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Convergence of the Variational Iteration Method for Solving a First-Order Linear System of PDEs with Constant Coefficients

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Abstract : In this paper, we derive the distribution of a desease-model which is not possible to have backward transitons. The distribution is the sums of gamma distributions. In special cases, the results reduce to some AIDS medels and uniform forward model.

Keywords : Systems of PDEs, Variational iteration method, Lagrange multiplier, Convergence, Matrix exponential.
2002 Mathematics Subject Classification : 35E15, 40A30.

1 Introduction

The variational iteration method (VIM) was first proposed by He [1],[2] for solving a wide range of problems whose mathematical models yield ordinary differential equations or systems of ordinary differential equations. The idea of VIM is to construct a correction functional using generalized Lagrange multipliers. The multipliers can be defined using the calculus of variations. The multipliers in the correction functional should be chosen such that the corrected solution is superior to the initial approximation (trial function) and is the best for the given type of trial function. The trial function can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions. The method has been found to give rapidly convergent successive approximations for the solution if such a solution exists. VIM has successfully been applied to many situations. For example, Wazwaz [3] used VIM to solve linear and nonlinear Schrodinger equations. Dehghan and Shakeri [4] applied VIM to solve the Cauchy

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reaction-diffusion problem. Jinbo and Jiang [5] used VIM to solve an inverse parabolic equation. Ramos [6] applied VIM to solve nonlinear partial differential equations. Das [7] used VIM to obtain the solution of a fractional-diffusion equation. Assas [8] applied VIM to solve coupled-KdV equations. Inc [9] used VIM to solve space- and time-fractional Burgers equations with initial conditions. Javidi and Jalilian [10] applied VIM to calculate the wave solution of the Boussinesq equation. Wazwaz [11] applied VIM to some linear and nonlinear systems of PDEs. Batiha et al. [12] used VIM to solve systems of PDEs. Salkuyeh [13] showed convergence of VIM for linear systems of ODEs with constant coefficients. In this paper, we consider linear systems of partial differential equations (PDEs) with constant coefficients of the form:

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} + \frac{\partial U(x,t)}{\partial x} + AU(x,t) &= F(x,t), 0 \le x \le L,\ 0 \le t \le T \\ U(x,0) &= U_0 \end{aligned} \tag{1.1}$$

where, $U(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))^T$ and $u_i(x,t),\ i = 1,2,\ldots,n$ are unknown real functions of the variables x and t, $A \in \mathbf{R}^{n\times n}$ is a constant square matrix, $F(x,t) = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))^T$ are known real functions of x and t and $U_0(x)$ is a given vector function of $x \in \mathbf{R}^n$. We apply VIM for computing an approximate-analytical solution to (1.1) and consider convergence of the variational iteration method.

2 VIM for a system of PDEs

Consider a general system of PDEs,

$$\begin{aligned} L_1(u_1,u_2,\ldots,u_n) + N_1(u_1,u_2,\ldots,u_n) &= f_1 \\ L_2(u_1,u_2,\ldots,u_n) + N_2(u_1,u_2,\ldots,u_n) &= f_2 \\ &\vdots \\ L_n(u_1,u_2,\ldots,u_n) + N_n(u_1,u_2,\ldots,u_n) &= f_n \end{aligned}$$

where $L_1, L_2, \ldots, L_n$ are linear operators, $N_1, N_2, \ldots N_n$ are nonlinear operators, $u_i = u_i(x,t),\ i = 1,2,\ldots,n$ and $f_n = f_n(x,t)$. According to VIM [14]-[18], we can construct a correction functional as follows:

$$
\begin{aligned}
u_{1,m+1} &= u_{1,m} + \int_0^t \lambda_1(s)\{L_1(u_{1,m}, u_{2,m}, \ldots, u_{n,m}) \\
&\qquad + N_1(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \ldots, \tilde{u}_{n,m}) - f_1\}ds \\
u_{2,m+1} &= u_{2,m} + \int_0^t \lambda_2(s)\{L_2(u_{1,m}, u_{2,m}, \ldots, u_{n,m}) \\
&\qquad + N_2(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \ldots, \tilde{u}_{n,m}) - f_2\}ds \\
&\vdots \\
u_{n,m+1} &= u_{n,m} + \int_0^t \lambda_n(s)\{L_1(u_{1,m}, u_{2,m}, \ldots, u_{n,m}) \\
&\qquad + N_n(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \ldots, \tilde{u}_{n,m}) - f_n\}ds \qquad (2.1)
\end{aligned}
$$

where $\tilde{u}(x,t)$ is considered as a restricted variation [14]-[15], i.e., $\delta\tilde{u}_m(x,t) = 0$, the subscript n denotes the nth-order approximation. The optimal value of the generalized Lagrange multipliers $\lambda_1(s), \lambda_2(s), \ldots, \lambda_n(s)$ can be identified optimally [19] via the variational theory by using the stationary conditions in the following form:

$$
\begin{aligned}
\delta u_{1,m+1} &= \delta u_{1,m} + \delta\int_0^t \lambda_1(s)\{L_1(u_{1,m}, u_{2,m}, \ldots, u_{n,m}) \\
&\qquad + N_1(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \ldots, \tilde{u}_{n,m}) - f_1\}ds \\
\delta u_{2,m+1} &= \delta u_{2,m} + \delta\int_0^t \lambda_2(s)\{L_2(u_{1,m}, u_{2,m}, \ldots, u_{n,m}) \\
&\qquad + N_2(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \ldots, \tilde{u}_{n,m}) - f_2\}ds \\
&\vdots \\
\delta u_{n,m+1} &= \delta u_{n,m} + \delta\int_0^t \lambda_n(s)\{L_1(u_{1,m}, u_{2,m}, \ldots, u_{n,m}) \\
&\qquad + N_n(\tilde{u}_{1,m}, \tilde{u}_{2,m}, \ldots, \tilde{u}_{n,m}) - f_n\}ds
\end{aligned}
$$

Finally the solution of the differential equations is considered as the fixed point of the functional (2.1) for the optimal values of the multipliers and the suitable choice of the initial terms U_0.

3 The VIM for solving a linear PDE with constant coefficients

For solving the problem (1.1) by means of the variational iteration method, the matrix $A = (a_{ij})$ is decomposed into two matrices D and B such that $A = D + B$, where $D = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ and $B = A - D$. Then we construct the following correction functional for $U(x,t)$,

$$\begin{aligned} U_{m+1}(x,t) &= U_m(x,t) + \int_0^t \Lambda(s)\Big[\frac{\partial}{\partial s}U_m(x,s) + \frac{\partial}{\partial x}\tilde{U}_m(x,s) \\ &\quad -DU_m(x,s) - B\tilde{U}_m(x,s) - F(x,s)\Big]ds \end{aligned} \tag{3.1}$$

where $\Lambda(s) = \mathrm{diag}(\lambda_1(s), \lambda_2(s), \ldots, \lambda_n(s))$, with $\lambda_i(s), i = 1, 2 \ldots, n$ being the Lagrange multipliers and $\tilde{U}_m(x,t)$ being the restricted variation, i.e., $\delta\tilde{U}_m(x,t) = 0$. Note that although $B\tilde{U}(x,t)$ is not a nonlinear term, we consider it in the same way that the nonlinear term is treated in (2.1). The variation of (3.1) is then:

$$\begin{aligned} \delta U_{m+1}(x,t) &= \delta U_m(x,t) + \delta\int_0^t \Lambda(s)\Big[\frac{\partial}{\partial s}U_m(x,s) \\ &\quad +\frac{\partial}{\partial x}\tilde{U}_m(x,s) - DU_m(x,s) - B\tilde{U}_m(x,s) - F(x,s)\Big]ds \end{aligned}$$

By using integration by parts, we have

$$\begin{aligned} \delta U_{m+1}(x,t) &= \delta U_m(x,t) + \Lambda(s)\delta U_m(x,s)|_{s=t} \\ &\quad +\delta\int_0^t (-\Lambda'(s)U_m(x,s) - \Lambda(s)DU_m(x,s))ds \\ &= (I + \Lambda(s)|_{s=t})\delta U_m(x,t) - \delta\int_0^t (\Lambda' + \Lambda D)U_m(x,s)ds \end{aligned} \tag{3.2}$$

and the stationary conditions are as follows:

$$\begin{aligned} \Lambda(s)' + \Lambda(s)D &= 0, \\ I + \Lambda(s)|_{s=t} &= 0 \end{aligned} \tag{3.3}$$

Here, the prime stands for differentiation with respect to s. Equation (3.3) can be written as

$$\begin{aligned} \lambda_i(s)' + a_{ii}\lambda(s) &= 0, \\ 1 + \lambda(s)|_{s=t} &= 0 \end{aligned} \tag{3.4}$$

We consider two cases. For fixed i, if $a_{ii} = 0$ then it follows that $\lambda_i(s) = -1$ and if $a_{ii} \neq 0$, then $\lambda_i(s) = -e^{-a_{ii}(s-t)}$. Then we have the diagonal exponential matrix $\Lambda(s) = -e^{-(s-t)D}$. Therefore, from (3.1) the following formula for computing $U_m(x,t)$ may be obtained

$$\begin{aligned} U_{m+1}(x,t) &= U_m(x,t) - \int_0^t e^{-(s-t)D}\Big[\frac{\partial}{\partial s}U_m(x,s) \\ &\quad +\frac{\partial}{\partial x}U_m(x,s) - AU_m(x,s) - F(x,s)\Big]ds,\ m = 1, 2, 3, \ldots \end{aligned} \tag{3.5}$$

Now, we show that the sequence $\{U_m(x,t)\}_{m=1}^{\infty}$ given by (3.5) with $U_0(x,t) = U_0$ converges to the exact solution of (1.1). To do this we state and prove the following theorem (4.1).

4 Convergence of the variational iteration method

Theorem 4.1. *Let $U(x,t) \in (C^1(R))^n,\ (x,t) \in R = [0,L] \times [0,T]$ be the exact solution of (1.1) and $U_m(x,t) \in (C^1(R))^n$ be the solutions of the sequence*

$$\begin{aligned} U_{m+1}(x,t) &= U_m(x,t) - \int_0^t e^{-(s-t)D}\Big[\frac{\partial}{\partial s}U_m(x,s) \\ &\quad + \frac{\partial}{\partial x}U_m(x,s) - AU_m(x,s) - F(x,s)\Big]ds \end{aligned} \tag{4.1}$$

with $U_0 = (x,t) = U_0$.

If $E_m(x,t) = U_m(x,t) - U(x,t)$ and $\left\|\frac{\partial}{\partial x}E_m(x,t)\right\|_2 \le \|E_m(x,t)\|_2$ then the functional sequence $\{U_m(x,t)\}_{m=1}^{\infty}$ defined by (4.1) converges to $U(x,t)$.

Proof. Since $U(x,t)$ is the exact solution of (3.5), it is obvious that.

$$\begin{aligned} U(x,t) &= U(x,t) - \int_0^t e^{-(s-t)D}\Big[\frac{\partial}{\partial s}U(x,s) + \frac{\partial}{\partial x}U(x,s) \\ &\quad - AU(x,s) - F(x,s)\Big]ds \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), the error function is

$$\begin{aligned} E_{m+1}(x,t) &= E_m(x,t) - \int_0^t e^{-(s-t)D}\left[\frac{\partial}{\partial s}E_m(x,s) - AE_m(x,s)\right]ds \\ &\quad - \int_0^t e^{-(s-t)D}\frac{\partial}{\partial x}E_m(x,s)ds \end{aligned}$$

By using integration by parts we conclude that

$$\begin{aligned} E_{m+1}(x,t) &= E_m(x,t) - \Big\{e^{-(s-t)D}E_m(x,s)|_0^t + \int_0^t e^{-(s-t)D}DE_m(x,s)ds \\ &\quad - \int_0^t e^{-(s-t)D}AE_m(x,s)ds\Big\} - \int_0^t e^{-(s-t)D}\frac{\partial}{\partial x}E_m(x,s)ds \end{aligned}$$

We know that $E_m(x,0) = 0,\ m = 0,1,\ldots,$ and therefore:

$$E_{m+1}(x,t) = \int_0^t e^{-(s-t)D}BE_m(x,s)ds - \int_0^t e^{-(s-t)D}\frac{\partial}{\partial x}E_m(x,s)ds$$

Therefore

$$\begin{aligned}\Big\|E_{m+1}(x,t)\Big\|_2 &\leq \int_0^t \Big\|e^{-(s-t)D}\Big\|_2\Big\|B\Big\|_2\Big\|E_m(x,s)\Big\|_2 ds \\ &\quad + \int_0^t \Big\|e^{-(s-t)D}\Big\|_2 \Big\|\frac{\partial}{\partial x}E_m(x,s)\Big\|_2 ds \\ &= \Big\|B\Big\|_2 \int_0^t \Big\|e^{-(s-t)D}\Big\|_2\Big\|E_m(x,s)\Big\|_2 ds \\ &\quad + \int_0^t \Big\|e^{-(s-t)D}\Big\|_2 \Big\|\frac{\partial}{\partial x}E_m(x,s)\Big\|_2 ds\end{aligned}$$

Since $\left\|\frac{\partial}{\partial x}E_m(x,t)\right\|_2 \leq \|E_m(x,t)\|_2$ we have

$$\begin{aligned}\|E_{m+1}(x,t)\|_2 \leq \|B\|_2 \int_0^t \Big\|e^{-(s-t)D}\Big\|_2 \|E_m(x,s)\|_2\, ds \\ + \int_0^t \Big\|e^{-(s-t)D}\Big\|_2 \|E_m(x,s)\|_2\, ds\end{aligned} \tag{4.3}$$

Then for $s \leq t \leq T$ we obtain

$$\Big\|e^{-(s-t)D}\Big\|_2 \leq e^{\|-(s-t)D\|_2} = e^{\|s-t\|_2\|D\|_2} \leq e^{2t\max\limits_i|a_{ii}|} \leq e^{2T\max\limits_i|a_{ii}|}$$

Let $M = \left(\|B\|_2 e^{2T\max\limits_i|a_{ii}|} + e^{2T\max\limits_i|a_{ii}|}\right)$, from (4.3) we obtain

$$\Big\|E_{m+1}(x,t)\Big\|_2 \leqslant M \int_0^t \Big\|E_m(x,s)\Big\|_2 ds$$

Now we proceed as follows

$$\begin{aligned}\Big\|E_1(x,t)\Big\|_2 &\leqslant M \int_0^t \Big\|E_0(x,s)\Big\|_2 ds \leqslant M \max_{x\in[0,L],s\in[0,T]}\Big\|E_0(x,s)\Big\|_2 \int_0^t ds \\ &\leqslant M \max_{x\in[0,L],s\in[0,T]}\Big\|E_0(x,s)\Big\|_2 t,\end{aligned}$$

$$\begin{aligned}\Big\|E_2(x,t)\Big\|_2 &\leqslant M \int_0^t \Big\|E_1(x,s)\Big\|_2 ds \leqslant M^2 \int_0^t \max_{x\in[0,L],s\in[0,T]}\Big\|E_0(x,s)\Big\|_2 s ds \\ &= M^2 \max_{x\in[0,L],s\in[0,T]}\Big\|E_0(x,s)\Big\|_2 \frac{t^2}{2!}\end{aligned}$$

$$\Big\|E_3(x,t)\Big\|_2 \leqslant M \int_0^t \Big\|E_2(x,s)\Big\|_2 ds \leqslant M^3 \int_0^t \max_{x\in[0,L],s\in[0,T]}\Big\|E_0(x,s)\Big\|_2 \frac{s^2}{2!} ds$$

$$
\begin{aligned}
&= M^3 \max_{x\in[0,L],s\in[0,T]} \Big\|E_0(x,s)\Big\|_2 \frac{t^3}{3} \\
&\vdots \\
\Big\|E_m(x,t)\Big\|_2 &\leqslant M \int_0^t \Big\|E_{m-1}(x,s)\Big\|_2 ds \\
&\leqslant M^m \int_0^t \max_{x\in[0,L]s\in[0,T]} \Big\|E_0(x,s)\Big\|_2 \frac{s^{m-1}}{(m-1)!} ds \\
&= \max_{x\in[0,L],s\in[0,T]} \Big\|E_0(x,s)\Big\|_2 \frac{(Mt)^m}{m!}
\end{aligned}
$$

We have

$$
\max_{x\in[0,L],s\in[0,T]} \Big\|E_0(x,s)\Big\|_2 \frac{(Mt)^m}{m!} \leqslant \max_{x\in[0,L],s\in[0,T]} \Big\|E_0(x,s)\Big\|_2 \frac{(MT)^m}{m!}
$$

as $m \to \infty$. Therefore $\Big\|E_m(x,t)\Big\|_2 \to 0$ and hence the functional sequence $\{U_m(x,t)\}_{m=1}^{\infty}$ converges to $U(x,t)$.

5 Examples for VIM

To illustrate the effectiveness of the present method, several test examples are considered in this section. The accuracy of the method is assessed by comparison with the exact solutions.

Example 5.1.

Consider the 2×2 linear system of (1)

$$
\begin{aligned}
u_t + v_x &= 0 \\
v_t - u_x &= 0
\end{aligned}
$$

with the initial conditions $u(x,0) = e^x, v(x,0) = 0$ We can transform the linear system into matrix equations of the form of (1.1). We obtain:
$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $F(x,t) = (0,0)^T$, and $U_0 = (e^x, 0)^T$. The exact solution is

$$
U(x,t) = \begin{bmatrix} e^x \cos(t) \\ e^x \sin(t) \end{bmatrix}
$$

By using the proposed variational iteration method which the Lagrange multipliers are $\lambda_1(s), \lambda_2(s) = -1$, we obtain the following sequence of approximating functions $U_n(x,t)$:

$$\begin{aligned} U_1(x,t) &= \begin{bmatrix} e^x \\ e^x t \end{bmatrix}, \qquad\qquad U_2(x,t) = \begin{bmatrix} e^x - \frac{t^2}{2}e^2 \\ e^x t \end{bmatrix} \\ U_3(x,t) &= \begin{bmatrix} e^x - \frac{1}{2}e^x t^2 \\ te^x - \frac{1}{6}e^x t^3 \end{bmatrix} \\ &\vdots \\ U_n(x,t) &= \begin{bmatrix} e^x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots + \frac{t^n}{n!}) \\ e^x(t - \frac{t^3}{3!} + \frac{t^5}{5!} - + \cdots - \frac{t^{n-1}}{(n-1)!}) \end{bmatrix} \end{aligned}$$

Then $\lim_{n\to\infty} U_n(x,t)$ gives the exact solution

$$U(x,t) = \begin{bmatrix} e^x \cos(t) \\ e^x \sin(t) \end{bmatrix}$$

The exact and approximate solutions are depicted in Fig. 1, 2. As we see, there is very good agreement between the approximate solution obtained by 3-iterate the variational iteration method and the exact solution which show in Table 1, 2.

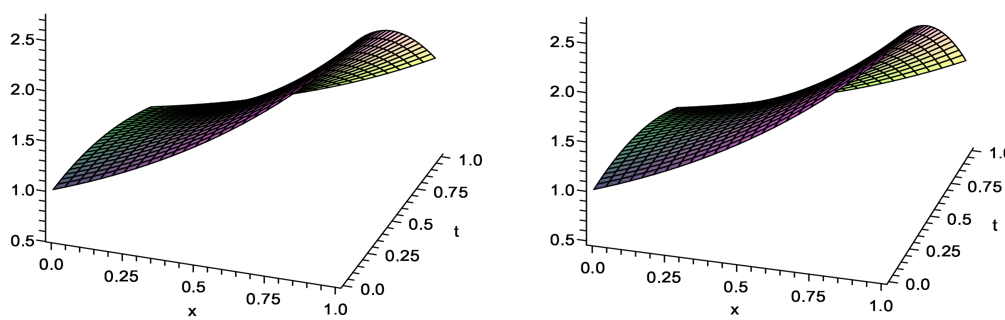

Figure 1: The exact solution for $u(x,t)$ (left) and 3-iterate VIM for $u(x,t)$ (right)

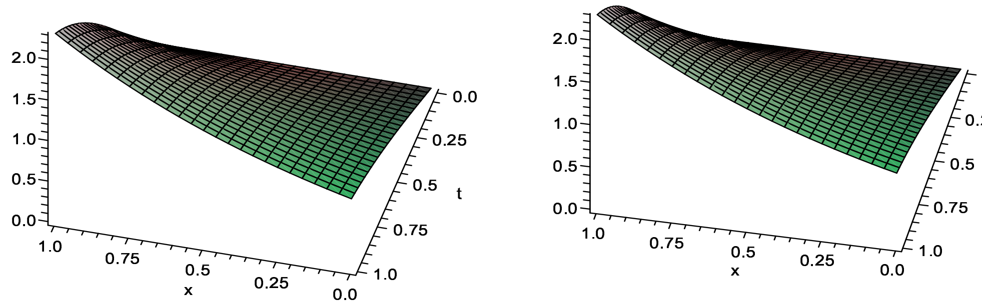

Figure 2: The exact solution for $v(x, t)$ (left) and 3-iterate VIM for $v(x, t)$ (right)

Example 5.2.

Consider the linear system of

$$\begin{aligned} u_t + v_x + u + v &= 0 \\ v_t + u_x - u - v &= 0 \end{aligned}$$

and $u(x, 0) = e^x, v(x, 0) = e^{-x}$. We can transform the linear system into the matrix equation form of (1.1).

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ F(x,t) = (0,0)^T \text{ and } U_0 = \begin{bmatrix} e^x \\ e^{-x} \end{bmatrix}.$$

The exact solution of this problem is

$$U(x,t) = \begin{bmatrix} e^{x-t} \\ e^{-x+t} \end{bmatrix} \tag{5.1}$$

Further calculations will confirm that $U_1 = U_2 = \cdots = \begin{bmatrix} e^{x-t} \\ e^{-x+t} \end{bmatrix}$.

Table 1: The numerical results of example1 for the approximate analytical solutions $u(x,t)$ obtained by 3-iterate VIM for with the exact solutions

x	t	VIM(U_3)	Exact solution	Absolute error
0.0	0.02	0.9998000000	0.9998000067	6.7000e-09
	0.04	0.9992000000	0.9992001067	1.0670e-07
	0.06	0.9982000000	0.9982005399	5.3990e-07
	0.08	0.9968000000	0.9968017063	1.7063e-06
	0.10	0.9950000000	0.9950041653	4.1656e-06
0.2	0.02	1.2211584770	1.2211584860	9.0000e-09
	0.04	1.2204256360	1.2204257660	1.3000e-07
	0.06	1.2192042330	1.2192048920	6.5900e-07
	0.08	1.2174942690	1.2174963530	2.0840e-06
	0.10	1.2152957440	1.2153008320	5.0880e-06
0.4	0.02	1.4915263330	1.4915263430	1.0000e-08
	0.04	1.4906312380	1.4906313970	1.5900e-07
	0.06	1.4891394140	1.4891402190	8.0500e-07
	0.08	1.4870508590	1.4870534040	2.5450e-06
	0.10	1.4843655750	1.4843717880	6.2130e-06
0.6	0.02	1.8217543760	1.8217543880	1.2000e-08
	0.04	1.8206611050	1.8206612990	1.9400e-07
	0.06	1.8188389860	1.8188399700	9.8400e-07
	0.08	1.8162880200	1.8162911290	3.1090e-06
	0.10	1.8130082060	1.8130157960	7.5900e-06
0.8	0.02	2.2250958350	2.2250958350	1.5000e-08
	0.04	2.2237604950	2.2237607330	2.3800e-07
	0.06	2.2215349540	2.2215361560	1.2020e-06
	0.08	2.2184191970	2.2184229940	3.7970e-06
	0.10	2.2144132230	2.2144224930	9.2700e-06
1.0	0.02	2.7177381720	2.7177381900	1.8212e-08
	0.04	2.7161072030	2.7161074930	2.9004e-07
	0.06	2.7133889210	2.7133903880	1.4676e-06
	0.08	2.7095833260	2.7095879640	4.6382e-06
	0.10	2.7046904190	2.7047017410	1.1322e-05

Table 2: The numerical results of example1 for the approximate analytical solutions $v(x,t)$ obtained by 3-iterate VIM for with the exact solutions

x	t	VIM	Exact solution	Absolute error
0	0.02	0.0199986666	0.0199986666	2.0000e-11
	0.04	0.0399893333	0.0399893341	8.6000e-10
	0.06	0.0599640000	0.0599400648	6.4800e-09
	0.08	0.0799146666	0.0799146939	2.7300e-08
	0.1	0.0998333333	0.0998334166	8.3320e-08
0.2	0.02	0.0244264266	0.0244264266	2.0000e-11
	0.04	0.0488430820	0.0488430830	1.0500e-09
	0.06	0.0732401949	0.0732402029	7.9200e-09
	0.08	0.0976079928	0.0976080276	3.3340e-08
	0.10	0.1219367087	0.1219368104	1.0170e-07
0.4	0.02	0.0298345048	0.0298345049	3.0000e-11
	0.04	0.0596570751	0.0596570764	1.2800e-09
	0.06	0.0894557781	0.0894557858	9.6700e-09
	0.08	0.1192186735	0.1192187142	4.0700e-08
	0.10	0.1489338323	0.1488339566	1.2430e-07
0.6	0.02	0.0364399465	0.0364399465	4.0000e-11
	0.04	0.0728653160	0.0728653176	1.5700e-09
	0.06	0.1092615317	0.1092615435	1.1800e-08
	0.08	0.1456140165	0.1456140663	4.9800e-08
	0.10	0.1819081935	0.1819083453	1.5180e-07
0.8	0.02	0.0445078511	0.0445078512	4.1000e-11
	0.04	0.0889978980	0.0889978992	1.9100e-09
	0.06	0.1334523362	0.1334523506	1.4400e-08
	0.08	0.1778533614	0.1778534222	6.0800e-08
	0.1	0.2221831693	0.2221833547	1.8540e-07
1.0	0.02	0.0543020121	0.0543620122	5.4365e-11
	0.04	0.1087022781	0.1087022804	2.3377e-09
	0.06	0.1629990515	0.1629990691	1.7614e-08
	0.08	0.2172305862	0.2172306604	7.4209e-08
	0.1	0.2713751358	0.2713753623	2.2648e-07

6 Conclusion

We have proved a convergence theorem for the variational iteration method of solving a system of partial differential equations with constant coefficients. We have used VIM to solve two examples with known exact solutions. Comparisons with the exact solutions reveal that VIM is very effective and convenient for these examples. In the second example the first iteration of the variational iteration method gave the exact solution. The method can also be implemented easily.

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References

[1] J.H. He, Variational iteration method kind of non-linear analytical technique: Some examples, Int. J. Non-Linear Mech., 34 (1999), 699-708.

[2] J.H. He, Variational iteration method for delay differential equations, Commun. Nonlinear Sci. Numer. Simul., 2 (4) (1997), 235-236.

[3] Abdul-Majid Wazwaz, A study on linear and nonlinear Schrodinger equations by the variational iteration method, Chaos, Solitons and Fractals, 37 (2008), 1136-1142.

[4] Mehdi Dehghan, Fatemeh Shakeri, Application of Hes variational iteration method for solving the Cauchy reactiondiffusion problem, Journal of Computational and Applied Mathematics., 214 (2008) ,435-446.

[5] Liu Jinbo , Tang Jiang, Variational iteration method for solving an inverse parabolic equation, Physics Letters A., 372 (2008) ,3569-3572.

[6] J.I. Ramos, On the variational iteration method and other iterative techniques for nonlinear differential equations, Applied Mathematics and Computation, 199 (2008), 39-69.

[7] S. Das, Analytical solution of a fractional diffusion equation by variational iteration method, Computers and Mathematics with Applications, 57 (2009), 483-487.

[8] Laila M.B. Assas, Variational iteration method for solving coupled-KdV equations, Chaos, Solitons and Fractals, 38 (2008), 1225-1228.

[9] Mustafa Inc, The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method, J. Math. Anal. Appl., 345 (2008), 476-484.

[10] M. Javidi , Y. Jalilian, Exact solitary wave solution of Boussinesq equation by VIM, Chaos, Solitons and Fractals, 36 (2008), 1256-1260.

[11] Abdul-Majid Wazwaz, The variational iteration method for solving linear and nonlinear systems of PDEs, Computers and Mathematics with Applications ,54 (2007), 895-902.

[12] B. Batiha , M.S.M. Noorani , I. Hashim , K. Batiha, Numerical simulations of systems of PDEs by variational iteration method, Physics Letters A ,372 (2008), 822-829.

[13] Davod Khojasteh Salkuyeh, Convergence of the variational iteration method for solving linear systems of ODEs with constant coefficients, Computers and Mathematics with Applications, 56 (2008), 2027-2033.

[14] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput., 114 (23) (2000), 115-123.

[15] J.H. He, Y.Q.Wan, Q. Guo, An iteration formulation for normalized diode characteristics, Int. J. Circuit Theory Appl., 32 (6) (2004), 629-632.

[16] J.H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Modern Phys. B, 20 (10) (2006), 1141-1199.

[17] J.H. He, Non-perturbative methods for strongly nonlinear problems, Dissertation, de-Verlag im Internet GmbH, Berlin, 2006.

[18] J.H. He, Variational iteration methodSome recent results and new interpretations. J. Comput. Appl. Math. (in press).

[19] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Nassed (Ed.),Variational Method in the Mechanics of Solids, Pergamon Press, 1978, pp. 156-162.

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