d}\right)=0$, for each $d>0$.

Similarly we define $\overline{c}(M,\Delta^m)$ and $\overline{\ell}_\infty(M,\Delta^m)$ as a subspace of $c(M,\Delta^m)$ and $\ell_\infty(M,\Delta^m)$ respectively. The topology of $\overline{X}(M,\Delta^m)$ is the one it inherits from $\|\bullet\|_{\Delta^m}$.

It is obvious that $\overline{c}_0(M,\Delta^m) \subset \overline{c}(M,\Delta^m) \subset \overline{\ell}_\infty(M,\Delta^m)$.

Also as above we can show that $\overline{c}_0(M,\Delta^m)$, $\overline{c}(M,\Delta^m)$ and $\overline{\ell}_\infty(M,\Delta^m)$ are isometrically isomorphic to $\overline{c}_0(M)$, $\overline{c}(M)$ and $\overline{\ell}_\infty(M)$ respectively.

Moreover $X(M,\Delta^i) \subset X(M,\Delta^m)$ and $\overline{X}(M,\Delta^i) \subset \overline{X}(M,\Delta^m)$ for $i = 0, 1, ..., m-1$. which can be shown by repeated application of the following inequality.

$$M\left(\frac{|\Delta^m x_k|}{2\rho}\right) \leq \frac{1}{2}M\left(\frac{|\Delta^{m-1} x_k|}{\rho}\right) + \frac{1}{2}M\left(\frac{|\Delta^{m-1} x_{k+1}|}{\rho}\right).$$

2 Some Properties of the Spaces

Theorem 2.1. *If M satisfies the Δ_2-condition then we have $X(M,\Delta^m) = \overline{X}(M,\Delta^m)$ for $X = c_0$, c and ℓ_∞.*

Proof. We give the proof for $X = \ell_\infty$ and for other spaces it will follow on applying similar arguments.

To prove the theorem it is enough to show that $\ell_\infty(M,\Delta^m)$ is a subspace of $\overline{\ell}_\infty(M,\Delta^m)$. Let $x \in \ell_\infty(M,\Delta^m)$, then for some $\rho > 0$,
$\sup_k M\left(\frac{|\Delta^m x_k|}{\rho}\right) < \infty$ and $M\left(\frac{|\Delta^m x_k|}{\rho}\right) < \infty$ for every $k \geq 1$
Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$ then $M\left(\frac{|\Delta^m x_k|}{\eta}\right) < \infty$ for every $k \geq 1$. Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$.
Since M satisfies the Δ_2-condition, there exists a constant K such that
$M\left(\frac{|\Delta^m x_k|}{\eta}\right) \leq K\frac{\rho}{\eta}M\left(\frac{|\Delta^m x_k|}{\rho}\right) < \infty$ for every $k \geq 1$ and hence
$\sup_k M\left(\frac{|\Delta^m x_k|}{\eta}\right) < \infty$ for every $\eta > 0$ □

Theorem 2.2. (i) *$c_0(M,\Delta^m)$, $c(M,\Delta^m)$ and $\ell_\infty(M,\Delta^m)$ are convex sets.*
(ii) If M satisfies the Δ_2-condition then $\overline{c}_0(M,\Delta^m)$, $\overline{c}(M,\Delta^m)$ and $\overline{\ell}_\infty(M,\Delta^m)$ are convex sets.

Proof. (i) We prove the Theorem for $c_0(M,\Delta^m)$ and for other cases it will follow on applying similar arguments.
Let $x, y \in c_0(M,\Delta^m)$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\lim_k M\left(\frac{|\Delta^m x_k|}{\rho_1}\right) = 0 \quad \text{and} \quad \lim_k M\left(\frac{|\Delta^m y_k|}{\rho_2}\right) = 0$$

For $\lambda = 0$ or $\lambda = 1$, the result is obvious. Let $0 < \lambda < 1$. Considering $\rho = \max(|\lambda|\rho_1, |1-\lambda|\rho_2)$, we have

$$M\left(\frac{|\Delta^m(\lambda x_k + (1-\lambda)y_k)|}{2\rho}\right) \leq \frac{1}{2}M\left(\frac{|\Delta^m(\lambda x_k)|}{\rho}\right) + \frac{1}{2}M\left(\frac{|\Delta^m(1-\lambda)y_k)|}{\rho}\right)$$

$$\leq \frac{1}{2}M\left(\frac{|\Delta^m x_k|}{\rho_1}\right) + \frac{1}{2}M\left(\frac{|\Delta^m y_k|}{\rho_2}\right)$$

This completes the proof.
(ii) Proof follows from (i) using Theorem 2.1. □

Theorem 2.3. *(i) $c_0(M,\Delta^m)$ and $c(M,\Delta^m)$ are nowhere dense subsets of $\ell_\infty(M,\Delta^m)$.*
(ii) $\overline{c}_0(M,\Delta^m)$ and $\overline{c}(M,\Delta^m)$ are are nowhere dense subsets of $\overline{\ell}_\infty(M,\Delta^m)$.

Proof. The proof follows from (1.1) and (1.2). □

3 Köthe Toeplitz and Null Dual Spaces

In this section we give the $\alpha-$dual and $N-$(or null) dual of the sequence spaces $c_0(M,\Delta^m)$, $c(M,\Delta^m)$, $\ell_\infty(M,\Delta^m)$, $\overline{c}_0(M,\Delta^m)$, $\overline{c}(M,\Delta^m)$ and $\overline{\ell}_\infty(M,\Delta^m)$.

Let E and F be two sequence spaces. Then the F dual of E is defined as
$E^F = \{(x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E\}$.

For $F = \ell_1$ and c_o , the duals are termed as $\alpha-$dual (Köthe Toeplitz dual) and $N-$(or null) dual of E and denoted by E^α and E^N respectively. If $X \subset Y$, then $Y^z \subset X^z$ for $z = \alpha, N$.

Lemma 3.1. [9] *Let m be a positive integer. Then there exists positive constants C_1 and C_2 such that*

$$C_1 k^m \leq \binom{m+k}{k} \leq C_2 k^m, \qquad k = 0,1,2,\cdots$$

Lemma 3.2. *$x \in \ell_\infty(M,\Delta^m)$ implies $\sup_k M\left(\frac{|k^{-1}\Delta^{m-1}x_k|}{\rho}\right) < \infty$, for some $\rho > 0$.*

Proof. Let $x \in \ell_\infty(M,\Delta^m)$, then

$$\sup_k M\left(\frac{|\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}|}{\rho}\right) < \infty, \text{ for some } \rho > 0.$$

Then there exists a $U > 0$ such that

$$M\left(\frac{|\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}|}{\rho}\right) < U, \text{ for all } k \in N.$$

On taking $\eta = k\rho$, $k > 1$ being arbitrary fixed number, we have

$$M\left(\frac{|\Delta^{m-1}x_1 - \Delta^{m-1}x_{k+1}|}{\eta}\right) = M\left(\frac{|\sum_{i=1}^{k}\Delta^{m-1}x_i - \Delta^{m-1}x_{i+1}|}{k\rho}\right)$$

$$\leq M\left(\frac{\sum_{i=1}^{k}|\Delta^{m-1}x_i - \Delta^{m-1}x_{i+1}|}{k\rho}\right)$$

$$\le \frac{1}{k}M\left(\frac{|\Delta^{m-1}x_1-\Delta^{m-1}x_2|}{\rho}\right)+\cdots+\frac{1}{k}M\left(\frac{|\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}|}{\rho}\right)$$

$$\le \frac{1}{k}U+\frac{1}{k}U+\cdots+\frac{1}{k}U \ = O(k)$$

Now the result follows from the following inequality using the convexity of M:

$$|\Delta^{m-1}x_{k+1}| \le |\Delta^{m-1}x_1| + |\Delta^{m-1}x_1 - \Delta^{m-1}x_{k+1}|.$$

□

Lemma 3.3. *(i)* $\sup_k M\left(\frac{|k^{-1}\Delta^m x_k|}{\rho}\right) < \infty$ *implies* $\sup_k M\left(\frac{|k^{-m}x_k|}{\rho}\right) < \infty$ *for some* $\rho > 0$
(ii) $x \in \ell_\infty(M,\Delta^m)$ *implies* $\sup_k M\left(\frac{|k^{-m}x_k|}{\rho}\right) < \infty,$ *for some* $\rho > 0,$
(iii) $x \in \ell_\infty(M,\Delta^m)$ *implies* $\sup_k |k^{-m}x_k| < \infty.$

Proof. (i) Proof follows from Lemma 3.2 by repeated application of the same arguments.
(ii) Combining the Lemma 3.2 and part (i).
(iii) Proof follows from part (i). □

Remark 1. *Similar results as in Lemma 3.3 hold for* $\overline{\ell}_\infty(M,\Delta^m)$ *also, where the statement "for some* $\rho > 0$*" should be replaced by "for every* $\rho > 0$*"*

Theorem 3.4. *Let* M *be an Orlicz function. Then*
(i) $[c_0(M,\Delta^m)]^\alpha = [c(M,\Delta^m)]^\alpha = [\ell_\infty(M,\Delta^m)]^\alpha = D_1,$
(ii) $D_1^\alpha = D_2,$ *where*
$D_1 = \left\{a=(a_k) : \sum_{k=1}^{\infty} |k^m a_k| < \infty\right\},$
$D_2 = \left\{b=(b_k) : \sup_k |k^{-m} b_k| < \infty\right\}.$

Proof. (i) Let $a \in D_1$, then $\sum_{k=1}^{\infty} |k^m a_k| < \infty$. Now for any $x \in \ell_\infty(M,\Delta^m)$ we have $\sup_k |k^{-m}x_k| < \infty$. Then we have

$$\sum_{k=1}^{\infty} |a_k x_k| \le \sup_k |k^{-m}x_k| \ \sum_{k=1}^{\infty} |k^m a_k| < \infty.$$

Hence $a \in [\ell_\infty(M,\Delta^m)]^\alpha$. Conversely suppose that $a \in [X(M,\Delta^m)]^\alpha$, for $X = c$ and ℓ_∞. Then $\sum_{k=1}^{\infty} |a_k x_k| < \infty$, for each $x \in X(M,\Delta^m)$. So we take $x_k = k^m$, $k \ge 1$ then $\sum_{k=1}^{\infty} |k^m a_k| = \sum_{k=1}^{\infty} |a_k x_k| < \infty$. This implies that $a \in D_1$.

Again suppose that $a \in [c_0(M, \Delta^m)]^\alpha$ and $a \notin D_1$. Then there exists a strictly increasing sequence (n_i) of positive integers n_i with $n_1 < n_2 < \cdots$, such that

$$\sum_{k=n_i+1}^{n_{i+1}} |k^m a_k| > i.$$

Define $x \in c_0(M, \Delta^m)$ by

$$\begin{aligned} x_k &= 0, & 1 \le k \le n_1 \\ &= k^m \ \frac{sgn\ a_k}{i}, & n_i < k \le n_{i+1} \end{aligned}$$

Then we have $\sum\limits_{k=1}^{\infty} |a_k x_k| = \sum\limits_{k=n_1+1}^{n_2} |a_k x_k| + \cdots + \sum\limits_{k=n_i+1}^{n_{i+1}} |a_k x_k| + \cdots$

$$= \sum_{k=n_1+1}^{n_2} |k^m a_k| + \cdots + \frac{1}{i} \sum_{k=n_i+1}^{n_{i+1}} |k^m a_k| + \cdots$$

$$> 1 + 1 + \cdots \quad = \infty.$$

This contradicts to $a \in [c_0(M, \Delta^m)]^\alpha$. Hence $a \in D_1$. This completes the proof of (i).

(ii) proof is similar to proof of part (i). □

Theorem 3.5. *Let M be an Orlicz function. Then*
(i) $[\overline{c}_0(M, \Delta^m)]^\alpha = [\overline{c}(M, \Delta^m)]^\alpha = \left[\overline{\ell}_\infty(M, \Delta^m)\right]^\alpha = D_1$,
(ii) $D_1^\alpha = D_2$, where

$$D_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |k^m a_k| < \infty \right\},$$

$$D_2 = \left\{ b = (b_k) : \sup_k |k^{-m} b_k| < \infty \right\}.$$

Proof. The proof is similar to that of Theorem 4. □

If we take $m = 0$ in Theorem 3.4 and Theorem 3.5 then we obtain the following Corollary.

Corollary 3.6. *For $X = c_0$, c and ℓ_∞.*
(i) $[X(M)]^\alpha = \left[\overline{X}(M)\right]^\alpha = H_1$,
(ii) $H_1^\alpha = H_2$, where

$$H_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| < \infty \right\},$$

$$H_2 = \left\{ b = (b_k) : \sup_k |b_k| < \infty \right\}.$$

Lemma 3.7. *If $x \in c_0(M, \Delta^m)$, then $\lim\limits_k M\left(\frac{\left|\binom{m+k}{k}^{-1} x_k\right|}{\rho}\right) = 0$, for some $\rho > 0$.*

Proof. The proof follows from Lemma 3.1.

Theorem 3.8. *Let M and Φ be mutually complemetary Orlicz functions. Then*
(i) $[c(M,\Delta^m)]^N = [\ell_\infty(M,\Delta^m)]^N = G_1$,
(ii) $[c_0(M,\Delta^m)]^N = \overline{G}_2$, *where*
$G_1 = \{a=(a_k) : \lim_k k^m a_k = 0\}$,
$\overline{G}_2 = \left\{a=(a_k) : \lim_k \Phi\left(\frac{\left|\binom{m+k}{k}a_k\right|}{d}\right) = 0,\ \text{for every } d>0\right\}$.

Proof. (i) Proof is immediate using Lemma 3.3(iii).
(ii) Let $a \in \overline{G}_2$, then $\lim_k \Phi\left(\frac{\left|\binom{m+k}{k}a_k\right|}{d}\right) = 0$ for every $d > 0$. Now for any $x \in c_0(M,\Delta^m)$ we have
$\lim_k M\left(\frac{\left|\binom{m+k}{k}^{-1}x_k\right|}{\rho}\right) = 0$ for some $\rho > 0$. Then using the inequality

$$|a_k x_k| \le M\left(\left|\frac{\binom{m+k}{k}^{-1}x_k}{\rho}\right|\right) + \Phi\left(\rho\left|\binom{m+k}{k}a_k\right|\right),$$

we have $\lim_k a_k x_k = 0$ and hence $a \in [c_0(M,\Delta^m)]^N$.

Conversely suppose that $a \in [c_0(M,\Delta^m)]^N$. Then $\lim_k a_k x_k = 0$, for each $x \in c_0(M,\Delta^m)$. So we take $x_k = \binom{m+k}{k}$, $k \ge 1$, then $\lim_k \binom{m+k}{k}a_k = \lim_k a_k x_k = 0$ and so

$$\lim_k \Phi\left(\frac{\left|\binom{m+k}{k}a_k\right|}{d}\right) = 0$$

for every $d > 0$. Hence $a \in \overline{G}_2$. This completes the proof. □

Theorem 3.9. *Let M and Φ be mutually complemetary Orlicz functions. Then*
(i) $[\overline{c}(M,\Delta^m)]^N = \left[\overline{\ell}_\infty(M,\Delta^m)\right]^N = G_1$,
(ii) $[\overline{c}_0(M,\Delta^m)]^N = G_2$, *where*
$G_1 = \{a=(a_k) : \lim_k k^m a_k = 0\}$,
$G_2 = \left\{a=(a_k) : \lim_k \Phi\left(\frac{\left|\binom{m+k}{k}a_k\right|}{d}\right) = 0, \text{for some } d>0\right\}$.

Proof. Proof is similar to that of Theorem 3.8. □

If we take $m = 0$ in Theorem 3.8 and Theorem 3.9 then we obtain the following Corollary.

Corollary 3.10. *For $X = c$ and ℓ_∞,*
(i) $[X(M)]^N = \left[\overline{X}(M)\right]^N = L_1$,
(ii) $[c_0(M)]^N = \overline{L}_2$,
(iii) $[\overline{c}_0(M)]^N = L_2$, *where*
$L_1 = \left\{a=(a_k) : \lim_k a_k = 0\right\}$,

$\overline{L}_2 = \{a = (a_k) : \lim_k \Phi(\frac{|a_k|}{d}) = 0, \text{ for every } d > 0\}$,
and
$L_2 = \{a = (a_k) : \lim_k \Phi(\frac{|a_k|}{d}) = 0, \text{ for some } d > 0\}$.

References

[1] P. Chandra and B.C. Tripathy, On generalized Köthe-Toeplitz duals of some sequence spaces, *Indian J. Pure Appl. Math.*, **33**(8) (2002), 1301-1306.

[2] M. Et, On some topological properties of generalized difference sequence spaces, *Internat. J. Math. and Math. Sci.*, **24**(11) (2000),785-791.

[3] M. Et , R.Colak, On generalized difference sequence spaces, *Soochow J. Math.*, **21**(4) (1995), 377-386.

[4] H. Kizmaz, On certain sequence spaces, *Canad. Math. Bull.*, **24**(2) (1981), 169-176.

[5] P.K. Kamthan and M. Gupta, Sequence Spaces and Series, *Marcel Dekker Inc., New York*, (1981).

[6] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[7] E. Kreyszig, Introductory Functional Analysis with Applications, *Jhon Wiley and Sons, New York* (1978).

[8] I.J. Maddox, Elements of Functional Analysis, *Universal Book Stall, New Delhi* (1989).

[9] E. Malkowsky and S.D. Parasar, Matrix transformation in spaces of bounded and convergent difference sequences of order m, *Analysis*, **17** (1997), 87-97.

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