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Extension Condition to Evaluate Solution of the Regularized Long-Wave Equation

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Abstract : Extend the condition to constructing the three-solitary wave solution of the regularized long-wave (RLW) equation which applies the Hirota direct method to obtain analytic solutions of the RLW equation. Considering a transformation of the RLW equation to the Hirota bilinear form and applying the Hirota perturbation to this equation. The exact three-solitary wave solutions of the RLW obtained results.

Keywords : Hirota direct method; Hirota bilinear form; Hirota D-operator; Regularized long- wave equation; Solitary wave **2000 Mathematics Subject Classification :** 47H09; 47H10 (2000 MSC)

1 Introduction

The regularized long-wave (RLW) equation is a nonlinear partial differential equation which is non-integrable equation. In 1834, Scott Russell [6] observed the occurring of a solitary-wave. Many people described the phenomenon behavior of the nonlinear wave equation. Such as, in 1845, Biddell Airy [3] found speed of wave formula that relating its height and amplitude and concluded that the solitary wave solution does not exist. In 1895, Korteweg -de Vries (KdV) [2] derived a nonlinear wave equation is called KdV equation that is a shallow water wave and exists the solitary wave solution too. In 1971, Ryogo Hirota [5] discovered the Hirota direct method, who applied the method to construct multi-solitons of the KdV equation. In 2006, Ali Pekcan [1] applied the Hirota direct method to find solutions of non-integrable equations and described the extensions of many equations. In this paper, we applied the Hirota method to construct three-solitary wave solution of the RLW equation which in Sirirat Suksai[7] showed one and two-solitary wave solutions. This equation is

$$u_t + uu_x - u_{xxt} = 0 \tag{1.1}$$

In section 2, we describe background of the Hirota method, show theorem to satisfy the solution of this equation and transform this equation to the Hirota bilinear form. In section 3, we find the three-solitary wave solution of the RLW equation. The last section, we conclude progress of research.

2 Preliminaries

2.1 The Hirota Direct Method

We reviewed the Hirota direct method in four steps by following Hietarinta's article [2] and Peckan [1] . Let $F[u] = F(u, u_x, u_t) = 0$ be a nonlinear partial differential equation.

Corollary 2.1. Let S a space of differentiable functions from $\mathbb{C}^2 \to \mathbb{C}$. Then Hirota D-operator $D : S \to S$ is defined as

$$[D_x^m D_t^n]\{f \cdot g\} = [(\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n] f(x,t) \cdot g(x', t')|_{x'=x, t'=t} \tag{2.1}$$

where m, n are positive integers and x, t are independent variables.

By using some sort of combination of the Hirota D-operator, we try to write the bilinear form of $F[U]$ as a polynomial of D-operator. We call this polynomial $P(D)$.

2.2 The Hirota Perturbation and the Multi-Solitary Wave Solutions

Here we consider the nonlinear partial differential or difference equation $F[u] = 0$ whose the Hirota bilinear form is in the form $P(D)\{f \cdot f\} = 0$ and we give the steps involved in finding exact solutions of $F[u] = 0$ by using its Hirota bilinear form. We shall use the perturbation expansions. For this purpose, let us write $f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \ldots$ where f_0 is a constant, $f_m, m = 1, 2, \ldots$ are functions of x, t, and ε is a constant called the perturbation parameter. Without loss of generality, we take $f_0 = 1$. So the product $f \cdot f$ becomes

$$\begin{aligned} f \cdot f &= 1 \cdot 1 + \varepsilon(1 \cdot f_1 + f_1 \cdot 1) + \varepsilon^2 (f_1 \cdot f_1 + 1 \cdot f_2 + f_2 \cdot 1) \\ &\quad + \varepsilon^3 (1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) + \ldots \end{aligned} \tag{2.2}$$

Substituting this expression into $P(D)\{f \cdot f\} = 0$ and using the linearity of the polynomial $P(D)$, we get

$$\begin{aligned} P(D)\{f \cdot f\} &= P(D)\{1 \cdot 1\} + \varepsilon P(D)\{f_1 \cdot 1 + 1 \cdot f_1\} + \varepsilon^2 P(D)\{f_2 \cdot 1 + f_1 \cdot f_1 \\ &\quad + 1 \cdot f_2\} + \varepsilon^3 P(D)\{1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1\} \\ &\quad + \ldots = 0. \end{aligned} \tag{2.3}$$

To satisfy this equation we make the coefficients of $\varepsilon^m, m = 0, 1, 2, ...$ to vanish. The coefficient of ε^0 is trivially zero. From the coefficient of ε^1 we have

$$P(D)\{f_1 \cdot 1 + 1 \cdot f_1\} = 2P(\partial)f_1 = 0. \tag{2.4}$$

One of the solutions of this equation is the exponential function. While we are applying the Hirota direct method we take f_1 as exponential function and so the other f_i's will also come as exponential functions. The effectiveness of the Hirota direct method reveals at this point. Since we will write f as a polynomial of exponential functions when we consider s-solitary wave solutions of an equation $F[u] = 0$,f_j for all $j \geq s + 1$ will be zero. So here after while we are construction s-solitary wave solutions of an equation we will assume that $f_j = 0$ for all $j \geq s+1$.

3 Main Results

Theorem 3.1. *Let $u = T[f(x, t)]$ be a bilinearizing transformation of a nonlinear partial differential or difference equation $F[u] = 0$, which can be written in the Hirota bilinear form $P(D)\{f \cdot f\} = 0$. Then if $F[u] = 0$ satisfies the three-solitary wave solution condition which is*

$$\sum_{\delta_i=\pm 1} P(\delta_1 p_1 + \delta_2 p_2 + \delta_3 p_3)P(\delta_1 p_1 - \delta_2 p_2)P(\delta_2 p_2 - \delta_3 p_3)P(\delta_3 p_3 - \delta_1 p_1) = 0 \tag{3.1}$$

with $P(p_i) = 0$, i=1,2,3 then its three-solitary wave solution is

$$\begin{aligned} u &= T[f(x, t)] \\ &= T[1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1, 2)e^{\theta_1+\theta_2} + A(1, 3)e^{\theta_1+\theta_3} \\ &\quad + A(2, 3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}] \end{aligned} \tag{3.2}$$

where $\theta_i x + \omega_i t + \alpha_i$, i=1,2,3. Here $A(i, j) = -\frac{P(p_i-p_j)}{P(p_i+p_j)}$ for $i, j = 1, 2, 3, i < j$ and $B = A(1, 2)A(1, 3)A(2, 3)$.

Proof. To construct three-solitary wave solution of $F[u] = 0$ we take $f = 1+\varepsilon f_1+ \varepsilon^2 f_2 + \varepsilon^3 f_3$ where $f_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3}$ with $\theta_i = k_i x + \omega_i t + \alpha_i$, $i = 1, 2, 3$. Note that $f_j = 0$ for all $j \geq 4$. Now we insert f into the equation (3.1.2) and we make the coefficients of $\varepsilon^m, m = 0, 1, 2, ..., 6$ to vanish. The coefficient of ε^0 is

$$P(D)\{1 \cdot 1\} = P(0, 0)1 = 0 \tag{3.3}$$

and it trivially zero. From the coefficient of ε^1 which is

$$\begin{aligned} P(D)\{1 \cdot f_1 + f_1 \cdot 1\} &= 0 \\ 2P(\partial)f_1 &= 0 \\ 2P(\partial)\{e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\} &= 0 \\ 2[P(\partial)e^{\theta_1} + P(\partial)e^{\theta_2} + P(\partial)e^{\theta_3}] &= 0 \\ 2[P(p_1)e^{\theta_1} + P(p_2)e^{\theta_2} + P(p_3)e^{\theta_3}] &= 0 \end{aligned} \tag{3.4}$$

we have $P(p_i) = 0$ for $i = 1, 2, 3$. From the coefficient of ε^2, we get

$$\begin{aligned} P(D)\{1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1\} &= 0 \\ P(D)\{1 \cdot f_2 + f_2 \cdot 1\} + P(D)\{f_1 \cdot f_1\} &= 0 \\ 2P(\partial)f_2 + P(D)\{f_1 \cdot f_1\} &= 0 \\ -2P(\partial)f_2 &= P(D)\{f_1 \cdot f_1\} \qquad (3.5) \end{aligned}$$

where

$$\begin{aligned} f_1 \cdot f_1 &= (e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \cdot (e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \\ &= e^{\theta_1} \cdot e^{\theta_1} + e^{\theta_1} \cdot e^{\theta_2} + e^{\theta_1} \cdot e^{\theta_3} + e^{\theta_2} \cdot e^{\theta_1} + e^{\theta_2} \cdot e^{\theta_2} \\ &\quad + e^{\theta_2} \cdot e^{\theta_3} + e^{\theta_3} \cdot e^{\theta_1} + e^{\theta_3} \cdot e^{\theta_2} + e^{\theta_3} \cdot e^{\theta_3} \\ &= e^{\theta_1} \cdot e^{\theta_1} + e^{\theta_2} \cdot e^{\theta_2} + e^{\theta_3} \cdot e^{\theta_3} + 2e^{\theta_1} \cdot e^{\theta_2} + 2e^{\theta_1} \cdot e^{\theta_3} + 2e^{\theta_2} \cdot e^{\theta_3}. \end{aligned}$$

Inserting this expression into the equation (3.2.12) we obtain

$$\begin{aligned} -2P(\partial)f_2 &= P(D)\{e^{\theta_1} \cdot e^{\theta_1}\} + P(D)\{e^{\theta_2} \cdot e^{\theta_2}\} + P(D)\{e^{\theta_3} \cdot e^{\theta_3}\} \\ &\quad + 2P(D)\{e^{\theta_1} \cdot e^{\theta_2}\} + 2P(D)\{e^{\theta_1} \cdot e^{\theta_3}\} + 2P(D)\{e^{\theta_2} \cdot e^{\theta_3}\} \\ &= P(p_1 - p_1)e^{\theta_1+\theta_1} + P(p_2 - p_2)e^{\theta_2+\theta_2} + P(p_3 - p_3)e^{\theta_3+\theta_3} \\ &\quad + 2[P(p_1 - p_2)e^{\theta_1+\theta_2} + P(p_1 - p_3)e^{\theta_1+\theta_3} + P(p_2 - p_3)e^{\theta_2+\theta_3}] \\ &= 2[P(p_1 - p_2)e^{\theta_1+\theta_2} + P(p_1 - p_3)e^{\theta_1+\theta_3} + P(p_2 - p_3)e^{\theta_2+\theta_3}]. \end{aligned} \qquad (3.6)$$

Hence f_2 has the form $f_2 = A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}$. After substituting f_2 into the equation (2.2.13), we define $A(i,j)$ as

$$\begin{aligned} &-2P(\partial)[A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}] \\ &\qquad = 2[P(p_1 - p_2)e^{\theta_1+\theta_2} + P(p_1 - p_3)e^{\theta_1+\theta_3} + P(p_2 - p_3)e^{\theta_2+\theta_3}] \end{aligned}$$

$$\begin{aligned} &A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3} \\ &\qquad = -\frac{[P(p_1 - p_2)e^{\theta_1+\theta_2} + P(p_1 - p_3)e^{\theta_1+\theta_3} + P(p_2 - p_3)e^{\theta_2+\theta_3}]}{P(\partial)} \\ &\qquad = -\frac{P(p_1 - p_2)}{P(\partial)}e^{\theta_1+\theta_2} - \frac{P(p_1 - p_3)}{P(\partial)}e^{\theta_1+\theta_3} - \frac{P(p_2 - p_3)}{P(\partial)}e^{\theta_2+\theta_3} \\ &\qquad = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)}e^{\theta_1+\theta_2} - \frac{P(p_1 - p_3)}{P(p_1 + p_3)}e^{\theta_1+\theta_3} - \frac{P(p_2 - p_3)}{P(p_2 + p_3)}e^{\theta_2+\theta_3} \end{aligned}$$

therefore, we get

$$A(1,2) = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)},$$

$$A(1,3) = -\frac{P(p_1 - p_3)}{P(p_1 + p_3)},$$

and

$$A(2,3) = -\frac{P(p_2 - p_3)}{P(p_2 + p_3)}.$$

Hence, there are in this form

$$A(i,j) = -\frac{P(p_i - p_j)}{P(p_i + p_j)} \tag{3.7}$$

for $i, j = 1, 2, 3, i < j$. The coefficient of ε^3 gives us

$$\begin{aligned} P(D)\{1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1\} &= 0 \\ P(D)\{1 \cdot f_3 + f_3 \cdot 1\} + P(D)\{f_1 \cdot f_2 + f_2 \cdot f_1\} &= 0 \end{aligned}$$

thus, we obtain

$$\begin{aligned} 2P(\partial)f_3 + 2P(D)\{f_1 \cdot f_2\} &= 0 \\ -P(\partial)f_3 &= P(D)\{f_1 \cdot f_2\} \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} f_1 \cdot f_2 &= (e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \cdot [A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}] \\ &= A(1,2)[e^{\theta_1} \cdot e^{\theta_1+\theta_2} + e^{\theta_2} \cdot e^{\theta_1+\theta_2} + e^{\theta_3} \cdot e^{\theta_1+\theta_2}] \\ &\quad + A(1,3)[e^{\theta_1} \cdot e^{\theta_1+\theta_3} + e^{\theta_2} \cdot e^{\theta_1+\theta_3} + e^{\theta_3} \cdot e^{\theta_1+\theta_3}] \\ &\quad + A(2,3)[e^{\theta_1} \cdot e^{\theta_2+\theta_3} + e^{\theta_2} \cdot e^{\theta_2+\theta_3} + e^{\theta_3} \cdot e^{\theta_2+\theta_3}] \\ P(D)\{f_1 \cdot f_2\} &= A(1,2)P(D)[e^{\theta_1} \cdot e^{\theta_1+\theta_2} + e^{\theta_2} \cdot e^{\theta_1+\theta_2} + e^{\theta_3} \cdot e^{\theta_1+\theta_2}] \\ &\quad + A(1,3)P(D)[e^{\theta_1} \cdot e^{\theta_1+\theta_3} + e^{\theta_2} \cdot e^{\theta_1+\theta_3} + e^{\theta_3} \cdot e^{\theta_1+\theta_3}] \\ &\quad + A(2,3)P(D)[e^{\theta_1} \cdot e^{\theta_2+\theta_3} + e^{\theta_2} \cdot e^{\theta_2+\theta_3} + e^{\theta_3} \cdot e^{\theta_2+\theta_3}]. \end{aligned} \tag{3.9}$$

Hence from equation (3.2.15), becomes

$$\begin{aligned} -P(\partial)f_3 &= A(1,2)P(D)[e^{\theta_1} \cdot e^{\theta_1+\theta_2} + e^{\theta_2} \cdot e^{\theta_1+\theta_2} + e^{\theta_3} \cdot e^{\theta_1+\theta_2}] \\ &\quad + A(1,3)P(D)[e^{\theta_1} \cdot e^{\theta_1+\theta_3} + e^{\theta_2} \cdot e^{\theta_1+\theta_3} + e^{\theta_3} \cdot e^{\theta_1+\theta_3}] \\ &\quad + A(2,3)P(D)[e^{\theta_1} \cdot e^{\theta_2+\theta_3} + e^{\theta_2} \cdot e^{\theta_2+\theta_3} + e^{\theta_3} \cdot e^{\theta_2+\theta_3}] \\ &= A(1,2)P(D)[e^{\theta_3} \cdot e^{\theta_1+\theta_2}] + A(1,3)P(D)[e^{\theta_2} \cdot e^{\theta_1+\theta_3}] \\ &\quad + A(2,3)P(D)[e^{\theta_1} \cdot e^{\theta_2+\theta_3}] \\ &= [A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) \\ &\quad + A(2,3)P(p_1 - p_2 - p_3)]e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{3.10}$$

Note that f_3 should have the form $f_3 = Be^{\theta_1+\theta_2+\theta_3}$. We determine B from the above equation as

$$B = -\frac{A(1,2)P(p_3 - p_1 - p_2) + A(1,3)P(p_2 - p_1 - p_3) + A(2,3)P(p_1 - p_2 - p_3)}{P(p_1 + p_2 + p_3)}. \tag{3.11}$$

Since $f_4 = 0$, the coefficient of ε^4 becomes

$$\begin{aligned} \{1 \cdot f_4 + f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1 + f_4 \cdot 1\} &= 0 \\ P(D)\{f_1 \cdot f_3 + f_3 \cdot f_1 + f_2 \cdot f_2\} &= 0 \\ 2P(D)\{f_1 \cdot f_3\} + P(D)\{f_2 \cdot f_2\} &= 0 \qquad (3.12) \end{aligned}$$

where $P(D)\{f_1 \cdot f_3\}$ and $P(D)\{f_2 \cdot f_2\}$ are simplified as

$$\begin{aligned} f_1 \cdot f_3 &= (e^{\theta_1} + e^{\theta_2} + e^{\theta_3}) \cdot (Be^{\theta_1+\theta_2+\theta_3}) \\ &= B(e^{\theta_1} \cdot e^{\theta_1+\theta_2+\theta_3} + e^{\theta_2} \cdot e^{\theta_1+\theta_2+\theta_3} + e^{\theta_3} \cdot e^{\theta_1+\theta_2+\theta_3}) \end{aligned}$$

$$\begin{aligned} P(D)\{f_1 \cdot f_3\} &= BP(D)(e^{\theta_1} \cdot e^{\theta_1+\theta_2+\theta_3} + e^{\theta_2} \cdot e^{\theta_1+\theta_2+\theta_3} + e^{\theta_3} \cdot e^{\theta_1+\theta_2+\theta_3}) \\ &= B[P(p_1 - (p_1 - p_2 - p_3))e^{2\theta_1+\theta_2+\theta_3} \qquad (3.13) \\ &\quad + P(p_2 - (p_2 - p_1 - p_3))e^{\theta_1+2\theta_2+\theta_3} \\ &\quad + P(p_3 - (p_3 - p_1 - p_2))e^{\theta_1+\theta_2+2\theta_3}] \\ &= B[P(p_2 + p_3)e^{2\theta_1+\theta_2+\theta_3} + P(p_1 + p_3)e^{\theta_1+2\theta_2+\theta_3} \\ &\quad + P(p_1 + p_2)e^{\theta_1+\theta_2+2\theta_3}] \qquad (3.14) \end{aligned}$$

and

$$\begin{aligned} f_2 \cdot f_2 &= [A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}] \cdot \\ &\quad [A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} + A(2,3)e^{\theta_2+\theta_3}] \\ &= A^2(1,2)e^{2\theta_1+2\theta_2} + 2A(1,2)A(1,3)e^{2\theta_1+\theta_2+\theta_3} \\ &\quad + 2A(1,2)A(2,3)e^{\theta_1+2\theta_2+\theta_3} + A^2(1,3)e^{2\theta_1+2\theta_3} \\ &\quad + 2A(1,3)A(2,3)e^{\theta_1+\theta_2+2\theta_3} + A^2(2,3)e^{2\theta_2+2\theta_3} \end{aligned}$$

$$\begin{aligned} P(D)\{f_2 \cdot f_2\} &= A^2(1,2)P(D)e^{2\theta_1+2\theta_2} + 2A(1,2)A(1,3)P(D)e^{2\theta_1+\theta_2+\theta_3} \\ &\quad + 2A(1,2)A(2,3)P(D)e^{\theta_1+2\theta_2+\theta_3} + A^2(1,3)P(D)e^{2\theta_1+2\theta_3} \\ &\quad + 2A(1,3)A(2,3)P(D)e^{\theta_1+\theta_2+2\theta_3} + A^2(2,3)P(D)e^{2\theta_2+2\theta_3} \\ &= 2[A(1,2)A(1,3)P(D)e^{2\theta_1+\theta_2+\theta_3} \qquad (3.15) \\ &\quad + A(1,2)A(2,3)P(D)e^{\theta_1+2\theta_2+\theta_3} \\ &\quad + A(1,3)A(2,3)P(D)e^{\theta_1+\theta_2+2\theta_3}] \\ &= 2[A(1,2)A(1,3)P(p_2 - p_3)e^{\theta_1+\theta_2+2\theta_3} \qquad (3.16) \\ &\quad + A(1,2)A(2,3)P(p_1 - p_3)e^{\theta_1+2\theta_2+\theta_3} \\ &\quad + A(1,3)A(2,3)P(p_1 - p_2)e^{\theta_1+\theta_2+2\theta_3}]. \qquad (3.17) \end{aligned}$$

Hence when we use these in the equation (3.2.19) we get

$$2B[P(p_2+p_3)e^{2\theta_1+\theta_2+\theta_3}+P(p_1+p_3)e^{\theta_1+2\theta_2+\theta_3}+P(p_1+p_2)e^{\theta_1+\theta_2+2\theta_3}]$$
$$+2[A(1,2)A(1,3)P(p_2-p_3)e^{2\theta_1+\theta_2+\theta_3}+A(1,2)A(2,3)P(p_1-p_3)e^{\theta_1+2\theta_2+\theta_3}$$
$$+A(1,3)A(2,3)P(p_1-p_2)e^{\theta_1+\theta_2+2\theta_3}]=0$$

$$B[P(p_2+p_3)e^{2\theta_1+\theta_2+\theta_3}+P(p_1+p_3)e^{\theta_1+2\theta_2+\theta_3}+P(p_1+p_2)e^{\theta_1+\theta_2+2\theta_3}]$$
$$=-[A(1,2)A(1,3)P(p_2-p_3)e^{2\theta_1+\theta_2+\theta_3}+A(1,2)A(2,3)P(p_1-p_3)e^{\theta_1+2\theta_2+\theta_3}$$
$$+A(1,3)A(2,3)P(p_1-p_2)e^{\theta_1+\theta_2+2\theta_3}] \tag{3.18}$$

To satisfy the above equation, the coefficients of the exponential terms should vanish. So we find that

$$B=A(1,2)A(1,3)A(2,3). \tag{3.19}$$

Remember that when we are analyzing the coefficients of ε^3, we have found another expression for the coefficient B. To be consistent these expressions for B should be equivalent i.e.

$$\begin{aligned} B &= -\frac{A(1,2)P(p_3-p_1-p_2)+A(1,3)P(p_2-p_1-p_3)+A(2,3)P(p_1-p_2-p_3)}{P(p_1+p_2+p_3)} \\ &= A(1,2)A(1,3)A(2,3). \end{aligned} \tag{3.20}$$

When we insert the formulas for $A(1,2)A(1,3)$ and $A(2,3)$ in that equation, we obtain a relation that is

$$-[A(1,2)P(p_3-p_1-p_2)+A(1,3)P(p_2-p_1-p_3)+A(2,3)P(p_1-p_2-p_3)]$$
$$=A(1,2)A(1,3)A(2,3)P(p_1+p_2+p_3)$$

$$-[-\frac{P(p_1-p_2)}{P(p_1+p_2)}P(p_3-p_1-p_2)-\frac{P(p_1-p_3)}{P(p_1+p_3)}P(p_2-p_1-p_3)$$
$$-\frac{P(p_2-p_3)}{P(p_2+p_3)}P(p_1-p_2-p_3)]$$
$$=[-\frac{P(p_1-p_2)}{P(p_1+p_2)}][-\frac{P(p_1-p_3)}{P(p_1+p_3)}][-\frac{P(p_2-p_3)}{P(p_2+p_3)}]P(p_1+p_2+p_3)$$

$$P(p_1-p_2)P(p_1+p_3)P(p_2+p_3)P(p_3-p_1-p_2)$$
$$+P(p_1-p_3)P(p_1+p_2)P(p_2+p_3)P(p_2-p_1-p_3)$$
$$+P(p_2-p_3)P(p_1+p_2)P(p_1+p_3)P(p_1-p_2-p_3)$$
$$=-P(p_1-p_2)P(p_1-p_3)P(p_2-p_3)P(p_2+p_3)P(p_1+p_2+p_3)$$

$$P(p_1-p_2)P(p_1+p_3)P(p_2+p_3)P(p_3-p_1-p_2)$$
$$+P(p_1-p_3)P(p_1+p_2)P(p_2+p_3)P(p_2-p_1-p_3)$$
$$+P(p_2-p_3)P(p_1+p_2)P(p_1+p_3)P(p_1-p_2-p_3)$$
$$+P(p_1-p_2)P(p_1-p_3)P(p_2-p_3)P(p_2+p_3)P(p_1+p_2+p_3)=0. \tag{3.21}$$

By writing the above equation in a more appropriate form we can conclude that to have three-solitary wave solution, nonlinear partial differential and difference equations which have the Hirota bilinear form $P(D)\{f \cdot f\} = 0$ should satisfy the condition which we call the three-solitary wave condition:

$$\sum_{\delta_i=\pm 1} P(\delta_1 p_1+\delta_2 p_2+\delta_3 p_3)P(\delta_1 p_1-\delta_2 p_2)P(\delta_2 p_2-\delta_3 p_3)P(\delta_3 p_3-\delta_1 p_1) = 0 \quad (3.22)$$

with the dispersion relation $P(p_i) = 0$ for $i = 1, 2, 3$. An equation $F[u] = 0$ satisfying condition possesses three-solitary wave solution given by

$$\begin{aligned} u &= T[f(x,t)] \\ &= T[1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} \\ &\quad + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}] \end{aligned} \quad (3.23)$$

where $\theta_i = k_i x + \omega_i t + \alpha_i,\ i = 1, 2, 3$. Here $A(i,j) = -\frac{P(p_i-p_j)}{P(p_i+p_j)}$ for $i,j = 1, 2, 3, i < j$ and $B = A(1,2)A(1,3)A(2,3)$. □

From Suksai [7], we transform the RLW equation in the Hirota bilinear form as,

$$(D_x^2 + 3D_x D_t - D_x^4)\{f \cdot f\} = 0. \quad (3.24)$$

Hence three-solitary wave solution of the RLW equation is

$$\begin{aligned} u(x,t) &= 4\frac{\partial^2}{\partial x^2} \ln[1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} \\ &\quad + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}] \\ &= 4\frac{\partial}{\partial x}\left[\frac{P(x,t)}{f}\right] \\ &= 4\left[\frac{f(x,t)Q(x,t) - P^2(x,t)}{N(x,t)}\right] \\ &= 4\frac{M(x,t)}{N(x,t)} \end{aligned} \quad (3.25)$$

where

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} \\ &\quad + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3} \\ P(x,t) &= k_1e^{\theta_1} + k_2e^{\theta_2} + k_3e^{\theta_3} + A(1,2)(k_1+k_2)e^{\theta_1+\theta_2} \\ &\quad + A(1,3)(k_1+k_3)e^{\theta_1+\theta_3} + A(2,3)(k_2+k_3)e^{\theta_2+\theta_3} \\ &\quad + B(k_1+k_2+k_3)e^{\theta_1+\theta_2+\theta_3} \\ Q(x,t) &= k_1^2e^{\theta_1} + k_2^2e^{\theta_2} + k_3^2e^{\theta_3} + A(1,2)(k_1+k_2)^2e^{\theta_1+\theta_2} \\ &\quad + A(1,3)(k_1+k_3)^2e^{\theta_1+\theta_3} + A(2,3)(k_2+k_3)^2e^{\theta_2+\theta_3} \\ &\quad + B(k_1+k_2+k_3)^2e^{\theta_1+\theta_2+\theta_3} \end{aligned}$$

$$
\begin{aligned}
f(x,t)Q(x,t) &= k_1^2e^{\theta_1}+k_2^2e^{\theta_2}+k_3^2e^{\theta_3}+k_1^2e^{2\theta_1}+k_2^2e^{2\theta_2}+k_3^2e^{2\theta_3}\\
&+e^{\theta_1+\theta_2}\{k_1^2+k_2^2+A(1,2)[(k_1+k_2)^2+(k_1+k_2)^2e^{\theta_1}\\
&+(k_1+k_2)^2e^{\theta_2}+k_1^2e^{\theta_1}+k_2^2e^{\theta_2}]\}+e^{\theta_1+\theta_3}\{k_1^2+k_3^2\\
&+A(1,3)[(k_1+k_3)^2+(k_1+k_3)^2e^{\theta_1}+(k_1+k_3)^2e^{\theta_3}\\
&+k_1^2e^{\theta_1}+k_3^2e^{\theta_3}]\}+e^{\theta_2+\theta_3}\{k_2^2+k_3^2+A(2,3)[(k_2+k_3)^2\\
&+(k_2+k_3)^2e^{\theta_2}+(k_1+k_3)^2e^{\theta_3}+k_2^2e^{\theta_2}+k_3^2e^{\theta_3}]\}\\
&+e^{\theta_1+\theta_2+\theta_3}\{A(1,2)[k_3^2+(k_1+k_2)^2+A(1,3)(k_1+k_3)^2e^{\theta_1}\\
&+A(2,3)(k_2+k_3)^2e^{\theta_2}]+A(1,3)[k_3^2+(k_1+k_3)^2\\
&+A(1,2)(k_1+k_2)^2e^{\theta_1}+A(2,3)(k_2+k_3)^2e^{\theta_3}]\\
&+A(2,3)[k_1^2+(k_2+k_3)^2+A(1,2)(k_1+k_2)^2e^{\theta_2}\\
&+A(1,3)(k_1+k_3)^2e^{\theta_3}]+B(k_1+k_2+k_3)^2\}\\
&+Be^{\theta_1+\theta_2+\theta_3}\{(k_1+k_2+k_3)^2e^{\theta_1}+(k_1+k_2+k_3)^2e^{\theta_2}\\
&+(k_1+k_2+k_3)^2e^{\theta_3}+A(1,2)[(k_1+k_2+k_3)^2e^{\theta_1+\theta_2}\\
&+(k_1+k_2)^2e^{\theta_1+\theta_2}]+A(1,3)[(k_1+k_2+k_3)^2e^{\theta_1+\theta_3}\\
&+(k_1+k_3)^2e^{\theta_1+\theta_3}]+A(2,3)[(k_1+k_2+k_3)^2e^{\theta_2+\theta_3}\\
&+(k_2+k_3)^2e^{\theta_2+\theta_3}]\}+A^2(1,2)(k_1+k_2)^2e^{2\theta_1+2\theta_2}\\
&+A^2(1,3)(k_1+k_3)^2e^{2\theta_1+2\theta_3}+A^2(2,3)(k_2+k_3)^2e^{2\theta_2+2\theta_3}\\
&+B^2(k_1+k_2+k_3)^2e^{2\theta_1+2\theta_2+2\theta_3}\\
P^2(x,t) &= k_1^2e^{\theta_1}+k_2^2e^{\theta_2}+k_3^2e^{\theta_3}+e^{\theta_1+\theta_2}\{2k_1k_2+A(1,2)[2(k_1+k_2)^2k_1e^{\theta_1}\\
&+2(k_1+k_2)^2k_2e^{\theta_2}]\}+e^{\theta_1+\theta_3}\{2k_1k_3+A(1,3)[2(k_1+k_3)^2k_1e^{\theta_1}\\
&+2(k_1+k_3)^2k_3e^{\theta_3}]\}+e^{\theta_2+\theta_3}\{2k_2k_3+A(2,3)[2(k_2+k_3)^2k_2e^{\theta_2}\\
&+2(k_2+k_3)^2k_3e^{\theta_3}]\}+e^{\theta_1+\theta_2+\theta_3}\{A(1,2)[2(k_1+k_2)k_3\\
&+2A(1,3)(k_1+k_2)(k_1+k_3)e^{\theta_1}+2A(2,3)(k_1+k_2)(k_2+k_3)e^{\theta_2}]\\
&+A(1,3)[2(k_1+k_3)k_2+2A(2,3)(k_1+k_3)(k_2+k_3)e^{\theta_3}]\\
&+A(2,3)[2(k_2+k_3)k_1]\}+Be^{\theta_1+\theta_2+\theta_3}\{2(k_1+k_2+k_3)k_1e^{\theta_1}\\
&+2(k_1+k_2+k_3)k_2e^{\theta_2}+2(k_1+k_2+k_3)k_3e^{\theta_3}\\
&+2A(1,2)(k_1+k_2)(k_1+k_2+k_3)e^{\theta_1+\theta_2}\\
&+2A(1,3)(k_1+k_3)(k_1+k_2+k_3)e^{\theta_1+\theta_3}\\
&+2A(2,3)(k_2+k_3)(k_1+k_2+k_3)e^{\theta_2+\theta_3}\}\\
&+A^2(1,2)(k_1+k_2)^2e^{2\theta_1+2\theta_2}+A^2(1,3)(k_1+k_3)^2e^{2\theta_1+2\theta_3}\\
&+A^2(2,3)(k_2+k_3)^2e^{2\theta_2+2\theta_3}+B^2(k_1+k_2+k_3)^2e^{2\theta_1+2\theta_2+2\theta_3}
\end{aligned}
\tag{3.26}
$$

$$\begin{aligned} M(x,t) &= f(x,t)Q(x,t) - P^2(x,t) \\ &= k_1^2 e^{\theta_1} + k_2^2 e^{\theta_2} + k_3^2 e^{\theta_3} + e^{\theta_1+\theta_2}\{(k_1-k_2)^2[1 + A(1,3)A(2,3)e^{2\theta_3}] \\ &\quad + A(1,2)[(k_1+k_2)^2 + k_2^2 e^{\theta_1} + k_1^2 e^{\theta_2}]\} + e^{\theta_1+\theta_3}\{(k_1-k_3)^2[1 \\ &\quad + A(1,2)A(2,3)e^{2\theta_2}] + A(1,3)[(k_1+k_3)^2 + k_3^2 e^{\theta_1} + k_1^2 e^{\theta_3}]\} \\ &\quad + e^{\theta_2+\theta_3}\{(k_2-k_3)^2[1 + A(1,2)A(1,3)e^{2\theta_1}] + A(2,3)[(k_2+k_3)^2 \\ &\quad + k_3^2 e^{\theta_2} + k_2^2 e^{\theta_3}]\} + e^{\theta_1+\theta_2+\theta_3}[A(1,2)(k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 - 2k_1k_3 \\ &\quad - 2k_2k_3) + A(1,3)(k_1^2 + k_2^2 + k_3^2 - 2k_1k_2 + 2k_1k_3 - 2k_2k_3) \\ &\quad + A(2,3)(k_1^2 + k_2^2 + k_3^2 - 2k_1k_2 - 2k_1k_3 + 2k_2k_3) + B(k_1^2 + k_2^2 + k_3^2 \\ &\quad + 2k_1k_2 + 2k_1k_3 + 2k_2k_3)] + Be^{\theta_1+\theta_2+\theta_3}[(k_2+k_3)^2 e^{\theta_1} + (k_1+k_3)^2 e^{\theta_2} \\ &\quad + (k_1+k_2)^2 e^{\theta_3} + A(1,2)k_3^2 e^{\theta_1+\theta_2} + A(1,3)k_2^2 e^{\theta_1+\theta_3} + A(2,3)k_1^2 e^{\theta_2+\theta_3} \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} N(x,t) &= [1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + A(1,2)e^{\theta_1+\theta_2} + A(1,3)e^{\theta_1+\theta_3} \\ &\quad + A(2,3)e^{\theta_2+\theta_3} + Be^{\theta_1+\theta_2+\theta_3}]^2 \end{aligned} \quad (3.28)$$

for $\theta_i = k_i x + (k_i^3 - k_i)t + \alpha_i,\ A(i,j) = \frac{(k_i-k_j)^2}{(k_i+k_j)^2},\ i,j = 1,2,3,\ i < j$ and $B = A(1,2)A(1,3)A(2,3)$.

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