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Some Results on Generalized Sasakian-Space-Forms

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Abstract : The object of the present paper is to study locally $\phi-$symmetric generalized Sasakian-space-forms and generalized Sasakian-space-forms with $\eta-$recurrent Ricci tensor. Such space-forms with three-dimensional quasi-Sasakian structure are also considered.

Keywords : generalized Sasakian-space-forms, conformally flat, locally $\phi-$symmetric, $\eta-$recurrent, $\eta-$parallel, $\eta-$Einstein manifold, scalar curvature, conformal transformation, quasi-Sasakian.
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1 Introduction

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real-space-form and its curvature tensor is given by

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

A Sasakian manifold with constant $\phi-$sectional curvature is a Sasakian-space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame P. Alegre, D. E. Blair and A. Carriazo introduced the notion of generalized Sasakian-space-forms in 2004 [1]. In this connection it should be mentioned that in 1989 Z. Olszak [12] studied generalized complex-space-forms and proved its existence. A generalized Sasakian-space-form is defined as follows [1]:

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Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. In [1] the authors cited several examples of such manifolds. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$ then a generalized Sasakian-space-form with Sasakian structure becomes Sasakian-space-form.

Generalized Sasakian-space-forms and Sasakian-space-forms have been studied by several authors, viz., [1], [2], [4], [13]. Symmetry of a manifold is the most important property among its all geometrical properties. Symmetry property of manifolds have been studied by many authors, viz., [8], [9]. As a weaker notion of locally symmetric manifolds T. Takahashi [16] introduced and studied locally $\phi-$symmetric Sasakian manifolds. Symmetry of a manifold primarily depends on curvature tensor and Ricci tensor of the manifold. Motivated by these facts, we study $\phi-$symmetry of the space-form by using curvature tensor. Again using Ricci tensor of the space-form we characterize such space-forms to have $\eta-$recurrent and $\eta-$parallel Ricci tensor. How a three-dimensional generalized Sasakian-space-form behaves with quasi-Sasakian structure is also discussed. The paper is organized as follows:

Section 2 of this paper contains some preliminary results. In Section 3 we study locally $\phi-$symmetric generalized Sasakian-space-forms, Section 4 consists of generalized Sasakian-space-forms with $\eta-$recurrent Ricci tensor. The last section is devoted to study three-dimensional generalized Sasakian-space-forms with quasi-Sasakian structure.

2 Preliminaries

This section contains some basic results and formulas which will be used in our main results.

A $(2n+1)-$dimensional Riemannian manifold (M, g) is called an almost contact manifold if the following results hold [2]:

$$\phi^2(X) = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0, \tag{2.4}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \tag{2.5}$$

An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X,\phi Y).$$

Φ is called the fundamental two form of the manifold. If in addition ξ is a Killing vector the manifold is called a $K-$contact manifold. It is well known that a contact metric manifold is $K-$contact if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M. On the other hand a normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X,$$

for any vector fields X, Y. In 1967 D. E. Blair introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosymplectic manifolds [6]. Again in 1986 Z. Olszak introduced and characterized three-dimensional quasi-Sasakian manifolds [10]. An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta\phi X, \tag{2.6}$$

for $X \in TM$ and a function β such that $\xi\beta = 0$. Here ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection on the manifold. As a consequence of (2.6) we get

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y), \tag{2.7}$$

$$(\nabla_X \eta)\xi = -\beta\eta(\phi X) = 0. \tag{2.8}$$

Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. It is known that [7] for a three-dimensional quasi-Sasakian manifold the Riemannian curvature tensor satisfies

$$R(X,Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y) + d\beta(Y)\phi X - d\beta(X)\phi Y. \tag{2.9}$$

For a $(2n+1)-$dimensional generalized Sasakian-space-form we have [1]

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{2.10}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y). \tag{2.11}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3. \tag{2.12}$$

Here S is the Ricci tensor and r is the scalar curvature of the space-form.

A generalized Sasakian-space-form of dimension greater than three is said to be conformally flat if its Weyl conformal curvature tensor vanishes. It is known that [13] a $(2n+1)-$dimensional $(n > 1)$ generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is conformally flat if and only if $f_2 = 0$.

3 Locally $\phi-$symmetric generalized Sasakian space-forms

Definition 3.1. A generalized Sasakian space form is said to be locally $\phi-$symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z = 0,$$

for all vector fields X, Y, Z orthogonal to ξ. This notion was introduced by T. Takahashi for Sasakian manifolds [16].

Differentiating (2.10) covariantly with respect to W we get

$$\begin{aligned}
(\nabla_W R)(X,Y)Z &= df_1(W)\{g(Y,Z)X - g(X,Z)Y\} \\
&+ df_2(W)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\
&+ f_2\{g(X,\phi Z)(\nabla_W \phi)Y + g(X,(\nabla_W \phi)Z)\phi Y \\
&- g(Y,\phi Z)(\nabla_W \phi X) - g(Y,(\nabla_W \phi)Z)\phi X \\
&+ 2g(X,\phi Y)(\nabla_W \phi)Z + 2g(X,(\nabla_W \phi)Y)\phi Z\} \\
&+ df_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
&+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \\
&+ f_3\{(\nabla_W \eta)(X)\eta(Z)Y + \eta(X)(\nabla_W \eta)(Z)Y \\
&- (\nabla_W \eta)(Y)\eta(Z)X - \eta(Y)(\nabla_W \eta)(Z)X \\
&+ g(X,Z)(\nabla_W \eta)(Y)\xi + g(X,Z)\eta(Y)(\nabla_W \xi) \\
&- g(Y,Z)X(\nabla_W \eta)(X)\xi - g(Y,Z)\eta(X)(\nabla_W \xi)\}. \qquad (3.1)
\end{aligned}$$

Taking X, Y, Z orthogonal to ξ, applying ϕ^2 on both sides of (3.1) and using (2.1) we get

$$\begin{aligned}
\phi^2(\nabla_W R)(X,Y)Z &= df_1(W)\{g(X,Z)Y - g(Y,Z)X\} \\
&+ df_2\{g(Y,\phi Z)\phi X - 2g(X,\phi Y)\phi Z - g(X,\phi Z)\phi Y\} \\
&+ f_2\{g(X,\phi Z)\phi^2((\nabla_W \phi)Y) - g(X,(\nabla_W \phi)\phi Z)\phi Y \\
&- g(Y,\phi Z)\phi^2((\nabla_W \phi)X) + g(Y,(\nabla_W \phi)Z)\phi X \\
&+ 2g(X,\phi Y)\phi^2((\nabla_W \phi)Z) - 2g(X,(\nabla_W \phi)Y)\phi Z\}. \qquad (3.2)
\end{aligned}$$

If the manifold is conformally flat then $f_2 = 0$. Therefore, (3.2) yields

$$\phi^2(\nabla_W R)(X,Y)Z = df_1\{g(X,Z)Y - g(Y,Z)X\}, \qquad (3.3)$$

where X, Y, Z are orthogonal to ξ. In view of (3.3) we obtain the following:

Theorem 3.1. A $(2n+1)-$dimensional $(n > 1)$ conformally flat generalized Sasakian-space-form is locally $\phi-$symmetric if and only if f_1 is constant.

Remark 3.1. In [13], U. K. Kim studied generalized Sasakian-space-forms and proved that if a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ of dimension greater than three is conformally flat and ξ is Killing, then it is locally symmetric. Moreover, if $M(f_1, f_2, f_3)$ is locally symmetric, then $f_1 - f_3$ is constant. In the above theorem it is shown that a conformally flat generalized Sasakian-space-form of dimension greater than three is locally $\phi-$symmetric if and only if f_1 is constant. Thus we observe the difference between locally symmetric generalized Sasakian-space forms and locally $\phi-$symmetric generalized Sasakian-space-forms.

Example 3.1. In [1] it is shown that $\mathbb{R}\times_f \mathbb{C}^m$ is a generalized Sasakian-space-form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f = f(t)$. If we choose $f(t) = e^t$, then it follows that f_1 is constant. Hence by Theorem 3.1 this manifold becomes locally $\phi-$symmetric.

4 $\eta-$recurrent generalized Sasakian-space-forms

Definition 4.1. A $(2n+1)-$dimensional generalized Sasakian-space-form is said to have $\eta-$recurrent Ricci tensor if there exists a non-zero $1-$form $A(X)$ such that

$$(\nabla_X S)(\phi Y, \phi Z) = A(X)S(Y,Z). \tag{4.1}$$

If the $1-$form vanishes on M then the space-form is said to have $\eta-$parallel Ricci tensor. The notion of $\eta-$parallel Ricci tensor was introduced by Kon in the context of Sasakian geometry [14].

From (2.11) we have

$$\begin{aligned}(\nabla_W S)(\phi X, \phi Y) &= d(2nf_1 + 3f_2 - f_3)(W)(g(X,Y) - \eta(X)\eta(Y)) \\ &- (2nf_1 + 3f_2 - f_3)((\nabla_W \eta)(X)\eta(Y) \\ &+ \eta(X)(\nabla_W \eta)(Y)).\end{aligned} \tag{4.2}$$

Suppose that the space-form has $\eta-$recurrent Ricci tensor. Then in view of (4.1) and (4.2) it follows that

$$\begin{aligned}&d(2nf_1 + 3f_2 - f_3)(W)(g(X,Y) - \eta(X)\eta(Y)) \\ -\ &(2nf_1 + 3f_2 - f_3)((\nabla_W \eta)(X)\eta(Y) + \eta(X)(\nabla_W \eta)(Y)) \\ =\ &A(W)((2nf_1 + 3f_2 - f_3)g(X,Y) \\ -\ &(3f_2 + (2n-1)f_3)\eta(X)\eta(Y)).\end{aligned} \tag{4.3}$$

In (4.3) replacing X by ϕX and Y by ϕY we have

$$d(2nf_1 + 3f_2 - f_3)(W) = A(W)(2nf_1 + 3f_2 - f_3). \tag{4.4}$$

Let $2nf_1 + 3f_2 - f_3 = f$. Then (4.4) reduces to

$$fA(W) = df(W). \tag{4.5}$$

From (4.5) we get

$$df(Y)A(W) + (\nabla_Y A)(W)f = d^2 f(W,Y). \tag{4.6}$$

Interchanging Y and W we get from the above equation

$$df(W)A(Y) + (\nabla_W A)(Y)f = d^2 f(Y, W). \tag{4.7}$$

Subtracting (4.7) from (4.6) we get

$$(\nabla_W A)(Y) - (\nabla_Y A)(W) = 0.$$

Hence the $1-$form $A(W)$ is closed.

Thus we have the following:

Theorem 4.1. In an $\eta-$recurrent generalized Sasakian-space-form the $1-$form A is closed.

Since $A(W)$ is non-zero, equation (4.4) leads us to state the following:

Theorem 4.2. If a $(2n+1)-$dimensional generalized Sasakian space form has $\eta-$recurrent Ricci tensor then $2nf_1 + 3f_2 - f_3$ can never be a non-zero constant.

In view of (4.4) we also have

Theorem 4.3. A $(2n+1)-$dimensional generalized Sasakian-space-form $M(f_1, f_2, f_3)$ has $\eta-$parallel Ricci tensor if and only if $2nf_1 + 3f_2 - f_3$ is constant.

We know that if a $(2n+1)-$dimensional generalized Sasakian-space-form admits contact metric structure then, $f_1 - f_3$ is constant [1]. Hence f_3 is constant if and only if f_1 is constant. If $f_2 = 0$ then we see that $2nf_1 - f_3$ is constant if and only if f_1 is constant.

Thus combining Theorem 3.1 and Theorem 4.3 we get

Corollary 4.1. A $(2n+1)-$dimensional $(n > 1)$ conformally flat contact metric generalized Sasakian space-form has $\eta-$parallel Ricci tensor if and only if it is locally $\phi-$symmetric.

For three-dimensional generalized Sasakian-space-form we obtain from (2.12)

$$r = 2(2f_1 + 3f_2 - f_3) + 2(f_1 - f_3).$$

If the manifold admits contact metric structure, then $f_1 - f_3$ is constant. Hence r is constant if and only if $2f_1 + 3f_2 - f_3$ is constant. From [2] it is known that if the contact metric is non-Sasakian, then $2f_1 + 3f_2 - f_3 = 0$. Hence for a three-dimensional non-Sasakian contact metric generalized Sasakian-space-form the scalar curvature is always constant. Now we are in a position to state the following:

Corollary 4.2. A three-dimensional generalized Sasakian-space-form with non-Sasakian contact metric has $\eta-$parallel Ricci tensor if and only if its scalar curvature is constant.

In case of Sasakian manifold the above corollary was proved by Kon [14] in another way.

From (2.11) we see that

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{4.8}$$

where $a = 2nf_1 + 3f_2 - f_3$ and $b = 3f_2 + (2n-1)f_3$. If l^2 denotes the square of the length of the Ricci tensor, then

$$l^2 = \sum_{i=1}^{2n+1} S(Qe_i, e_i), \tag{4.9}$$

where Q is the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor S and $\{e_i\}$, $i = 1, 2, ..., 2n+1$, is an orthonormal basis of the tangent space at each point of the manifold. If we put $X = Y = e_i$ in (4.8) we get

$$r = (2n+1)a + b, \tag{4.10}$$

where r is the scalar curvature of the manifold. Again from (4.8)

$$S(\xi, \xi) = a + b. \tag{4.11}$$

From (4.9), (4.10) and (4.11) it follows that

$$l^2 = 2na^2 + (a+b)^2. \tag{4.12}$$

If the manifold has $\eta-$parallel Ricci tensor then by Theorem 4.3 we have

$$2nf_1 + 3f_2 - f_3 \tag{4.13}$$

is constant. Hence a is constant. If the space-form admits contact metric structure then

$$f_3 - k = f_1, \tag{4.14}$$

where k is a constant. In view of (4.13) and (4.14) we obtain $3f_2 + (2n-1)f_3$ is constant. That is b is constant. Then it follows from (4.10) that r is constant. Also from (4.12) we see that l^2 is constant. Then consequently $\pounds_X l^2 = 0$, where $\pounds_X$ denotes Lie differentiation. Now we know that if a compact Riemannian manifold of dimension greater than 2 with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\pounds_X l^2 = 0$, then it is an Einstein manifold [18]. Thus we see, from (4.8), that if a generalized Sasakian-space-form admits an infinitesimal non-isometric conformal transformation X, then $b = 0$. From this we can conclude the following:

Theorem 4.4. If a $(2n+1)-$dimensional contact metric generalized Sasakian-space-form with $\eta-$parallel Ricci tensor admits an infinitesimal non-isometric conformal transformation, then $f_3 = \frac{3f_2}{1-2n}$.

It is known that a $(2n+1)-$dimensional generalized Sasakian-space-form is Ricci semisymmetric if and only if $f_3 = \frac{3f_2}{1-2n}$ [15]. Thus we are in a position to state the following:

Corollary 4.3. If a $(2n+1)-$dimensional contact metric generalized Sasakian-space-form with $\eta-$parallel Ricci tensor admits an infinitesimal non-isometric conformal transformation then it is Ricci semisymmetric.

5 Three-dimensional quasi-Sasakian generalized Sasakian space-form.

Let us assume that a three-dimensional generalized Sasakian-space-form admits quasi-Sasakian structure. In view of (2.9) we obtain

$$R(X,\xi)\xi = \beta^2(X - \eta(X)\xi). \tag{5.1}$$

But in view of (2.10) it follows that

$$R(X,\xi)\xi = (f_1 - f_3)X. \tag{5.2}$$

$$R(X,\phi X)\xi = 0. \tag{5.3}$$

From (5.1) and (5.2) we see that

$$(f_1 - f_3)X = \beta^2(X - \eta(X)\xi). \tag{5.4}$$

Changing X by ϕX we get from (5.4)

$$(f_1 - f_3) = \beta^2. \tag{5.5}$$

In view of (5.5) we have

Theorem 5.1. In a three-dimensional quasi-Sasakian generalized Sasakian-space-form $f_1 - f_3$ is constant if and only if β is constant.

If $\beta = 0$ the quasi-Sasakian structure becomes cosymplectic. Hence we can state the following corollary:

Corollary 5.1. In a three dimensional cosymplectic generalized Sasakian-space-form $f_1 = f_3$.

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