Thai Journal of Mathematics
Volume 7 (2009) Number 2 : 405–415

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Γ-Group Congruences on *E*-Inversive Γ-Semigroups

S. Sattayaporn

Abstract : In this paper, we give a characterization and some properties of E-inversive Γ-semigroup. Moveover, we also introduce a Γ-group congruence on any E-inversive Γ-semigroup and give its characterizations. Our main results improve and extend many results obtained by Seth [1].

Keywords : E-inversive Γ-semigroup, Γ-group congruence.
2000 Mathematics Subject Classification : 20M10.

1 Introduction

The characterization of Γ-semigroup has been studied by Sen and Saha [3], they gave some characterizations of orthodox Γ-semigroups and extended different results of orthodox semigroups to orthodox Γ-semigroups. They also studied some properties of orthodox Γ-semigroups interm of (α, β)-inverse and regular Γ-semigroups. In 1992, Seth [1] gave the sufficient condition of being Γ-group congruences and the least Γ-group congruence on regular Γ-semigroups. In 2005, Chattopadhyay [4] introduced the concept of right (left) orthodox Γ-semigroup and gave some interesting results of this kind of Γ-semigroup. In this paper, we extend some results of E-inversive semigroup as in [5] to E-inversive Γ-semigroup.

Sen and Saha [3] defined the concepts of Γ-semigroup and regular Γ-semigroup as follows : For two non-empty sets S and Γ, S is said to be a *Γ-semigroup* if for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, (i) $a\alpha b \in S$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$. A Γ-semigroup S is called a *regular Γ-semigroup* if for any $a \in S$ there exist $a' \in S, \alpha, \beta \in \Gamma$ such that $a = a\alpha a'\beta a$. An element $a' \in S$ is called an *(α, β)-inverse* of an element $a \in S$ if $a = a\alpha a'\beta a$ and

$a' = a'\beta a\alpha a'$. In this case, $a' \in V_\alpha^\beta(a)$. If S is regular Γ-semigroup then $V_\alpha^\beta(a) \neq \emptyset$ for some $\alpha, \beta \in \Gamma$. An element $e \in S$ is called an *α-idempotent*, where $\alpha \in \Gamma$, if $e\alpha e = e$. We denote the set of all α-idempotents of S by E_α. Now, for any $a \in S, \alpha, \beta \in \Gamma$ if $a' \in V_\alpha^\beta(a)$ then $a\alpha a'$ is β-idempotent and $a'\beta a$ is α-idempotent. A non-empty set H of Γ-semigroup S is said to be a *Γ-subsemigroup* of S if $H\Gamma H \subseteq H$. In 2005, Siripitukdet and Sattayaporn [5] showed that every regular, orthodox and inverse semigroups are E-inversive semigroup.

A Γ-semigroup S is said to be an *E-inversive Γ-semigroup* if for all $a \in S$ there exist $x \in S, \alpha, \beta \in \Gamma$ such that $a\alpha x$ is β-idempotent. In this research, for $\alpha, \beta \in \Gamma$ we define weak (α, β)-inverse of an element $a \in S$ as follows : $W_\alpha^\beta(a) := \{x \in S \mid x = x\beta a\alpha x\}$ the set of all weak (α, β)-inverses of an element a. In this paper, we replace regular Γ-semigroup in [1] by E-inversive Γ-semigroup and replace the set of all (α, β)-inverses $V_\alpha^\beta(a)$ by the set of all weak (α, β)-inverses $W_\alpha^\beta(a)$ of an E-inversive Γ-semigroup.

Example 1.1. *Let Q^* be the set of all non-zero rational numbers and Γ be the set of all positive integers ($\mathbb{Z}^+$). For $a, b \in Q^*$ and $\alpha \in \Gamma$, we define, $a\alpha b = |a|\alpha b$. We will show that Q^* is Γ-semigroup.*

Let $\frac{p}{q} \in Q^*, p \neq 0, q \neq 0$ and $|p|, |q| \in \Gamma$. Then $E_{\beta=|p|} = \{-\frac{1}{p}, \frac{1}{p}\}$ and $E_{\alpha=|q|} = \{-\frac{1}{q}, \frac{1}{q}\}$. Hence $\frac{1}{|p|} \in Q^*$ and $\frac{p}{q}|q|\frac{1}{|p|} = \frac{|p|}{|q|}|q|\frac{1}{|p|} = 1 \in E_{(1)}$. Therefore Q^* is E-inversive Γ-semigroup and $\frac{1}{|p|} \in W_{|q|}^1(\frac{p}{q})$ where $|p|, |q|, 1 \in \Gamma$.

2 Some Auxiliary Results

In this section, we give some conditions and some results of E-inversive Γ-semigroups.

Proposition 2.1. *S is an E-inversive Γ-semigroup if and only if $W_\alpha^\beta(a) \neq \emptyset$ for all $a \in S$ and for some $\alpha, \beta \in \Gamma$.*

Proof. Suppose that S is an E-inversive Γ-semigroup and $a \in S$. Then there exist $x \in S, \alpha, \beta \in \Gamma$ such that $a\alpha x \in E_\beta$. Thus

$$\begin{aligned} (a\alpha x)\beta(a\alpha x) &= a\alpha x \\ (x\beta a\alpha x)\beta a\alpha(x\beta a\alpha x) &= x\beta(a\alpha x\beta a\alpha x\beta a\alpha x) \;=\; x\beta a\alpha x. \end{aligned}$$

Therefore $x\beta a\alpha x \in W_\alpha^\beta(a)$.
Now, let $a \in S$ and $x \in W_\alpha^\beta(a)$ for some $\alpha, \beta \in \Gamma$. Then $x = x\beta a\alpha x$ and $a\alpha x = (a\alpha x)\beta(a\alpha x)$, hence $a\alpha x \in E_\beta$ and so S is an E-inversive Γ-semigroup. □

Proposition 2.2. *Every regular* Γ*-semigroup is an E-inversive* Γ*-semigroup.*

Proof. Let S be regular Γ-semigroup and $a \in S$. Then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $V_\alpha^\beta(a) \neq \emptyset$. Note that $V_\alpha^\beta(a) \subseteq W_\alpha^\beta(a)$, we have $W_\alpha^\beta(a) \neq \emptyset$. The result is obtained by Theorem 2.1. □

Theorem 2.3. [1] *A regular* Γ*-semigroup S is* Γ*-group if and only if for all* $\alpha, \beta \in \Gamma, e\alpha f = f\alpha e = f$ *and* $e\beta f = f\beta e = e$ *for any* $e \in E_\alpha$ *and* $f \in E_\beta$.

The following definition is needed for our consideration.

Definition 2.4. Let S be a Γ-semigroup and $\alpha \in \Gamma$, and let $H := \{H_\alpha \mid \alpha \in \Gamma\}$ where H_α are subsets of S, for all $\alpha \in \Gamma$. H is called *full and weakly-conjugate family of S* if

(1) $E_\alpha \subseteq H_\alpha$ for all $\alpha \in \Gamma$,
(2) for each $a \in H_\alpha$ and $b \in H_\beta, \alpha, \beta \in \Gamma, a\alpha b \in H_\beta$ and $a\beta b \in H_\alpha$,
(3) for each $a' \in W_\alpha^\beta(a)$ and $c \in H_\gamma, \alpha, \beta, \gamma \in \Gamma, a\alpha c\gamma a', a\gamma c\alpha a' \in H_\beta$ and $a'\beta c\gamma a, a'\gamma c\beta a \in H_\alpha$.

Example 2.5. *By Example 1.1, let* $\frac{p}{q} \in Q^*$ *where* $p, q \neq 0$. *We have* $|q|, |p|, 1 \in \Gamma$ *and*

(1) $E_{|q|} = \{-\frac{1}{q}, \frac{1}{q}\}, E_{|p|} = \{-\frac{1}{p}, \frac{1}{p}\}$. and $E_{(1)} = \{1\}$

$H_{|q|} = \{-\frac{1}{q}, \frac{1}{q}, 1\}$ and $H_{|p|} = \{-\frac{1}{p}, \frac{1}{p}, 1\}$.

Therefore $E_{|q|} \subseteq H_{|q|}, E_{|p|} \subseteq H_{|p|}$ and $H_{(1)} = \{1\} = E_{(1)}$.

(2) Let $\frac{1}{|q|} \in H_{|q|}$ and $\frac{1}{p} \in H_{|p|}$.

Then $\frac{1}{|q|}|q| \cdot \frac{1}{p} = \frac{1}{p} \in H_{|p|}$ and $\frac{1}{|q|}|p| \cdot \frac{1}{p} \in H_{|q|}$.

(3) Let $a = \frac{p}{q} \in Q^*$ where $p, q \neq 0$. Note that $\frac{1}{|p|} \in W_{|q|}^{(1)}(\frac{p}{q})$ and $\frac{1}{p} \in H_{|p|}$, we have $\alpha = |q|, \beta = 1, \gamma = |p|$. Choose $a' = \frac{1}{|p|}$ and $c = \frac{1}{p}$.

Hence $a\alpha c\gamma a', a\gamma c\alpha a' \in H_\beta$ and $a'\alpha c\gamma a, a'\gamma c\alpha a \in H_\alpha$. Let $H = \{H_{|q|}, H_{|p|}, H_{(1)}\}$. Then H is full and weakly-conjugate family of Q^*.

Proposition 2.6. *Let S be an E-inversive Γ-semigroup and $a, b \in S, \theta \in \Gamma$. If $x \in W_{\gamma}^{\delta}(a\theta b)$ for some $\gamma, \delta \in \Gamma$ then $b\gamma x\delta a$ is θ-idempotent of S.*

Proof. Let $x \in W_{\gamma}^{\delta}(a\theta b)$ for some $\gamma, \delta \in \Gamma$. Then $(b\gamma x\delta a)\theta(b\gamma x\delta a) = b\gamma(x\delta a\theta b\gamma x)\delta a = b\gamma x\delta a$, hence $b\gamma x\delta a \in E_{\theta}$. □

4 Main Results

The purpose of this section is to give some characterizations of Γ-group congruences on E-inversive Γ-semigroup and those of the least Γ-group congruence.

Theorem 4.1. *Let S be an E-inversive Γ-semigroup and $H := \{H_{\alpha}, \alpha \in \Gamma\}$ be full and weakly-conjugate family of S. Then*

$$\rho_H := \{(a, b) \in S \times S \mid a\alpha x = y\beta b \text{ for some } x \in H_{\alpha}, y \in H_{\beta} \text{ and } \alpha, \beta \in \Gamma\}$$

is a Γ-group congruence on S.

Proof. Let $a \in S$ and $a' \in W_{\alpha}^{\beta}(a)$ for some $\alpha, \beta \in \Gamma$. Now, $a\alpha(a'\beta a) = (a\alpha a')\beta a$. Since $a'\beta a \in E_{\alpha} \subseteq H_{\alpha}$ and $a\alpha a' \in E_{\beta} \subseteq H_{\beta}$, we have $(a, a) \in \rho_H$. Let $a, b \in S$ and $(a, b) \in \rho_H$. Then there exist $x \in H_{\alpha}$ and $y \in H_{\beta}$ where $\alpha, \beta \in \Gamma$ such that $a\alpha x = y\beta b$. Let $a' \in W_{\gamma}^{\delta}(a)$ and $b' \in W_{\theta}^{\phi}(b)$ for some $\gamma, \delta, \theta, \phi \in \Gamma$. Now, $b\theta[(b'\phi y\beta b)\gamma(a'\delta a)] = [(b\theta b')\phi(a\alpha x\gamma a')]\delta a$. Since $a'\delta a \in E_{\gamma} \subseteq H_{\gamma}$, by Definition 2.4(3), we have $b'\phi y\beta b \in H_{\theta}$, we get $(b'\phi y\beta b)\gamma(a'\delta a) \in H_{\theta}$. Again $b\theta b' \in E_{\phi} \subseteq H_{\phi}$ and by Definition 2.4(3), $a\alpha x\gamma a' \in H_{\delta}$, and by Definition 2.4(2), we have $(b\theta b')\phi(a\alpha x\gamma a') \in H_{\delta}$. Therefore $(b, a) \in \rho_H$.

Let $a, b, c \in S$ be such that $(a, b) \in \rho_H$ and $(b, c) \in \rho_H$. Then there exist $x \in H_{\alpha}, y \in H_{\beta}, z \in H_{\gamma}$ and $w \in H_{\delta}$ for some $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x = y\beta b$ and $b\gamma z = w\delta c$. Now, $a\alpha(x\gamma z) = (a\alpha x)\gamma z = (y\beta b)\gamma z = y\beta(b\gamma z) = y\beta(w\delta c) = (y\beta w)\delta c$. Since $x\gamma z \in H_{\alpha}$ and $y\beta w \in H_{\delta}$, so $(a, c) \in \rho_H$, hence ρ_H is an equivalence relation on S.

To show that ρ_H is compatible, let $(a, b) \in \rho_H$ and $\theta \in \Gamma, c \in S$. Then there exist $x \in H_{\alpha}$ and $y \in H_{\beta}$ for some $\alpha, \beta \in \Gamma$ such that $a\alpha x = y\beta b$. Let $c' \in W_{\gamma}^{\delta}(c)$ and $g \in W_{\gamma_1}^{\delta_1}(b\theta c), h \in W_{\gamma_2}^{\delta_2}(a\theta c)$. By Proposition 2.6, $(c\gamma_2 h\delta_2 a) \in E_{\theta} \subseteq H_{\theta}$, so $(c\gamma_2 h\delta_2 a)\alpha x \in H_{\theta}$ and by Definition 2.4(3), $c'\delta[c\gamma_2 h\delta_2 a\alpha x]\theta c \in H_{\gamma}$. Again $g\delta_1(b\theta c) \in E_{\gamma_1} \subseteq H_{\gamma_1}$ and by Definition 2.4(3), $c'\delta[c\gamma_2 h\delta_2 a\alpha x\theta c]\gamma_1(g\delta_1 b\theta c) \in H_{\gamma}$. Similarly, since $c'\delta c \in E_{\gamma} \subseteq H_{\gamma}$ and by Definition 2.4(2), $y\beta[(b\theta c)\gamma_1 g] \in H_{\delta_1}$ and so $((a\theta c)\gamma c'\delta c\gamma_2 h)\delta_2(y\beta b\theta c\gamma_1 g) \in H_{\delta_1}$.

Now, $(a\theta c)\gamma[c'\delta c\gamma_2 h\delta_2 a\alpha x\theta c\gamma_1 g\delta_1 b\theta c] = [a\theta c\gamma c'\delta c\gamma_2 h\delta_2 y\beta b\theta c\gamma_1 g]\delta_1(b\theta c)$. Therefore $(a\theta c, b\theta c) \in \rho_H$.

Next, we show that $(c\theta a, c\theta b) \in \rho_H$. Let $c' \in W^{\delta}_{\gamma}(c), \theta \in \Gamma$ and $w \in W^{\delta_1}_{\gamma_1}(c\theta b), z \in W^{\delta_2}_{\gamma_2}(c\theta a)$ for some $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \Gamma$. Since $z\delta_2(c\theta a) \in E_{\gamma_2} \subseteq H_{\gamma_2}$ and by Definition 2.4(2), $z\delta_2 c\theta a\alpha x \in H_{\gamma_2}, c\gamma c' \in E_\delta \subseteq H_\delta$, by Definition 2.4(3), $w\delta_1(c\gamma c')\delta(c\theta b) \in H_{\gamma_1}$.

Then $(z\delta_2 c\theta a\alpha x)\gamma_1(w\delta_1(c\gamma c')\delta(c\theta b)) \in H_{\gamma_2}$. Similarly, by Proposition 2.6, $a\gamma_2 z\delta_2 c \in E_\theta \subseteq H_\theta$ and by Definition 2.4(2), $(a\gamma_2 z\delta_2 c)\theta y \in H_\beta$ because $y \in H_\beta$. Again by Proposition 2.6, $b\gamma_1 w\delta_1 c \in E_\theta \subseteq H_\theta$, then $(a\gamma_2 z\delta_2 c\theta y)\beta(b\gamma_1 w\delta_1 c) \in H_\theta$ and so $c\theta(a\gamma_2 z\delta_2 c\theta y\beta b\gamma_1 w\delta_1 c)\gamma c' \in H_\delta$. Now, $(c\theta a)\gamma_2[z\delta_2 c\theta a\alpha x\gamma_1 w\delta_1 c\gamma c'\delta c\theta b] = [c\theta a\gamma_2 z\delta_2 c\theta y\beta b\gamma_1 w\delta_1 c\gamma c']\delta(c\theta b)$. Hence $(c\theta a, c\theta b) \in \rho_H$ and so ρ_H is a congruence on S.

To show that S/ρ_H is Γ-group, we will show that S/ρ_H is a regular Γ-semigroup. Let $a' \in W^{\beta}_{\alpha}(a)$ where $\alpha, \beta \in \Gamma$. Then
$a\alpha(a'\beta a) = a\alpha(a'\beta a\alpha a')\beta a = (a\alpha a')\beta(a\alpha a'\beta a)$. Since $a'\beta a \in E_\alpha \subseteq H_\alpha$ and $a\alpha a' \in E_\beta \subseteq H_\beta$, we get that $(a, a\alpha a'\beta a) \in \rho_H$. Hence S/ρ_H is a regular Γ-semigroup.

Let $\alpha, \beta \in \Gamma$ and $e \in E_\alpha, f \in E_\beta$. Since $E_\alpha \subseteq H_\alpha$ and $E_\beta \subseteq H_\beta$ by Definition 2.4(2), we get $e\alpha f, f\alpha e \in H_\beta$. Now, $(e\alpha f)\beta f = (e\alpha f)\beta f$, hence $(e\alpha f, f) \in \rho_H$ and $(f\alpha e)\beta f = (f\alpha e)\beta f$, hence $(f\alpha e, f) \in \rho_H$. Thus $(e\rho_H)\alpha(f\rho_H) = f\rho_H = (f\rho_H)\alpha(e\rho_H)$. Similarly, we can show that $(e\beta f)\alpha e = (e\beta f)\alpha e$, hence $(e\beta f, e) \in \rho_H$ and $(f\beta e)\alpha e = (f\beta e)\alpha e$, hence $(f\beta e, e) \in \rho_H$. Thus $(e\rho_H)\beta(f\rho_H) = e\rho_H = (f\rho_H)\beta(e\rho_H)$. Therefore S/ρ_H is a Γ-group, and ρ_H is a Γ-group congruence on S. □

The following theorem, we give some characterizations of Γ-group congruence on E-inversive Γ-semigroup S by using a full and weakly-conjugate family of S and the following concept.

Definition 4.2. Let S be Γ-semigroup. If $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of subset of S, the *closure* H_ω of H is the family defined by

$$\begin{aligned} H_\omega :=\ & \{(H_\omega)_\gamma \mid \gamma \in \Gamma\} \quad \text{where} \\ (H_\omega)_\gamma =& \{a \in S \mid h\alpha a \in H_\gamma \text{ for some } h \in H_\alpha, \alpha \in \Gamma\}. \end{aligned}$$

Then H is *closed* if $H = H_\omega$.

Remark. Let S be a Γ-semigroup with $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of S. Then for all $e \in E_\alpha, \alpha \in \Gamma, e\alpha e = e \in E_\alpha \subseteq H_\alpha$, hence $e \in (H_\omega)_\alpha$ and for all $h \in H_\alpha$, if $h\alpha h \in H_\alpha$, we get $H_\alpha \subseteq (H_\omega)_\alpha$ for all $\alpha \in \Gamma$.

Theorem 4.3. *Let S be an E-inversive Γ-semigroup such that $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of S. Then*

$$\rho_H^* := \{(a,b) \in S \times S \mid a\gamma b' \in (H_\omega)_\delta \text{ for some } b' \in W_\gamma^\delta(b)\},$$

hence $\rho_H^* = \rho_H$.

Proof. By Theorem 4.1, $\rho_H := \{(a,b) \in S \times S \mid a\alpha x = y\beta b$ for some $x \in H_\alpha, y \in H_\beta$ and $\alpha, \beta \in \Gamma\}$.

Let $(a,b) \in \rho_H$. Then $(b,a) \in \rho_H$ and there exist $x \in H_\alpha, y \in H_\beta, \alpha, \beta \in \Gamma$ such that $b\alpha x = y\beta a$. Let $b' \in W_\gamma^\delta(b), \gamma, \delta \in \Gamma$. Then $(y\beta a)\gamma b' = (b\alpha x)\gamma b'$. By Definition 2.4(3), $b\alpha x\gamma b' \in H_\delta$. Since $y \in H_\beta$ and $y\beta(a\gamma b') \in H_\delta$, we get that $a\gamma b' \in (H_\omega)_\delta$ and so $(a,b) \in \rho_H^*$, hence $\rho_H \subseteq \rho_H^*$. Let $(a,b) \in \rho_H^*$. Then there exist $b' \in W_\gamma^\delta(b), \gamma, \delta \in \Gamma$ such that $a\gamma b' \in (H_\omega)_\delta$. Then there exist $h \in H, \alpha \in \Gamma$ such that $h\alpha(a\gamma b') \in H_\delta$. Put $f = h\alpha a\gamma b' \in H_\delta$. Note that, $b\theta(a'\phi h\alpha a\gamma b')\delta a = b\theta a'\phi f\delta a$ for some $a' \in W_\theta^\phi(a)$. Since $a'\phi h\alpha a \in H_\theta, h \in H_\alpha, b\theta(a'\phi h\alpha a)\gamma b' \in H_\delta$ and $a'\phi f\delta a \in H_\theta$, it follows that $b\theta(a'\phi f\delta a) = (b\theta a'\phi h\alpha a\gamma b')\delta a$. Hence $(b,a) \in \rho_H$ and $(a,b) \in \rho_H$. Therefore $\rho_H^* \subseteq \rho_H$ and consequently $\rho_H^* = \rho_H$. □

Now, we introduce the concept of the set $Ker\rho$.

Definition 4.4. [1] Let ρ be a congruence on Γ-semigroup S, and let $Ker\rho := \{(Ker\rho)_\alpha, \alpha \in \Gamma\}$ where $(Ker\rho)_\alpha := \{a \in S \mid e\rho a$ for some $e \in E_\alpha\}$.

Example 4.5. *Let ρ be a congruence on Γ-semigroup S with $E_\alpha \neq \emptyset$ for some $\alpha \in \Gamma$. Let $e \in E_\alpha$. Then $e\rho e$ for all $e \in E_\alpha$, and so $e \in (Ker\rho)_\alpha$. Therefore $(Ker\rho)_\alpha \neq \emptyset$.*

Theorem 4.6. *Let S be an E-inversive Γ-semigroup such that $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of S. Then $Ker\rho_H = H_\omega$ where ρ_H defined as in Theorem 4.1.*

Proof. To show that $(Ker\rho_H)_\alpha = (H_\omega)_\alpha$ for all $\alpha \in \Gamma$, let $x \in (Ker\rho_H)_\alpha$ for some $\alpha \in \Gamma$. Then $e\rho_H x$ for some $e \in E_\alpha$ and by Theorem 4.1, then exist $y \in H_\beta, z \in H_\gamma, \beta, \gamma \in \Gamma$ such that $e\beta y = z\gamma x$. Since $e\beta y \in H_\alpha$, we get that $z\gamma x \in H_\alpha$ and so $x \in (H_\omega)_\alpha$. Since $y \in (H_\omega)_\alpha, \alpha \in \Gamma$. then there exist $g \in H_\gamma, \gamma \in \Gamma$ such that $g\gamma y \in H_\alpha$. Now, for some $e \in E_\alpha, e\alpha(g\gamma y) = (e\alpha g)\gamma y$ where $g\gamma y \in H_\alpha$ and $e\alpha g \in H_\gamma$, it follows that $(e,y) \in \rho_H$ and by Definition 4.4, $y \in (Ker\rho_H)_\alpha$. Therefore $(Ker\rho_H)_\alpha = (H_\omega)_\alpha$ for all $\alpha \in \Gamma$. Hence $Ker\rho_H = H_\omega$. □

Theorem 4.7. *Let S be an E-inversive Γ-semigroup such that $H := \{H_\alpha, \alpha \in \Gamma\}$ is a full and weakly-conjugate family of S. Then $a\rho_H b$ if and only if one of the following equivalent conditions hold.*

(1) $a\gamma b' \in (H_\omega)_\delta$ *for some* $b' \in W^\delta_\gamma(b)$,

(2) $b'\delta a \in (H_\omega)_\gamma$ *for some* $b' \in W^\delta_\gamma(b)$,

(3) $a'\phi b \in (H_\omega)_\theta$ *for some* $a' \in W^\phi_\theta(b)$, *and*

(4) $b\theta a' \in (H_\omega)_\phi$ *for some* $a' \in W^\phi_\theta(b)$.

Proof. (1) $\Leftrightarrow$ (3) Let H be a full and weakly-conjugate family of S and suppose that $a\gamma b' \in (H_\omega)\delta$ for some $b' \in W^\delta_\gamma(b)$ where $\alpha, \delta \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h\alpha(a\gamma b') \in H_\delta$. Let $a' \in W^\phi_\theta(a)$ for some $\theta, \phi \in \Gamma$. Then $a'\phi(h\alpha a\gamma b')\delta a \in H_\theta$ and $(a'\phi h\alpha a\gamma b'\delta a)\theta a'\phi b = (a'\phi h\alpha a)\gamma b'\delta a\theta a'\phi b \in H_\theta$ Therefore $a'\phi b \in (H_\omega)_\theta$.

Suppose that $a'\phi b \in (H_\omega)_\theta$ for some $a' \in W^\phi_\theta(a), \theta, \phi \in \Gamma$. Then there exist $h \in H_\beta, \beta \in \Gamma$ such that $h\beta(a'\phi b) \in H_\theta$ and $a\theta(h\beta a'\phi b)\theta a' \in H_\phi$. Therefore for some $b' \in W^\delta_\gamma(b)$,

$$(a\theta h\beta a'\phi b\theta a')\phi(a\gamma b') = (a\theta h\beta a')\phi b\theta(a'\phi a)\gamma b' \in H_\delta.$$

Therefore $a\gamma b' \in (H_\omega)_\delta$.

To show (2) $\Leftrightarrow$ (4), let $b'\delta a \in (H_\omega)_\gamma$ for some $b' \in W^\delta_\gamma(b), \gamma, \delta \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h\alpha(b'\delta a) \in H_\gamma$. Let $a' \in W^\phi_\theta(a)$ for some $\theta, \phi \in \Gamma$. By Definition 2.4(3), $a\gamma(h\alpha b'\delta a)\theta a' \in H_\phi$ and $b'\delta a\theta a'\phi b \in H_\gamma, h\alpha(b'\delta a\theta a'\phi b) \in H_\gamma$, again $a\gamma(h\alpha b'\delta a\theta a'\phi b)\theta a' \in H_\phi$.

Now, $(a\gamma h\alpha b'\delta a\theta a')\phi(b\theta a') = a\gamma(h\alpha b'\delta a\theta a'\phi b)\theta a' \in H_\phi$. Therefore $b\theta a' \in (H_\omega)_\phi$.

Suppose that $b\theta a' \in (H_\omega)_\phi$ for some $a' \in W^\phi_\theta(a), \theta, \phi \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h\alpha(b\theta a') \in H_\phi$. Let $b' \in W^\delta_\gamma(b)$ for some $\gamma, \delta \in \Gamma$. Now, $(b'\delta h\alpha b\theta a'\phi b)\gamma(b'\delta a) = (b'\delta h\alpha b)\theta(a'\phi b\gamma b'\delta a)$. By Definition 2.4(3), $b'\delta(h\alpha b\theta a')\phi b \in H_\gamma$ and $b'\delta h\alpha b \in H_\gamma, a'\phi b\gamma b'\delta a \in H_\theta$. Thus $(b'\delta h\alpha b)\theta(a'\phi b\gamma b'\delta a) \in H_\gamma$, so $(b'\delta h\alpha b\theta a'\phi b)\gamma(b'\delta a) \in H_\gamma$, hence $b'\delta a \in (H_\omega)_\gamma$.

To show (4) $\Leftrightarrow$ (1), let $b\theta a' \in (H_\omega)_\phi$ for some $a' \in W^\phi_\theta(a), \theta, \phi \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h\alpha(b\theta a') \in H_\phi$. Let $b' \in W^\delta_\gamma(b)$ for some $\gamma, \delta \in \Gamma$. By Definition 2.4(3), $b\theta(a'\phi a)\gamma b' \in H_\delta$ and $h\alpha(b\theta a'\phi a\gamma b') \in H_\delta$. Now, $(h\alpha b\theta a')\phi(a\gamma b') = h\alpha(b\theta a'\phi a\gamma b') \in H_\delta$. Therefore $a\gamma b' \in (H_\omega)_\delta$.

Suppose that $a\gamma b' \in (H_\omega)_\delta$ for some $b' \in W^\delta_\gamma(b), \gamma, \delta \in \Gamma$. Then there exist $h \in H_\alpha, \alpha \in \Gamma$ such that $h\alpha(a\gamma b') \in H_\delta$. Let $a' \in W^\phi_\theta(a)$ for some $\theta, \phi \in \Gamma$. Since $b'\delta b \in E_\gamma \subseteq H_\gamma$ and by Definition 2.4(3), $a\gamma(b'\delta b)\theta a' \in H_\phi$ and $h\alpha(a\gamma b'\delta b\theta a' \in H_\phi$. Now $(h\alpha a\gamma b')\delta(b\theta a') = h\alpha(a\gamma b'\delta b\theta a') \in H_\phi$ for some $\theta, \phi \in \Gamma$. Therefore $b\theta a' \in (H_\omega)_\phi$.

Moreover, the symmetric property of ρ_H shows that $a\gamma b' \in (H_\omega)_\delta$ for some (all) $b' \in W_\gamma^\delta(b)$ if and only if $b\theta a' \in (H_\omega)_\phi$ for some (all) $a' \in W_\theta^\phi(a)$. Therefore the proof is completed. $\square$

To prove the least Γ-group congruence on E-inversive Γ-group S by using the smallest element of full and weakly-conjugate family of S. Now the following Lemma easily follows :

Lemma 4.8. *Let $\mathcal{C}$ be the collection of all full and weakly-conjugate families H_i of $S, (i \in \Lambda)$ where $H_i = \{H_{i\alpha}, \alpha \in \Gamma\}$.*

Let $U_\alpha := \bigcap_{i\in\Lambda} H_{i\alpha}$ and $U := \{U_\alpha \mid \alpha \in \Gamma\}$. Then U is a full and weakly-conjugate family of S and U is the smallest element in $\mathcal{C}$.

Proof. Clearly, $E_\alpha \subseteq U_\alpha$ for all $\alpha \in \Gamma$. Let $a \in U_\alpha$ and $b \in U_\beta, \alpha, \beta \in \Gamma$. Then $a \in H_{i\alpha}$ for all $i \in \Lambda$ and $b \in H_{i\beta}$ for all $i \in \Lambda$, and since $H_{i\alpha}, H_{i\beta} \in H_i$ for all $i \in \Lambda$, we get $a\alpha b \in H_{i\beta}$ and $a\beta b \in H_{i\alpha}$ for all $i \in \Lambda$, it implies $a\alpha b \in U_\beta$ and $a\beta b \in U_\alpha$.

If $a' \in W_\alpha^\beta(a)$ and $c \in U_\gamma, \alpha, \beta, \gamma \in \Gamma$, then $c \in H_{i\gamma}$ for all $i \in \Lambda$. Thus $a\alpha c\gamma a', a\gamma c\alpha a' \in H_{i\beta}$ for all $i \in \Lambda$ and $a'\beta c\gamma a, a'\gamma c\beta a \in H_{i\alpha}$ for all $i \in \Lambda$, hence $a\alpha c\gamma a', a\gamma c\alpha a' \in \bigcap_{i\in\Lambda} H_{i\beta} = U_\beta$ and $a'\beta c\gamma a, a'\gamma c\beta a \in \bigcap_{i\in\Lambda} H_{i\alpha} = U_\alpha$.

Therefore U is a full and weakly-conjugate family of S and U is the smallest element in $\mathcal{C}$. $\square$

Theorem 4.9. *Let S be an E-inversive Γ-semigroup. If σ is a Γ-group congruence on S, then $Ker\sigma$ is closed, full and weakly-conjugate of S. Moreover $\sigma = \rho_{Ker\sigma}$.*

Proof. Suppose that σ is a Γ-group congruence on S and let $K = \ker\sigma := \{(Ker\ \sigma)_\alpha, \alpha \in \Gamma\} = \{K_\alpha, \alpha \in \Gamma\}$ where $K_\alpha := \{a \in S \mid e\sigma a$ for some $e \in E_\alpha, \alpha \in \Gamma\}$. Let $e \in E_\alpha, \alpha \in \Gamma$. Then $e\sigma e$ and so $e \in K_\alpha$ for all $\alpha \in \Gamma$. Thus $E_\alpha \subseteq K_\alpha$ for all $\alpha \in \Gamma$. Let $a \in K_\alpha$ and $b \in K_\beta$ for some $\alpha, \beta \in \Gamma$. Then there exist $e \in E_\alpha$ and $f \in E_\beta$ such that $e\sigma a$ and $f\sigma b$. Thus $(a\alpha b)\sigma = (a\sigma)\alpha(b\sigma) = (e\sigma)\alpha(f\sigma) = (e\alpha f)\sigma = f\sigma$, because σ is Γ-group congruence. Then $(a\alpha b, f) \in \sigma$ where $f \in E_\beta$ and $a\alpha b \in K_\beta$. Thus $(a\beta b)\sigma = (a\sigma)\beta(b\sigma) = (e\sigma)\beta(f\sigma) = (e\beta b)\sigma = e\sigma$, because σ is Γ-group congruence. Therefore $(a\beta b, e) \in \sigma$ where $e \in E_\alpha$, hence $a\beta b \in K_\alpha$.

Next, let $a' \in W_\alpha^\beta(a)$ for some $\alpha, \beta \in \Gamma$ and $c \in K_\gamma, \gamma \in \Gamma$. Then there exists $g \in E_\gamma$ such that $(c, g) \in \sigma$. Thus $(a\alpha c\gamma a')\sigma = (a\sigma)\alpha(c\sigma)\gamma(a'\sigma) =$

$(a\sigma)\alpha((g\sigma)\gamma(a'\sigma)) = (a\sigma)\alpha(a'\sigma) = (a\alpha a')\sigma)$ because σ is Γ-group congruence. Therefore $(a\alpha c\gamma a', a\alpha a')$ where $a\alpha a' \in E_\beta$, so $a\alpha c\gamma a' \in K_\beta$. Similarly, we can show that $a\gamma c\alpha a' \in K_\beta$ and $a'\beta c\gamma a$, $a'\gamma c\beta a \in K_\alpha$. Therefore K is full and weakly-conjugate family of S.

To show that $K_\gamma = (K_\omega)_\gamma$ for all $r \in \Gamma$. Clearly, $K_\gamma \subseteq (K_\omega)_\gamma$, by Definition 4.2 and 4.4. To show that $(K_\omega)_\gamma \subseteq K_\gamma$, let $x \in (K_\omega)_\gamma$. Then there exist $h \in K_\alpha, \alpha \in \Gamma$ such that $h\alpha x \in K_\gamma$. Consequently, $(h\alpha x)\sigma = g\sigma$ where $g \in E_\gamma$ or $(h\sigma)\alpha(x\sigma) = g\sigma$. Since $h \in K_\alpha, \alpha \in \Gamma$, we get $(h, e) \in \sigma$ where $e \in E_\alpha$, so $h\sigma = e\sigma$ and $e\sigma$ is an identity of S/σ for all $\alpha, \alpha \in \Gamma$. Then $g\sigma = (h\sigma)\alpha(x\sigma) = (e\sigma)\alpha(x\sigma) = x\sigma$ because σ is Γ-group congruence. Thus $x \in K_\gamma$, hence $K_\gamma = (K_\omega)_\gamma$. To show that $\sigma = \rho_K$, by Theorem 4.3 and K is full and weakly-conjugate family of S, it follows that
$\rho_K := \{(a, b) \in S \times S \mid a\gamma b' \in (K_\omega)_\delta = K_\delta \text{ for some } b' \in W^\delta_\gamma(b), \gamma, \delta \in \Gamma\}$.
Let $(a, b) \in \rho_K$. Then $a\gamma b' \in K_\delta$ for some $b' \in W^\delta_\gamma(b), \gamma, \delta \in \Gamma$. It implies that $(a\gamma b', e) \in \sigma$ where $e \in E_\delta$ and $(a\gamma b'\delta b, e\delta b) \in \sigma$. Since $b'\delta b \in E_\gamma$, we get $a\sigma = (a\sigma)\gamma(b'\delta b)\sigma = (e\sigma)\delta(b\sigma) = b\sigma$, so $(a, b) \in \sigma$ and $\rho_K \subseteq \sigma$.

Finally, we shall show that $\sigma \subseteq \rho_K$, let $(a, b) \in \sigma$ and $b' \in W^\delta_\gamma(b)$ for some $\gamma, \delta \in \Gamma$. Then $(a\gamma b', b\gamma b') \in \sigma$. Since $b\gamma b' \in E_\delta$, we get $a\gamma b' \in E_\delta \subseteq K_\delta$. Thus $(a, b) \in \rho_K$. Therefore $\sigma = \rho_K$. □

Theorem 4.10. *Let S be an E-inversive Γ-semigroup with $H \in \mathcal{C}$ and let ρ_H be defined as in Theorem 4.1. Then ρ_U is the least Γ-group congruence on S and $Ker\rho_U = U_\omega$.*

Proof. Let σ be an arbitrary Γ-group congruence on S. By Theorem 4.9, we obtain $\sigma = \rho_K$ where $K = Ker\sigma$ and K is a full and weakly-conjugate family of S. Since U is the smallest full and weakly-conjugate family of S, we get that $U \subseteq K$.

Let $(a, b) \in \rho_U$. Then there exist $x \in U_\alpha \subseteq K_\alpha, \alpha \in \Gamma$ and $y \in U_\beta \subseteq K_\beta, \beta \in \Gamma$ such that $a\alpha x = y\beta b$. Thus $(a, b) \in \rho_K = \sigma$. Hence ρ_U is the least Γ-group congruence on S. By Theorem 4.6, $Ker\rho_U = U_\omega$. □

Now, we obtain the following theorems for characterizations of Γ-group congruences on E-inversive Γ-semigroups as obtained for regular Γ-semigroup in [1].

Theorem 4.11. *Let S be an E-inversive Γ-semigroup with ρ_H a Γ-group congruence on S where H is a full and weakly-conjugate family of S. The* *following statements are equivalent.*

(1) $a\rho_H b$,

(2) $a\mu x\gamma b' \in H_\delta$, *for some* $x \in H_\mu, \mu \in \Gamma$ *and for some (all)* $b' \in W_\gamma^\delta(b)$,

(3) $a'\phi x\mu b \in H_\theta$, *for some* $x \in H_\mu, \mu \in \Gamma$ *and for some (all)* $a' \in W_\theta^\phi(a)$,

(4) $b\mu x\theta a' \in H_\phi$, *for some* $x \in H_\mu, \mu \in \Gamma$ *and for some (all)* $a' \in W_\theta^\phi(a)$,

(5) $b'\delta x\mu a \in H_\gamma$, *for some* $x \in H_\mu, \mu \in \Gamma$ *and for some (all)* $b' \in W_\gamma^\delta(b)$,

(6) $a\alpha x = y\beta b$ *for some* $\alpha, \beta \in \Gamma$ *and for some* $x \in H_\alpha, y \in H_\beta$,

(7) $x\alpha a = b\beta y$ *for some* $\alpha, \beta \in \Gamma$ *and for some* $x \in H_\alpha, y \in H_\beta$, *and*

(8) $H_\beta\beta a\alpha H_\alpha \cap H_\beta\beta b\alpha H_\alpha \neq \emptyset$ *for some* $\alpha, \beta \in \Gamma$.

Proof. (2) $\Rightarrow$ (3) Suppose that $a\mu x\gamma b' \in H_\delta$ for some $x \in H_\mu$ and $b' \in W_\gamma^\delta(b), \gamma, \delta, \mu \in \Gamma$. If $a' \in W_\theta^\phi(a)$ for some $\theta, \phi \in \Gamma$, then $a'\phi a \in E_\theta \subseteq H_\theta$ and $b'\delta x\mu b \in H_\gamma$. Since $(a\mu x\gamma b')\delta x \in H_\mu$ and $x\gamma(b'\delta x\mu b) \in H_\mu$, we have $a'\phi(a\mu x\gamma b'\delta x)\mu b = (a'\phi a)\mu(x\gamma(b'\delta x\mu b)) \in H_\theta$.

(3) $\Rightarrow$ (6) Let $a'\phi x\mu b \in H_\theta$, for some $a' \in W_\theta^\phi(a)$ and $x \in H_\mu, \theta, \phi, \mu \in \Gamma$. Thus $a\theta(a'\phi x\mu b) = (a\theta a'\phi x)\mu b$, where $a'\phi x\mu b \in H_\theta$ and $a\theta a'\phi x \in H_\mu$. Hence (6) holds.

(6) $\Rightarrow$ (8) Let $a\alpha x = y\beta b$ for some $\alpha, \beta \in \Gamma$ and $x \in H_\alpha, y \in H_\beta$. Then $y\beta(a\alpha x) = (y\beta b)\alpha x$. Since $y\beta a\alpha x \in H_\beta\beta a\alpha H_\alpha$ and $y\beta b\alpha x \in H_\beta\beta b\alpha H_\alpha$, we get that $H_\beta\beta a\alpha H_\alpha \cap H_\beta\beta b\alpha H_\alpha \neq \emptyset$, for some $\alpha, \beta \in \Gamma$.

(8) $\Rightarrow$ (2) Let $H_\beta\beta a\alpha H_\alpha \cap H_\beta\beta b\alpha H_\alpha \neq \emptyset$ for some $\alpha, \beta \in \Gamma$. Then $x\beta a\alpha y = x_1\beta b\alpha y_1$ for some $x, x_1 \in H_\beta$ and $y, y_1 \in H_\alpha$. Let $a' \in W_\theta^\phi(a)$ for some $\theta, \phi \in \Gamma$ and $b' \in W_\gamma^\delta(b)$ for some $\gamma, \delta \in \Gamma$. Then $a'\phi x\beta a \in H_\theta$ and $(a'\phi x\beta a)\alpha y \in H_\theta$. Since $a\theta a' \in H_\phi, a\theta a'\phi x_1 \in H_\beta$ and $b\alpha y_1\gamma b' \in H_\delta$, we get that $(a\theta a'\phi x_1)\beta(b\alpha y_1\gamma b') \in H_\delta$. Then $a\theta(a'\phi x\beta a\alpha y)\gamma b' = (a\theta a')\phi(x\beta a\alpha y)\gamma b' = (a\theta a')\phi(x_1\beta b\alpha y_1)\gamma b' = (a\theta a')\phi x_1\beta(b\alpha y_1\gamma b') \in H_\delta$, hence (2), (3), (6) and (8) are equivalent.

(2) $\Rightarrow$ (4) Suppose that $a\mu x\gamma b' \in H_\delta$ for some $x \in H_\mu$ and $b' \in W_\gamma^\delta(b), \gamma, \delta, \mu \in \Gamma$. Let $b' \in W_\gamma^\delta(b)$ for $\gamma, \delta \in \Gamma$.
Now, $b\mu(x\theta a'\phi a\mu x\gamma b'\delta a)\theta a' = (b\mu x\theta a'\phi a\mu x\gamma b')\delta(a\theta a')$. By Definition 2.4 (3), we have $a'\phi(a\mu x\gamma b')\delta a \in H_\theta$, so $x\theta(a'\phi a\mu x\gamma b'\delta a) \in H_\mu$. Since $a\theta a' \in H_\phi$ and again, Definition 2.4(3), we have $b\mu x\theta a'\phi a\mu x\gamma b') \in H_\delta$, then $(b\mu x\theta a'\phi a\mu x\gamma b')\delta(a\theta a') \in H_\phi$. Hence $b\mu(x\theta a'\phi a\mu x\gamma b'\delta a)\theta a' \in H_\phi$.

(4) $\Rightarrow$ (5) Suppose that $b\mu x\theta a' \in H_\phi$ for some $x \in H_\mu$ and $a' \in W_\theta^\phi(a), \theta, \phi, \mu \in \Gamma$. Let $b' \in W_\gamma^\delta(b)$ for some $\gamma, \delta \in \Gamma$.
Now, $b'\delta(b\mu x\theta a'\phi x)\mu a = (b'\delta b)\mu x\theta(a'\phi x\mu a)$. Since $(b\mu x\theta a')\phi x \in H_\mu$ and $b'\delta b \in H_\gamma, a'\phi x\mu a \in H_\theta$, we get that $(b'\delta b)\mu[x\theta(a'\phi x\mu a)] \in H_\gamma$. Hence

$b'\delta(b\mu x\theta a'\phi x)\mu a \in H_\gamma$.

(5) $\Rightarrow$ (7) Let $b'\delta x\mu a \in H_\gamma$ for some $b' \in W_\gamma^\delta(b), x \in H_\mu$ and $\gamma, \delta, \mu \in \Gamma$. Now, $(b\gamma b'\delta x)\mu a = b\gamma(b'\delta x\mu a)$. Since $b\gamma b'\delta x \in H_\mu$ and $b'\delta x\mu a \in H_\gamma$, we have (7).

(7) $\Rightarrow$ (1) Let $x\alpha a = b\beta y$ for some $\alpha, \beta \in \Gamma$ and $x \in H_\alpha, y \in H_\beta$. Let $a' \in W_\theta^\phi(a)$ for some $\theta, \phi \in \Gamma$ and $b' \in W_\gamma^\delta(b)$. Now, $a\theta(a'\phi x\alpha a\gamma b'\delta b) = (a\theta a'\phi b\beta y\gamma b')\delta b$. Since $b'\delta b \in H_\gamma$ and $a'\phi x\alpha a \in H_\theta$, we have $(a\phi x\alpha a)\gamma(b'\delta b) \in H_\theta$. Since $a\theta a' \in H_\phi$ and $b\beta y\gamma b' \in H_\delta$, we have $(a\theta a')\phi(b\beta y\gamma b') \in H_\delta$. Then $(a, b) \in \rho_H$. Hence (2), (4), (5) and (7) are equivalent.

Also (1) $\Leftrightarrow$ (6) by Theorem 4.1. □

References

[1] A. Seth, Γ-group Congruences on Regular Γ-semigroups. *Internat. J. of Math. & Math. Sci.* Vol. 15 No. 1, (1992). 103-106.

[2] M.K. Sen, & N.K. Saha, Orthodox Γ-semigroups. *Internat. J. of Math. & Math. Sci.* Vol. 13 No. 3 (1990), 527-534.

[3] M.K. Sen, & N.K. Saha, On Γ-semigroup. *Bull. Cal. Math. Soc.* **78** (1986), 180-186.

[4] S. Chattopadhyay, Right Orthodox Γ-semigroup. *Southeast Asian Bull. of Math.*, **29** (2005), 23-30.

[5] M. Siripitukdet, & S. Sattayaporn, The Least Group Congruence on *E*-inversive semigroups and *E*-inversive *E*-semigroups. *Thai Journal of Mathematics.* Vol. 3 No. 2 (2005), 163-169.

(Received 25 August 2009)

Supavinee Sattayaporn
Department of Mathematics
Faculty Science and technology
Uttaradit Rajabhat University
Uttaradit 53000, Thailand.
e-mail : supavinee_uru@windowslive.com