Thai Journal of Mathematics
Volume 22 Number 3 (2024)
Pages 545–565

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On a High-Order Iterative Scheme for a Nonlinear Pseudoparabolic Equation with Viscoelastic Term

Nguyen Huu Nhan[1], Truong Thi Nhan[2], Le Thi Phuong Ngoc[3] and Nguyen Thanh Long[4,5,*]

[1] *Ho Chi Minh City University of Foreign Languages and Information Technology, 828 Su Van Hanh Str., Dist. 10, Ho Chi Minh City, Vietnam*
e-mail : nhannh1@huflit.edu.vn

[2] *The Faculty of Natural Basic Sciences, Vietnamese Naval Academy, 30 Tran Phu Str., Nha Trang City, Vietnam*
e-mail : nhanhaiquan@gmail.com

[3] *University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam*
e-mail : ngoc1966@gmail.com

[4] *Faculty of Mathematics and Computer Science, University of Science, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam*

[5] *Vietnam National University, Ho Chi Minh City, Vietnam*
e-mail : longnt2@gmail.com

Abstract In this paper, we study an initial boundary problem for a nonlinear pseudoparabolic equation with Robin-Dirichlet conditions. By constructing a high order iterative scheme, the local existence and uniqueness of solutions are proved. Moreover, we also show that the sequence associated with this high order iterative scheme converges at N-order rate, with $N \geq 2$, to the unique weak solution of the problem.

MSC: 34B60; 35K55; 35Q72; 80A

Keywords: nonlinear pseudoparabolic equation; Faedo-Galerkin approximation; high order iterative scheme

Submission date: 12.03.2020 / Acceptance date: 11.11.2023

1. Introduction

In this paper, we consider the following initial boundary problem for a nonlinear pseudoparabolic equation

$$\begin{cases} u_t + \left(1 + \dfrac{\partial}{\partial t}\right) Au - \int_0^t g(t-s)Au(s)ds = f\,(x,t,u),\ 1 < x < R, 0 < t < T, \\ u_x(1,t) - h_1 u(1,t) = u(R,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \end{cases} \tag{1.1}$$

*Corresponding author.

where $R > 1$, $h_1 \geq 0$ are given constants and f, g, $\tilde{u}_0$ are given functions satisfying conditions specified later; $Au \equiv -\left(u_{xx} + \frac{1}{x}u_x\right)$ with $u = u(x,t)$ is the unknown function.

Pseudoparabolic equations have been widely studied since the works of Ting [1], [2], such as [3]-[22] among others and the references given therein. In these works, the results of existence, asymptotic behavior, blow up and decay of solutions have been investigated. It takes into account that Eq. (1.1) is also regarded as a Sobolev-type equation or a Sobolev–Galpern type equation, and arisen in areas of mathematics and physics. One of the most important linear models in equations of this type is Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.2}$$

it was studied by Amick et al. in [3] with $\nu > 0$, $\alpha = 1$, $x \in \mathbb{R}$, $t \geq 0$, in which the solution of (1.2) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm *as* $t \to +\infty$. With $\nu > 0$, $\alpha > 0$, $x \in [0,1]$, $t \geq 0$, Eq. (1.2) was also investigated earlier by Bona and Dougalis [8], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu \geq 0$ and on $\alpha > 0$. Medeiros and Miranda [17] studied a nonlinear equation of Sobolev type, namely

$$u_t + f(u)_x - u_{xxt} = g(x,t), \tag{1.3}$$

where $u = u(x,t)$, $0 < x < 1$, and $t \geq 0$ is the time. They proved existence, uniqueness of solutions for f in C^1 and regularity in the case $f(s) = s^2/2$.

In [13], the well-poseness and solvability of solutions were established by Dai and Huang for the nonlinear pseudoparabolic equation

$$u_t + \left(a(x,t)u_{xt}\right)_x = F\left(x,t,u,u_x,u_{xx}\right),\ \alpha < x < \beta,\ 0 < t < T, \tag{1.4}$$

associated with the nonlocal moment boundary conditions

$$\int_\alpha^\beta u(x,t)dx = \int_\alpha^\beta xu(x,t)dx = 0,\ 0 \leq t \leq T. \tag{1.5}$$

In [20], Shang and Guo proved the existence, uniqueness, and regularities of the global strong solution and gave some conditions of the nonexistence of global solution of the nonlinear pseudoparabolic equation with Volterra integral term

$$\begin{aligned} u_t - f(u)_{xx} - u_{xxt} - \int_0^t \lambda(t-s)\left(\sigma\left(u(x,s), u_x(x,s)\right)\right)_x ds \\ = f\left(x,t,u,u_x\right), 0 < x < 1,\ t > 0. \end{aligned} \tag{1.6}$$

In [12], Y. Cao et al. established the global existence of classical solutions and the blow-up in a finite time for the viscous diffusion equation of higher order

$$\begin{cases} u_t + k_1 u_{xxxx} - k_2 u_{txx} - \left(\Phi\left(u_x\right)\right)_x + A(u) = 0,\ 0 < x < 1,\ t > 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0,\ t > 0, \\ u(x,0) = u_0(x),\ 0 < x < 1, \end{cases} \tag{1.7}$$

where $k_1 > 0$, $k_2 > 0$ and $\Phi(s)$, $A(s)$ are appropriately smooth, $u_0 \in C^{1+\beta}$ with $\beta \in (0,1)$ and $u_0(0) = u_0(1) = u_{0xx}(0) = u_{0xx}(1) = 0$.

For more physical explanations, it is well known that pseudo-parabolic equations describe a variety of important physical processes (see [11]), such as the seepage of homogeneous fluids through a fissured rock [6] (where k is a characteristic of the fissured rock, a decrease of k corresponds to a reduction in block dimension and an increase in the degree of fissuring), the unidirectional propagation of nonlinear, dispersive, long waves [7] (where u is typically the amplitude or velocity), and the aggregation of populations [18] (where u represents the population density). Also, Eq. (1.1) can be considered as a general model of third-grade fluid flows or second-grade fluid flows, of which the mathematical models can be found in [2], [4], [5], [14]-[16], [19] and references therein. In [5], the following equation of motions for the unsteady flow of third-grade fluid over the rigid plate with porous medium was investigated

$$\begin{aligned} \rho\frac{\partial U}{\partial t} &= \mu\frac{\partial^2 U}{\partial y^2} + \alpha_1\frac{\partial^3 U}{\partial y^2 \partial t} + 6\beta_3\left(\frac{\partial U}{\partial y}\right)^2\frac{\partial^2 U}{\partial y^2} \\ &\quad - \frac{\phi}{k}\left[\mu + \alpha_1\frac{\partial}{\partial t} + 2\beta_3\left(\frac{\partial U}{\partial y}\right)^2\right]U,\ y>0,\ t>0, \end{aligned} \tag{1.8}$$

where U is the velocity component, ρ is the density, μ the coefficient of viscosity, α_1 and β_3 are the material constants. In [14], some problems of second-grade unsteady fluid flows were also considered. These flows are generated by the sudden application of a constant pressure gradient or by the impulsive motion of a boundary. Here, the velocities of the flows are described by the partial differential equations, of which the exact analytic solutions are obtained. Suppose that the second-grade fluid is in a circular cylinder and is initially at rest, and the fluid starts suddenly due to the motion of the cylinder parallel to its length. The axis of the cylinder is chosen as the z-axis. Using cylindrical polar coordinates, the governing partial differential equation is

$$\begin{cases} \frac{\partial w}{\partial t} = (\nu + \alpha\frac{\partial}{\partial t})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w(r,t) - Nw,\ 0<r<a,\ t>0, \\ w(a,t) = W,\ t>0, \\ w(r,0) = 0,\ 0\le r<a, \end{cases} \tag{1.9}$$

where $w(r,t)$ is the velocity along the z-axis, ν is the kinematic viscosity, α is the material parameter, and N is the imposed magnetic field. In the boundary and initial conditions, W is the constant velocity at $r=a$ and a is the radius of the cylinder. In [16], A. Mahmood et. al. considered the longitudinal oscillatory motion of second-grade fluid between two infinite coaxial circular cylinders, oscillating along their common axis with given constant angular frequencies Ω_1 and Ω_2. Velocity field and associated tangential stress of the motion were determined by using Laplace and Hankel transforms. In order to find the exact analytic solutions for the flow of second-grade fluid between two longitudinally oscillating cylinders, the following problem was studied

$$\begin{cases} \frac{\partial v}{\partial t} = (\mu + \alpha\frac{\partial}{\partial t})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t),\ R_1<r<R_2,\ t>0, \\ v(R_1,t) = V_1\sin(\Omega_1 t),\ v(R_2,t) = V_2\sin(\Omega_2 t), \\ u(r,0) = 0,\ R_1\le r\le R_2, \end{cases} \tag{1.10}$$

where $0<R_1<R_2$, μ, α, V_2, Ω_1, Ω_2 are positive constants. The obtained exact solutions have been presented under series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$ and $Y_2(x)$, satisfying the governing equation and all imposed initial and boundary conditions.

As we know, the linearization method is one of useful methods to study the problems with a general nonlinear term, in which a linear or quadratic reccurent sequence is constructed and its convergence is respectively called 1-order convergence or 2-order convergence, for example [23], [24] and the references given therein. Later, the extending results of [23] and [24] have been continuously considered by Truong et. al [25], in which a N-order iterative scheme has been established in order to prove the existence and uniqueness of solutions for the following nonlinear wave equation of Kirchhoff-Carrier type

$$\begin{cases} u_{tt} - \mu\left(t, \|u(t)\|^2, \|u_x(t)\|^2\right) u_{xx} = f(x,t,u), \, 0 < x < 1, \, 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \; u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.11}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $h_0 > 0$, $h_1 \geq 0$ are given constants. In the paper, they associated with Eq. $(1.11)_1$ a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \mu\left(t, \|u_m(t)\|^2, \|u_{mx}(t)\|^2\right) u_{mxx} \\ &= \sum_{i=0}^{N} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i, \;\; 0 < x < 1, \;\; 0 < t < T, \end{aligned} \tag{1.12}$$

with u_m satisfying $(1.11)_{2,3}$. Recently, the authors in [26] have also used the similar method given in [25] to construct a N-order convergent recurrent sequence for a nolinear wave equation in annular of Carrier type associated with Robin-Dirichlet conditions as below

$$\begin{cases} u_{tt} - \mu\left(\|u(t)\|_0^2\right)\left(u_{xx} + \frac{1}{x}u_x\right) = f(x,t,u), \, \rho < x < 1, \, 0 < t < T, \\ u(\rho,t) = u_x(1,t) + \zeta u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \; u_t(x,0) = \tilde{u}_1(x), \end{cases} \tag{1.13}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and ρ, ζ are given constants, $0 < \rho < 1$.

The purpose of the present paper is devoted to studying the unique solvability of the problem (1.1). We first construct a recurrent sequence $\{u_m\}$ associated with Eq. $(1.1)_1$, which is defined by

$$\begin{aligned} &\frac{\partial u_m}{\partial t} - \left(\mu + \alpha\frac{\partial}{\partial t}\right) A u_m - \int_0^t g(t-s) A u_m(s) ds \\ &= \sum_{i=0}^{N} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i, \;\; 1 < x < R, \;\; 0 < t < T, \end{aligned} \tag{1.14}$$

and u_m satisfying $(1.1)_{2,3}$. Next, we prove the existence of the above sequence by using Galerkin method, in which the Banach fixed point theorem is applied to get the existence of Galerkin approximate solution. In last part, we prove that the sequence $\{u_m\}$ converges to a function u, and show that u is the weak solution of the problem (1.1). Moreover, we aslo establish an estimation of N-order convergence in the form $\|u_m - u\|_X \leq C \|u_{m-1} - u\|_X^N$, for some $C > 0$, all large positive integers N and X is a suitable space. To the best of our knowledge, there have been few works of high order iterative scheme for nonlinear pseudoparabolic equation with viscoelastic term. This paper consists of three sections. In Section 2, we present preliminaries. In Section 3, we present the main results of the local existence and uniqueness.

2. Preliminaries

Throughout this paper, we set $\Omega = (1, R)$ and use $L^2 = L^2(\Omega)$ to denote the Lebesgue space with the inner product defined by $(u, v) = \int_1^R u(x)v(x)dx$, L^2-norm of a function $u \in L^2$ is denoted by $\|u\| = \sqrt{(u, u)}$. We use $H^m = H^m(\Omega)$ to denote the Sobolev spaces with the norm $\|u\|_{H^m} = \left(\sum_{i=0}^{m} \left\|D^i u\right\|^2\right)^{1/2}$.

Moreover, we also introduce three weighted scalar products

$$\begin{aligned} \langle u, v\rangle &= \int_1^R xu(x)v(x)dx,\ u, v \in L^2, \\ \langle u, v\rangle_1 &= \langle u, v\rangle + \langle u_x, v_x\rangle,\ u, v \in H^1, \\ \langle u, v\rangle_2 &= \langle u, v\rangle + \langle u_x, v_x\rangle + \langle u_{xx}, v_{xx}\rangle,\ u, v \in H^2, \end{aligned} \tag{2.1}$$

then L^2, H^1, H^2 are the Hilbert spaces with respect to the above scalar products. We denote $\|u\|_0 = \sqrt{\langle u, u\rangle}$, $u \in L^2$; $\|u\|_1 = \sqrt{\langle u, u\rangle_1}$, $u \in H^1$; $\|u\|_2 = \sqrt{\langle u, u\rangle_2}$, $u \in H^2$.

Put

$$V = \left\{v \in H^1 : v(R) = 0\right\}. \tag{2.2}$$

The symmetric bilinear form $a(\cdot, \cdot)$ defined by

$$a(u, w) = \langle u_x, w_x\rangle + h_1 u(1)w(1), \text{ for all } u, w \in V, \tag{2.3}$$

with $h_1 \geq 0$ is a given constant and $\|v\|_a = \sqrt{a(v, v)}$.

Then, we have the following lemmas.

Lemma 2.1. *The imbeddings $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{aligned} &\text{(i)} \quad \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{R-1}\,\|v_x\|_0 \leq \sqrt{R-1}\,\|v\|_a \textit{ for all } v \in V, \\ &\text{(ii)} \quad \|v\|_0 \leq \tfrac{\sqrt{2R}(R-1)}{2}\,\|v_x\|_0 \textit{ for all } v \in V, \\ &\text{(iii)} \quad \|v_x\|_0 \leq \|v\|_a \leq \sqrt{1 + h_1(R-1)}\,\|v_x\|_0 \textit{ for all } v \in V. \end{aligned} \tag{2.4}$$

Lemma 2.2. *The symmetric bilinear form $a(\cdot, \cdot)$ is continuous on $V \times V$ and coercive on V, i.e., there exist two positive constants C_0, C_1 such that*

$$\begin{aligned} &\text{(i)} \quad |a(u, v)| \leq C_1 \|u_x\|_0 \|v_x\|_0, \\ &\text{(ii)} \quad a(v, v) \geq C_0 \|v_x\|_0^2, \end{aligned} \tag{2.5}$$

for all u, $v \in V$. Moreover, $C_1 = 1 + h_1(R-1)$ and $C_0 = 1$.

Lemma 2.3. *There exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue $\bar{\lambda}_j$ such that*

$$\begin{cases} 0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \cdots \leq \bar{\lambda}_j \leq \bar{\lambda}_{j+1} \leq \cdots,\ \lim\limits_{j \to +\infty} \bar{\lambda}_j = +\infty, \\ a(w_j, w) = \bar{\lambda}_j \langle w_j, w\rangle \ \textit{ for all } w \in V,\ j = 1, 2, \cdots. \end{cases} \tag{2.6}$$

Furthermore, the sequence $\{w_j/\sqrt{\bar{\lambda}_j}\}$ is also the Hilbert orthonormal base of V with respect to the scalar product $a(\cdot, \cdot)$. On the other hand, we have w_j satisfying the following boundary value problem

$$\begin{cases} Aw_j \equiv -\left(w_{jxx} + \frac{1}{x} w_{jx}\right) = -\frac{1}{x}\frac{\partial}{\partial x}\left(x w_{jx}\right) = \bar{\lambda}_j w_j, \textit{ in } (1, R), \\ w_{jx}(1) - h_1 w_j(1) = w_j(R) = 0,\ w_j \in C^\infty([1, R]). \end{cases} \tag{2.7}$$

The proof of Lemma 2.3 can be found in [[27], p.87, Theorem 7.7], with $H = L^2$ and $a(\cdot, \cdot)$ as defined by (2.3).

Lemma 2.4. *The operator $A : V \to V'$ in (2.7) is uniquely defined by Lax-Milgram's lemma, i.e.,*

$$a(u, v) = \langle Au, v \rangle, \text{ for all } u, \ v \in V. \tag{2.8}$$

The notation $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X. We denote by $L^p(0, T; X)$, $1 \le p \le \infty$ for the Banach space of real functions $u : (0, T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty \text{ for } 1 \le p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess \sup} \|u(t)\|_X \text{ for } p = \infty.$$

Denote $u(t) = u(x, t)$, $u'(t) = u_t(t) = \frac{\partial u}{\partial t}(x, t)$, $u''(t) = u_{tt}(t) = \frac{\partial^2 u}{\partial t^2}(x, t)$, $u_x(t) = \frac{\partial u}{\partial x}(x, t)$, $u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(x, t)$.

With $f \in C^N([0, 1] \times [0, T^*] \times \mathbb{R})$, $f = f(x, t, u)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial F}{\partial u}$ and $D^\alpha f = D_1^{\alpha_1} \cdots D_3^{\alpha_3} f$, $\alpha = (\alpha_1, \cdots, \alpha_3) \in \mathbb{Z}_+^3$, $|\alpha| = \alpha_1 + \cdots + \alpha_3 \le N$, $D^{(0,\cdots,0)} f = f$.

3. Main Results

In this section, the local solution of (1.1) is established by using linear approximate method and Faedo-Galerkin method. For a fixed constant $T^* > 0$, we make the following assumptions:

(H_1) $\tilde{u}_0 \in V \cap H^2$, $\tilde{u}_{0x}(1) - h_1 \tilde{u}_0(1) = 0$;
(H_2) $g \in L^2(0, T^*)$;
(H_3) $f \in C^1(\bar{\Omega} \times [0, T^*] \times \mathbb{R})$ satisfies the conditions:
$D_3^i f$, $D_2 D_3^j f \in C^0(\bar{\Omega} \times [0, T^*] \times \mathbb{R})$, $1 \le i \le N$, $1 \le j \le N - 1$.

Put

$$K_M(f) = \|f\|_{C^0(\bar{\Omega}_M)} + \sum\nolimits_{i=1}^N \|D_3^i f\|_{C^0(\bar{\Omega}_M)} + \sum\nolimits_{j=1}^{N-1} \left\| D_2 D_3^j f \right\|_{C^0(\bar{\Omega}_M)},$$

where

$$\begin{aligned} \|f\|_{C^0(\bar{\Omega}_M)} &= \sup\{|f(x, t, y)| : (x, t, y) \in \bar{\Omega}_M\}, \\ \bar{\Omega}_M &= [1, R] \times [0, T^*] \times [-\sqrt{R-1}M, \sqrt{R-1}M]. \end{aligned}$$

The weak solution of (1.1) is a function $u \in L^\infty(0,T;V\cap H^2)$ such that $u' \in L^\infty(0,T;V\cap H^2)$ and u satisfies the following variational equation

$$\begin{cases} \quad \langle u'(t),w\rangle + a\left(u'(t),w\right) + a\left(u(t),w\right) \\ = \int_0^t g(t-s)a\left(u(s),w\right)ds + \langle f[u](t),w\rangle, \text{ for all } w\in V, \text{ a.e., } t\in(0,T), \\ u(0) = \tilde{u}_0, \end{cases} \tag{3.1}$$

where $f[u](x,t) = f\left(x,t,u(x,t)\right)$.

For each $T\in(0,T^*]$, we introduce the space

$$W_T = \left\{v\in L^\infty(0,T;V\cap H^2) : v'\in L^\infty(0,T;V\cap H^2)\right\}.$$

Note that W_T is a Banach space with norm

$$\|v\|_{W_T} = \max\left\{\|v\|_{L^\infty(0,T;V\cap H^2)},\ \ \|v'\|_{L^\infty(0,T;V\cap H^2)}\right\}.$$

For $M>0$, we put

$$W(M,T) = \left\{v\in W_T : \|v\|_{W_T}\le M\right\}.$$

Now, we construct the recurrent sequence $\{u_m\}$ defined by $u_0\equiv 0$, and suppose that

$$u_{m-1}\in W\left(M,T\right). \tag{3.2}$$

Then u_m is found by the fact that $u_m\in W(M,T)$, $m\ge 1$ and provided

$$\begin{cases} \langle u_m'(t),w\rangle + a(u_m'(t),w) + a(u_m(t),w) \\ = \int_0^t g(t-s)a\left(u_m(s),w\right)ds + \left\langle F_m\left(t\right),w\right\rangle,\ \text{for all } w\in V,\ \text{a.e., } t\in(0,T), \\ u_m(0) = \tilde{u}_0, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} F_m\left(x,t\right) &= \sum_{i=0}^{N-1}\frac{1}{i!}D_3^i f\left[u_{m-1}\right](x,t)\left(u_m\left(x,t\right) - u_{m-1}\left(x,t\right)\right)^i \\ &= \sum_{j=0}^{N-1}\Phi_{mj}\left(x,t\right)u_m^j\left(x,t\right), \end{aligned} \tag{3.4}$$

and

$$\Phi_{mj}\left(x,t\right) = \sum_{i=j}^{N-1}\frac{(-1)^{i-j}}{j!(i-j)!}D_3^i f\left[u_{m-1}\right](x,t)\,u_{m-1}^{i-j}\left(x,t\right). \tag{3.5}$$

The first result of our paper is presented in the following theorem.

Theorem 3.1. *Assume that $\tilde{u}_0$, g, f satisfy $(H_1)-(H_3)$ respectively, then there exist the constants $M>0$ and $T>0$ such that the problem (3.3)-(3.4) admits $u_m\in W(M,T)$.*

Proof. Consider the basis $\{w_j\}$ for L^2 as in Lemma 2.3. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t)w_j,$$

where $c_{mj}^{(k)}$ is determined via the following system of nonlinear integrodifferential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + a\left(\dot{u}_m^{(k)}(t), w_j\right) + a\left(u_m^{(k)}(t), w_j\right) \\ = \int_0^t g(t-s)a\left(u_m^{(k)}(s), w_j\right)ds + \left\langle F_m^{(k)}(t), w_j\right\rangle, \ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \end{cases} \tag{3.6}$$

where

$$F_m^{(k)}(x,t) = \sum_{j=0}^{N-1} \Phi_{mj}(x,t)\left(u_m^{(k)}(x,t)\right)^j, \tag{3.7}$$

and $\tilde{u}_{0k}$ is satisfied the condition

$$\tilde{u}_{0k} = \sum\nolimits_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_0 \text{ strongly in } V \cap H^2. \tag{3.8}$$

The system (3.6) can be written in the form

$$\begin{cases} \dot{c}_{mj}^{(k)}(t) + \frac{\bar{\lambda}_j}{1+\bar{\lambda}_j}c_{mj}^{(k)}(t) = \frac{\bar{\lambda}_j}{1+\bar{\lambda}_j}\int_0^t g(t-s)\,c_{mj}^{(k)}(s)ds + \frac{1}{1+\bar{\lambda}_j}\left\langle F_m^{(k)}(t), w_j\right\rangle, \\ c_{mj}^{(k)}(0) = \beta_j^{(k)}, \ 1 \le j \le k. \end{cases} \tag{3.9}$$

After integrating, it can see that the system (3.9) is equivalent to the following system of intergal equations

$$\begin{aligned} c_{mj}^{(k)}(t) = \ & \beta_j^{(k)} e^{-\sigma_j t} + \frac{\bar{\lambda}_j}{1+\bar{\lambda}_j}\int_0^t H_j(t-s)\,c_{mj}^{(k)}(s)ds \\ & + \frac{1}{1+\bar{\lambda}_j}\int_0^t e^{-\sigma_j(t-s)}\left\langle F_m^{(k)}(s), w_j\right\rangle ds, \end{aligned} \tag{3.10}$$

with $1 \le j \le k$, where

$$H_j(t) = \int_0^t e^{-\sigma_j(t-s)} g(s)ds, \ \sigma_j = \frac{\bar{\lambda}_j}{1+\bar{\lambda}_j}, \ 1 \le j \le k. \tag{3.11}$$

By using the contraction mapping principle, it is not difficult to show that the existence of the approximate solution $u_m^{(k)}(t)$ of Eq. (3.10) on $[0, T_m^{(k)}] \subset [0, T]$.

In next step, we make some priori estimates that shows the bound of the approximate solution $u_m^{(k)}(t)$ on $[0, T_m^{(k)}]$. Then we can take $T_m^{(k)} = T$ independent of m and k, which permits that the approximate solution $u_m^{(k)}(t)$ of Eq. (3.10) can be extensively defined on the whole of $[0, T]$.

Put

$$\begin{aligned} S_m^{(k)}(t) = 2\int_0^t \left[\left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(s)\right\|_a^2 + \left\|\ddot{u}_m^{(k)}(s)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(s)\right\|_0^2\right] ds \\ + \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(t)\right\|_0^2, \end{aligned} \tag{3.12}$$

then we deduce from (3.6) and (3.12), that

$$\begin{aligned} S_m^{(k)}(t) &= \|\tilde{u}_{0k}\|_a^2 + \|A\tilde{u}_{0k}\|_0^2 + 2\int_0^t d\tau \int_0^\tau g(\tau - s) a\left(u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)\right) ds \\ &\quad + 2\int_0^t d\tau \int_0^\tau g(\tau - s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(\tau)\right\rangle ds \\ &\quad + \int_0^t g(t-s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(t)\right\rangle ds - \left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle \\ &\quad + 2\int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s) + A\dot{u}_m^{(k)}(s)\right\rangle ds + \left\langle F_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle \\ &= \|\tilde{u}_{0k}\|_a^2 + \|A\tilde{u}_{0k}\|_0^2 + \sum_{j=1}^6 I_j. \end{aligned} \tag{3.13}$$

We shall estimate the terms of the right-hand side of (3.13) as follows.

By the inequality $S_m^{(k)}(t) \geq \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(t)\right\|_0^2$, we estimate I_1, I_2, I_3 respectively as follows

$$\begin{aligned} I_1 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s) a(u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)) ds \\ &\leq 2\frac{C_1}{\sqrt{\mu}}\sqrt{T^*}\,\|g\|_{L^2(0,T^*)} \int_0^t S_m^{(k)}(\tau)\, d\tau; \\ I_2 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(\tau)\right\rangle ds \\ &\leq 2\sqrt{T^*}\,\|g\|_{L^2(0,T^*)} \int_0^t S_m^{(k)}(\tau)\, d\tau; \\ I_3 &= \int_0^t g(t-s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(t)\right\rangle ds \\ &\leq \frac{1}{4} S_m^{(k)}(t) + \|g\|_{L^2(0,T^*)}^2 \int_0^t S_m^{(k)}(s) ds. \end{aligned} \tag{3.14}$$

Using Cauchy-Schwarz, we get that

$$I_4 = \left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle \leq \left\|Au_m^{(k)}(t)\right\|_0^2 + \frac{1}{4} S_m^{(k)}(t). \tag{3.15}$$

The term $\left\|Au_m^{(k)}(t)\right\|_0^2$ is estimated as follows

$$\begin{aligned} \left\|Au_m^{(k)}(t)\right\|_0^2 &= \left\|A\tilde{u}_{0k} + \int_0^t A\dot{u}_m^{(k)}(s) ds\right\|_0^2 \\ &\leq \left(\|A\tilde{u}_{0k}\|_0 + \int_0^t \left\|A\dot{u}_m^{(k)}(s)\right\|_0 ds\right)^2 \\ &\leq 2\,\|A\tilde{u}_{0k}\|_0^2 + 2T^* \int_0^t S_m^{(k)}(s) ds. \end{aligned} \tag{3.16}$$

Then it follows from (3.15) and (3.16) that

$$I_4 = \left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle \leq \frac{1}{4}S_m^{(k)}(t) + 2\left\|A\tilde{u}_{0k}\right\|_0^2 + 2T^*\int_0^t S_m^{(k)}(s)ds. \tag{3.17}$$

In order to estimate the terms I_5 and I_6, we use the following lemma.

Lemma 3.2. *The terms* $\left\|F_m^{(k)}(t)\right\|_{L^\infty}$ *and* $\left\|\dot{F}_m^{(k)}(t)\right\|_{L^\infty}$ *are estimated as follows*

$$\begin{aligned} &\text{(i)} \quad \left\|F_m^{(k)}(t)\right\|_{L^\infty} \leq d_M^{(0)}\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right], \\ &\text{(ii)} \quad \left\|\dot{F}_m^{(k)}(t)\right\|_{L^\infty} \leq d_M^{(1)}\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right], \end{aligned} \tag{3.18}$$

where $d_M^{(0)}$ *and* $d_M^{(1)}$ *are defined by*

$$\begin{aligned} d_M^{(0)} &= \sum_{j=0}^{N-1}\bar{\alpha}_j(M)\left(\sqrt{R-1}\right)^j, \\ d_M^{(1)} &= \sum_{j=0}^{N-1}\left(\bar{\beta}_j(M)+j\bar{\alpha}_j(M)\right)\left(\sqrt{R-1}\right)^j, \\ \bar{\alpha}_j(M) &= K_M(f)\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left(\sqrt{R-1}M\right)^{i-j}, \\ \bar{\beta}_j(M) &= K_M(f)\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left(i-j+1+\sqrt{R-1}M\right)\left(\sqrt{R-1}M\right)^{i-j}, \\ &\quad j=\overline{0,N-1}. \end{aligned} \tag{3.19}$$

Proof.

(i) *Estimate of* $\left\|F_m^{(k)}(t)\right\|_{L^\infty}$. By using the inequality

$$|u_{m-1}(x,t)| \leq \|u_{m-1}(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{R-1}\,\|u_{m-1}(t)\|_a \leq \sqrt{R-1}M,$$

it follows from (3.5) that

$$\begin{aligned} |\Phi_{mj}(x,t)| &\leq \sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left|D_3^i f\left[u_{m-1}\right](x,t)\right|\left|u_{m-1}(x,t)\right|^{i-j} \\ &\leq K_M(f)\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left(\sqrt{R-1}\,\|u_{m-1}(t)\|_a\right)^{i-j} \\ &\leq K_M(f)\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left(\sqrt{R-1}M\right)^{i-j} = \bar{\alpha}_j(M). \end{aligned} \tag{3.20}$$

Due to $\left|u_m^{(k)}(x,t)\right| \leq \sqrt{R-1}\left\|u_m^{(k)}(t)\right\|_a \leq \sqrt{R-1}\sqrt{S_m^{(k)}(t)}$, we have

$$\begin{aligned} \left|F_m^{(k)}(x,t)\right| &\leq \sum_{j=0}^{N-1} \left|\Phi_{mj}(x,t)\right| \left|u_m^{(k)}(x,t)\right|^j \\ &\leq \sum_{j=0}^{N-1} \bar{\alpha}_j(M)\left(\sqrt{R-1}\right)^j \left[1+\left(\sqrt{\bar{S}_m^{(k)}(t)}\right)^{N-1}\right] \\ &= d_M^{(0)}\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right], \end{aligned} \tag{3.21}$$

where $d_M^{(0)} = \sum_{j=0}^{N-1} \bar{\alpha}_j(M)\left(\sqrt{R-1}\right)^j$. So (i) is proved.

(ii) *Estimate of* $\left\|\dot{F}_m^{(k)}(t)\right\|_{L^\infty}$. Note that

$$\begin{aligned} &\dot{F}_m^{(k)}(x,t) = \Phi'_{m0}(x,t) \\ &+\sum_{j=1}^{N-1}\left[\Phi'_{mj}(x,t)\left(u_m^{(k)}(x,t)\right)^j + j\Phi_{mj}(x,t)\left(u_m^{(k)}(x,t)\right)^{j-1}\dot{u}_m^{(k)}(x,t)\right]. \end{aligned} \tag{3.22}$$

On the other hand

$$\begin{aligned} &\Phi'_{mj}(x,t) \\ &= \sum_{i=j}^{N-1}\frac{(-1)^{i-j}}{j!(i-j)!}\left[D_2D_3^i f\left[u_{m-1}\right](x,t) + D_3^{i+1}f\left[u_{m-1}\right](x,t)\,u'_{m-1}(x,t)\right]u_{m-1}^{i-j}(x,t) \\ &\quad+\sum_{i=j}^{N-1}\frac{(-1)^{i-j}}{j!(i-j)!}(i-j)\left[D_3^i f\left[u_{m-1}\right](x,t)\,u_{m-1}^{i-j-1}(x,t)\,u'_{m-1}(x,t)\right], \end{aligned} \tag{3.23}$$

hence

$$\begin{aligned} &\left|\Phi'_{mj}(x,t)\right| \\ &\leq K_M(f)\sum_{i=j}^{N-1}\frac{C_i^j}{i!}\left(1+\left|u'_{m-1}(x,t)\right|\right)\left|u_{m-1}(x,t)\right|^{i-j} \\ &\quad+K_M(f)\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}(i-j)\left[\left|u_{m-1}(x,t)\right|^{i-j-1}\left|u'_{m-1}(x,t)\right|\right] \\ &\leq K_M(f)\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\left(i-j+1+\sqrt{R-1}M\right)\left(\sqrt{R-1}M\right)^{i-j} = \bar{\beta}_j(M). \end{aligned} \tag{3.24}$$

By $S_m^{(k)}(t) \geq \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|u_m^{(k)}(t)\right\|_a^2$, it follows from (3.20), (3.22) and (3.24) that

$$
\begin{aligned}
&\left|\dot{F}_m^{(k)}(x,t)\right| \\
&\leq \left|\Phi'_{m0}(x,t)\right| \\
&\quad + \sum_{j=1}^{N-1}\left[\left|\Phi'_{mj}(x,t)\right|\left|u_m^{(k)}(x,t)\right|^j + j\left|\Phi_{mj}(x,t)\right|\left|u_m^{(k)}(x,t)\right|^{j-1}\left|\dot{u}_m^{(k)}(x,t)\right|\right] \\
&\leq \bar{\beta}_0(M) + \sum_{j=1}^{N-1}\left(\sqrt{R-1}\right)^j\left[\bar{\beta}_j(M) + j\bar{\alpha}_j(M)\right]\left(\sqrt{S_m^{(k)}(t)}\right)^j \\
&= \sum_{j=0}^{N-1}\left(\sqrt{R-1}\right)^j\left[\bar{\beta}_j(M) + j\bar{\alpha}_j(M)\right]\left(\sqrt{S_m^{(k)}(t)}\right)^j
\end{aligned}
$$

thus

$$
\begin{aligned}
&\left|\dot{F}_m^{(k)}(x,t)\right| \\
&\leq \sum_{j=0}^{N-1}\left(\sqrt{R-1}\right)^j\left[\bar{\beta}_j(M) + j\bar{\alpha}_j(M)\right]\left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right] \\
&= d_M^{(1)}\left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right],
\end{aligned}
\tag{3.25}
$$

where $d_M^{(1)} = \sum_{j=0}^{N-1}\left(\sqrt{R-1}\right)^j\left[\bar{\beta}_j(M) + j\bar{\alpha}_j(M)\right]$.

Therefore, $(3.18)_{(ii)}$ follows. Lemma 3.2 is proved. □

In what follows, we estimate the integrals I_5 and I_6.

Estimate of I_5. By Lemma 3.2 (i) and the inequality $S_m^{(k)}(t) \geq \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(t)\right\|_0^2$, we obtain

$$
\begin{aligned}
I_5 &= \int_0^t \left\langle F_m^{(k)}(s), \dot{u}_m^{(k)}(s) + A\dot{u}_m^{(k)}(s)\right\rangle ds \\
&\leq 2\int_0^t \left\|F_m^{(k)}(s)\right\|_0\left[\left\|\dot{u}_m^{(k)}(s)\right\|_0 + \left\|A\dot{u}_m^{(k)}(s)\right\|_0\right] ds \\
&\leq 2\sqrt{2}\sqrt{\frac{R^2-1}{2}}d_M\int_0^t\left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]\sqrt{S_m^{(k)}(s)}ds \\
&\leq 4\sqrt{2}\sqrt{\frac{R^2-1}{2}}d_M\int_0^t\left[1 + \left(S_m^{(k)}(s)\right)^{N-1}\right]ds.
\end{aligned}
\tag{3.26}
$$

Estimate of I_6. Note that

$$
\begin{aligned}
F_m^{(k)}(x,0) &= \sum_{j=0}^{N-1}\Phi_{mj}(x,0)\,\tilde{u}_{0k}^j(x) \\
&= \sum_{j=0}^{N-1}\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}D_3^i f(x,0,\tilde{u}_0(x))\tilde{u}_0^{i-j}(x)\,\tilde{u}_{0k}^j(x),
\end{aligned}
$$

hence

$$\begin{aligned}&\left|F_m^{(k)}(x,0)\right|\\&\leq \sum_{j=0}^{N-1}\sum_{i=j}^{N-1}\frac{1}{j!(i-j)!}\sup_{(x,y)\in\Omega(\tilde{u}_0)}\left|D_3^i f(x,0,y)\right|\left(\sqrt{R-1}\right)^i\|\tilde{u}_0\|_a^{i-j}\|\tilde{u}_{0k}\|_a^j,\end{aligned}$$

where $\Omega(\tilde{u}_0)=[1,R]\times\left[-\sqrt{R-1}\,\|\tilde{u}_0\|_a,\sqrt{R-1}\,\|\tilde{u}_0\|_a\right]$.

Using the convergence given in (3.8), we get that there exists a constant $\tilde{F}_0>0$ independent of k and m such that

$$\left|F_m^{(k)}(x,0)\right|\leq\tilde{F}_0,\ \text{for all } x\in[1,R] \text{ and } m,\ k\in\mathbb{N}. \tag{3.27}$$

By Lemma 3.2 (ii), we obtain

$$\begin{aligned}\left\|F_m^{(k)}(t)\right\|_0&\leq\left\|F_m^{(k)}(0)\right\|_0+\int_0^t\left\|\dot{F}_m^{(k)}(s)\right\|_0 ds\\&\leq\tilde{F}_0+\sqrt{\frac{R^2-1}{2}}d_M^{(1)}\int_0^t\left[1+\left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]ds.\end{aligned}\tag{3.28}$$

By (3.27) and (3.28), it follows that

$$\begin{aligned}I_6&=\left\langle F_m^{(k)}(t),A\dot{u}_m^{(k)}(t)\right\rangle\\&\leq\frac{1}{4}\left\|A\dot{u}_m^{(k)}(t)\right\|_0^2+\left\|F_m^{(k)}(t)\right\|_0^2\\&=\frac{1}{4}S_m^{(k)}(t)+2\tilde{F}_0^2+2(R^2-1)\left(d_M^{(1)}\right)^2T^*\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds.\end{aligned}\tag{3.29}$$

Combining (3.14), (3.17), (3.26) and (3.29), it implies from (3.13) that

$$S_m^{(k)}(t)\leq\bar{S}_{0k}+\tilde{\gamma}_M\int_0^t\left[1+\left(S_m^{(k)}(s)\right)^{N-1}\right]ds,\tag{3.30}$$

where

$$\begin{aligned}\bar{S}_{0k}&=4\left(\|\tilde{u}_{0k}\|_a^2+\|A\tilde{u}_{0k}\|_0^2\right)+8\left(\|A\tilde{u}_{0k}\|_0^2+\tilde{F}_0^2\right),\\\tilde{\gamma}_M&=4\left[2\left(C_1+1\right)\sqrt{T^*}\,\|g\|_{L^2(0,T^*)}+\|g\|_{L^2(0,T^*)}^2+2T^*\right]\\&\quad+8\left[2\sqrt{2}\sqrt{\frac{R^2-1}{2}}d_M+(R^2-1)\left(d_M^{(1)}\right)^2T^*\right],\end{aligned}\tag{3.31}$$

Also, by using the convergences given in (3.8), we can deduce the existence of a constant $M>0$ independent of k and m such that

$$\bar{S}_{0k}\leq\frac{M^2}{2},\ \text{for all } m,\ k\in\mathbb{N}.\tag{3.32}$$

Finally, it follows from (3.30) and (3.32) that

$$S_m^{(k)}(t)\leq\frac{M^2}{2}+T\tilde{\gamma}_M+\tilde{\gamma}_M\int_0^t\left(S_m^{(k)}(s)\right)^{N-1}ds,\ 0\leq t\leq T_m^{(k)}\leq T.\tag{3.33}$$

Then, by solving the Volterra nonlinear integral inequality (3.33) (based on the methods in [28]), the following lemma is proved.

Lemma 3.3. *There exists a constant $T > 0$ independent of k and m such that*

$$S_m^{(k)}(t) \le M^2, \ \forall t \in [0,T], \ \forall m, k \in \mathbb{N}. \tag{3.34}$$

By Lemma 3.3, we can take constant $T_m^{(k)} = T$ for all k and $m \in \mathbb{N}$. Thus, we have

$$u_m^{(k)} \in W(M,T), \ \forall m, \ k \in \mathbb{N}. \tag{3.35}$$

From (3.35), we obtain that there exists a subsequence $\{u_m^{(k_j)}\}$ of $\{u_m^{(k)}\}$, still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T; V \cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T; V \cap H^2) \text{ weakly*}, \\ u_m \in W(M,T). \end{cases} \tag{3.36}$$

Using the compactness lemma of Lions ([29], p.57) and applying Fischer-Riesz theorem, from (3.36), there exists a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol satisfying

$$u_m^{(k)} \to u_m \text{ strong in } L^2(0,T;V) \text{ and a.e. in } Q_T. \tag{3.37}$$

On the other hand, by using the inequality

$$\left|a^j - b^j\right| \le j M_1^{j-1} |a-b|, \ \forall a, b \in [-M_1, M_1], \ \forall M_1 > 0, \ \forall j \in \mathbb{N}, \tag{3.38}$$

we deduce from (3.34) that

$$\begin{aligned} &\left|(u_m^{(k)}(x,t))^j - u_m^j(x,t)\right| \\ &\le j\left(\sqrt{R-1}M\right)^{j-1}\left|u_m^{(k)}(x,t) - u_m(x,t)\right|, \ 0 \le j \le N-1. \end{aligned} \tag{3.39}$$

Therefore, (3.37) and (3.39) imply

$$(u_m^{(k)})^j \to u_m^j \text{ strong in } L^2(Q_T). \tag{3.40}$$

By (3.4), (3.7) and (3.20), we get that

$$\begin{aligned} \left\|F_m^{(k)} - F_m\right\|_{L^2(Q_T)} &\le \sum_{j=0}^{N-1} \left\|\Phi_{mj}\left[\left(u_m^{(k)}\right)^j - u_m^j\right]\right\|_{L^2(Q_T)} \\ &\le \sum_{j=0}^{N-1} \bar{\alpha}_j(M) \left\|\left(u_m^{(k)}\right)^j - u_m^j\right\|_{L^2(Q_T)} \to 0. \end{aligned} \tag{3.41}$$

this leads to

$$F_m^{(k)} \to F_m \text{ strong in } L^2(Q_T). \tag{3.42}$$

Taking the limits in (3.7) and (3.8), we have u_m satisfying (3.3) and (3.4) in $L^2(0,T)$. Theorem 3.1 is proved. □

By using Theorem 3.1 and the compact imbedding theorems, we shall prove the existence and uniqueness of weak local in time solution for the problem (1.1).

First, we consider the space

$$W_1(T) = \left\{v \in L^\infty(0,T;V) : v' \in L^2(0,T;V)\right\}, \tag{3.43}$$

then $W_1(T)$ is a Banach space with the norm (see Lions [29])

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^2(0,T;V)}.$$

Theorem 3.4. *Let $(H_1)-(H_3)$ hold. Then, there exist constants $M>0$ and $T>0$ such that the problem (1.1) has a unique weak solution $u \in W(M,T)$, and the recurrent sequence $\{u_m\}$ defined by (3.2)-(3.5) strongly converges at a rate of order N to u in $W_1(T)$ in sense*

$$\|u_m - u\|_{W_1(T)} \le C\,\|u_{m-1}-u\|^N_{W_1(T)}, \tag{3.44}$$

for all $m \ge 1$, where C is a suitable constant. On the other hand, the following estimate is fulfilled

$$\|u_m - u\|_{W_1(T)} \le C_T\,(\beta_T)^{N^m}, \ \textit{for all } m \in \mathbb{N}, \tag{3.45}$$

where $C_T>0$ and $0<\beta_T<1$ are constants only depending on T.

Proof. We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$.

Indeed, we put $v_m = u_{m+1}-u_m$. Then v_m satisfies the variational problem

$$\begin{cases} \langle v_m'(t),w\rangle + a(v_m'(t),w) + a(v_m(t),w) \\ = \int_0^t g(t-s)a\left(v_m(s),w\right)ds + \left\langle F_{m+1}(t) - F_m(t), w\right\rangle, \ \forall w \in V, \\ v_m(0)=0, \end{cases} \tag{3.46}$$

where

$$F_m(t) = \sum\nolimits_{i=0}^{N-1} \frac{1}{i!} D_3^i f\left[u_{m-1}\right](x,t)\left(u_m(x,t)-u_{m-1}(x,t)\right)^i. \tag{3.47}$$

Taking $w=v_m'(t)$ in (3.46), after integrating in t, we have

$$\begin{aligned} Z_m(t) &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)a\left(v_m(s),v_m'(\tau)\right)ds \\ &\quad + 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), v_m'(s)\right\rangle ds \\ &= J_1 + J_2, \end{aligned} \tag{3.48}$$

where

$$Z_m(t) = 2\int_0^t \left(\|v_m'(s)\|_0^2 + \|v_m'(s)\|_a^2\right)ds + \|v_m(t)\|_a^2. \tag{3.49}$$

Next, we have to estimate the integrals on the right-hand side of (3.48).

By using the inequalitiy $2ab \le \alpha a^2 + \frac{1}{\alpha}b^2$, $\forall a,b \in \mathbb{R}$, $\alpha>0$, then J_1 is estimated as follows

$$\begin{aligned} J_1 &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)a\left(v_m(s),v_m'(\tau)\right)ds \\ &\le 2\int_0^t d\tau \int_0^\tau |g(\tau-s)|\,\|v_m(s)\|_a\,\|v_m'(\tau)\|_a\,ds d\tau \\ &\le 2\int_0^t d\tau \int_0^\tau |g(\tau-s)|\sqrt{Z_m(s)}\,\|v_m'(\tau)\|_a\,ds \\ &\le \frac{1}{4}Z_m(t) + 2T^*\|g\|^2_{L^2(0,T^*)}\int_0^t Z_m(s)\,ds. \end{aligned} \tag{3.50}$$

Using Taylor's expansion of the function $f(x,t,u_m)=f(x,t,u_{m-1}+v_{m-1})$ around the point u_{m-1} up to order N, we obtain

$$\begin{aligned} &f(x,t,u_m)-f(x,t,u_{m-1}) \\ &=\sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(x,t,u_{m-1})v_{m-1}^i+\frac{1}{N!}D_3^N f(x,t,\delta_m)v_{m-1}^N, \end{aligned} \tag{3.51}$$

where $\delta_m=\delta_m(x,t)=u_{m-1}+\theta v_{m-1}$, $0<\theta<1$.

Hence

$$\begin{aligned} &F_{m+1}(x,t)-F_m(x,t) \\ &=\sum_{i=1}^{N-1}\frac{1}{i!}D_3^i f(x,t,u_m(x,t))\left(v_m(x,t)\right)^i \\ &\quad+\frac{1}{N!}D_3^N f(x,t,\delta_m(x,t))v_{m-1}^N(x,t). \end{aligned} \tag{3.52}$$

Note that

$$\begin{aligned} |v_m(x,t)|^i &\le \left(\sqrt{R-1}\,\|v_m(t)\|_a\right)^i\le\left(\sqrt{R-1}\right)^i(2M)^{i-1}\sqrt{Z_m(t)}, \\ \left|v_{m-1}^N(x,t)\right| &\le \left(\sqrt{R-1}\,\|v_{m-1}(t)\|_a\right)^N\le\left(\sqrt{R-1}\right)^N\|v_{m-1}\|_{W_1(T)}^N. \end{aligned} \tag{3.53}$$

Therefore, we have

$$\begin{aligned} &\|F_{m+1}(t)-F_m(t)\|_0 \\ &\le\left(\frac{R^2-1}{2}\right)K_M(f)\sum_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{R-1}\right)^i(2M)^{i-1}\sqrt{Z_m(t)} \\ &\quad+\frac{1}{N!}\left(\frac{R^2-1}{2}\right)K_M(f)\left(\sqrt{R-1}\right)^N\|v_{m-1}\|_{W_1(T)}^N \\ &\equiv\bar\gamma_1(M)\sqrt{Z_m(t)}+\bar\gamma_2(M)\|v_{m-1}\|_{W_1(T)}^N, \end{aligned} \tag{3.54}$$

where

$$\begin{aligned} \bar\gamma_1(M)&=\left(\frac{R^2-1}{2}\right)K_M(f)\sum_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{R-1}\right)^i(2M)^{i-1}, \\ \bar\gamma_2(M)&=\frac{1}{N!}\left(\frac{R^2-1}{2}\right)K_M(f)\left(\sqrt{R-1}\right)^N. \end{aligned} \tag{3.55}$$

By using the inequality (3.54) above, J_2 can be estimated as follows

$$\begin{aligned} J_2&=2\int_0^t\langle F_{m+1}(s)-F_m(s),v_m'(s)\rangle\,ds \\ &\le 2\int_0^t\|F_{m+1}(s)-F_m(s)\|_0^2\,ds+\frac{1}{2}\int_0^t\|v_m'(s)\|_0^2\,ds \\ &\le 4T\bar\gamma_2^2(M)\|v_{m-1}\|_{W_1(T)}^{2N}+4\bar\gamma_1^2(M)\int_0^t Z_m(s)ds+\frac{1}{4}Z_m(t). \end{aligned} \tag{3.56}$$

By (3.48) and (3.50), it follows from (3.56) that

$$Z_m(t) \leq 8T\bar{\gamma}_2^2(M) \|v_{m-1}\|_{W_1(T)}^{2N} + 4\left(2\bar{\gamma}_1^2(M) + T^* \|g\|_{L^2(0,T^*)}^2\right) \int_0^t Z_m(s)\, ds. \tag{3.57}$$

Using Gronwall's Lemma, we have

$$Z_m(t) \leq 8T\bar{\gamma}_2^2(M) \exp\left[4T\left(2\bar{\gamma}_1^2(M) + T^* \|g\|_{L^2(0,T^*)}^2\right)\right] \|v_{m-1}\|_{W_1(T)}^{2N}.$$

This leads to

$$\|v_m\|_{W_1(T)} \leq \mu_T \|v_{m-1}\|_{W_1(T)}^N, \ \forall m \in \mathbb{N}, \tag{3.58}$$

where $\mu_T = \left(2+\sqrt{2}\right)\bar{\gamma}_2(M)\sqrt{2T}\exp\left[2T\left(2\bar{\gamma}_1^2(M) + T^* \|g\|_{L^2(0,T^*)}^2\right)\right]$.

Note that

$$\begin{aligned} &\|u_m - u_{m+p}\|_{W_1(T)} \\ &\leq \|u_m - u_{m+1}\|_{W_1(T)} + \|u_{m+1} - u_{m+2}\|_{W_1(T)} + \cdots + \|u_{m+p-1} - u_{m+p}\|_{W_1(T)} \\ &\equiv \|v_m\|_{W_1(T)} + \|v_{m+1}\|_{W_1(T)} + \cdots + \|v_{m+p-1}\|_{W_1(T)}, \ \forall m,\ p \in \mathbb{N} \end{aligned} \tag{3.59}$$

On the other hand, we obtain from (3.58) that

$$\begin{aligned} \|v_m\|_{W_1(T)} &\leq \mu_T \|v_{m-1}\|_{W_1(T)}^N \\ &\leq \mu_T\left(\mu_T \|v_{m-2}\|_{W_1(T)}^N\right)^N = \mu_T \mu_T^N \left(\|v_{m-2}\|_{W_1(T)}\right)^{N^2} \\ &\leq \mu_T \mu_T^N \left(\mu_T \|v_{m-3}\|_{W_1(T)}^N\right)^{N^2} = \mu_T \mu_T^N \mu_T^{N^2} \left(\|v_{m-3}\|_{W_1(T)}\right)^{N^3} \\ &\leq \cdots \leq \mu_T \mu_T^N \mu_T^{N^2} \mu_T^{N^3} \cdots \mu_T^{N^{m-1}} \left(\|v_0\|_{W_1(T)}\right)^{N^m} \\ &= \mu_T^{1+N+N^2+\cdots+N^{m-1}} \left(\|v_0\|_{W_1(T)}\right)^{N^m} = \mu_T^{\frac{1-N^m}{1-N}} \left(\|v_0\|_{W_1(T)}\right)^{N^m} \\ &= \mu_T^{\frac{-1}{N-1}} \left(\mu_T^{\frac{1}{N-1}} \|v_0\|_{W_1(T)}\right)^{N^m} \\ &\leq \mu_T^{\frac{-1}{N-1}} \left(M\mu_T^{\frac{1}{N-1}}\right)^{N^m} \equiv \mu_T^{\frac{-1}{N-1}} \left(\beta_T\right)^{N^m}, \ \forall m \in \mathbb{N} \end{aligned} \tag{3.60}$$

where $\beta_T = M\mu_T^{\frac{1}{N-1}}$.

Hence, applying (3.60) to (3.59) and using the fact that $\beta_T < 1$ with choosing $T > 0$ small enough, we have

$$
\begin{aligned}
&\|u_m - u_{m+p}\|_{W_1(T)} \\
&\leq \|u_m - u_{m+1}\|_{W_1(T)} + \|u_{m+1} - u_{m+2}\|_{W_1(T)} + \cdots + \|u_{m+p-1} - u_{m+p}\|_{W_1(T)} \\
&\equiv \|v_m\|_{W_1(T)} + \|v_{m+1}\|_{W_1(T)} + \cdots + \|v_{m+p-1}\|_{W_1(T)} \\
&\leq \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} + \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^{m+1}} + \ldots + \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^{m+p-1}} \\
&= \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \left[1 + (\beta_T)^{N^{m+1}-N^m} + (\beta_T)^{N^{m+2}-N^m} + \ldots + (\beta_T)^{N^{m+p-1}-N^m}\right] \\
&= \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \left[1 + (\beta_T)^{(N-1)N^m} + (\beta_T)^{(N^2-1)N^m} + \ldots + (\beta_T)^{(N^{p-1}-1)N^m}\right]
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\|u_m - u_{m+p}\|_{W_1(T)} \\
&\leq \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \left[1 + (\beta_T)^{N^m} + (\beta_T)^{2N^m} + \ldots + (\beta_T)^{(p-1)N^m}\right] \\
&= \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \left[1 + (\beta_T)^{N^m} + \left((\beta_T)^{N^m}\right)^2 + \ldots + \left((\beta_T)^{N^m}\right)^{p-1}\right] \\
&= \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \frac{1 - \left((\beta_T)^{N^m}\right)^p}{1 - (\beta_T)^{N^m}} \leq \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \frac{1}{1 - (\beta_T)^{N^m}} \\
&\leq \mu_T^{\frac{-1}{N-1}} (\beta_T)^{N^m} \frac{1}{1 - \beta_T} = \mu_T^{\frac{-1}{N-1}} (1 - \beta_T)^{-1} (\beta_T)^{N^m}, \text{ for all } m \text{ and } p.
\end{aligned}
\tag{3.61}
$$

The inequality (3.61) ensures that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \text{ strongly in } W_1(T). \tag{3.62}$$

Note that $u_m \in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$
\begin{cases}
u_{m_j} \to u & \text{in} \quad L^\infty(0,T; V \cap H^2) \text{ weakly*}, \\
u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T; V \cap H^2) \text{ weakly*}, \\
u_m \in W(M,T).
\end{cases}
\tag{3.63}
$$

We note that

$$
\begin{aligned}
|F_m(x,t) - f[u_{m-1}](x,t)| &\leq \sum_{i=1}^{N-1} \frac{1}{i!} \left|D_3^i f[u_{m-1}](x,t)\right| |u_m(x,t) - u_{m-1}(x,t)|^i \\
&\leq K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{R-1}\,\|u_m - u_{m-1}\|_{W_1(T)}\right)^i; \\
|f[u_{m-1}](x,t) - f[u](x,t)| &\leq K_M(f) |u_{m-1}(x,t) - u(x,t)| \\
&\leq K_M(f) \sqrt{R-1}\,\|u_{m-1} - u\|_{W_1(T)}.
\end{aligned}
$$

Hence

$$\|F_m - f[u_{m-1}]\|_{L^\infty(Q_T)} \leq K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{R-1}\,\|u_m - u_{m-1}\|_{W_1(T)}\right)^i \to 0,$$
$$\|f[u_{m-1}] - f[u]\|_{L^\infty(Q_T)} \leq K_M(f)\sqrt{R-1}\,\|u_{m-1} - u\|_{W_1(T)} \to 0, \tag{3.64}$$

it follows that

$$\|F_m - f[u]\|_{L^\infty(Q_T)} \leq \|F_m - f[u_{m-1}]\|_{L^\infty(Q_T)} + \|f[u_{m-1}] - f[u]\|_{L^\infty(Q_T)} \to 0. \tag{3.65}$$

The estimates (3.64) and (3.65) imply that

$$F_m \longrightarrow f[u] \text{ strongly in } L^\infty(Q_T). \tag{3.66}$$

Taking the limits in (3.3) and (3.4) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the equation

$$\langle u'(t), w\rangle + a(u'(t), w) + a(u(t), w) = \int_0^t g(t-s)a(u(s), w)\,ds + \langle f[u](t), w\rangle,$$

for all $w \in V$ and the initial condition $u(0) = \tilde{u}_0$.

Finally, letting $m = m_j \to \infty$ in (3.3), (3.4) and using (3.62), (3.63) and (3.66), we get that there exists $u \in W(M,T)$ satisfying (3.1). The proof of existence is completed.

Next, we are easy to prove that the uniqueness of solutions of (3.1). Afterward, by passing to the limit in (3.61) as $p \to \infty$ for fixed m, we get (3.45). Theorem 3.4 is proved completely. □

Acknowledgements

The authors wish to express their sincere thanks to the editor and the referees for the valuable comments and important remarks for the improvement of the paper. The first author (Nguyen Huu Nhan) would also like to express gratitude to the University of Foreign Languages - Information Technology of Ho Chi Minh City for funding the project under grant number **H2023-05**.

References

[1] R.E. Showalter, T.W. Ting , Pseudoparabolic partial differential equations, SIAM J. Math. Anal. 1 (1970) 1–26.

[2] T.W. Ting, Certain non-steady flows of second-order fluids, Arch. Ration. Mech. Anal. 14 (1963) 1–26.

[3] C.J. Amick, J.L. Bona, M.E. Schonbek, Decay of solutions of some nonlinear wave equations, J. Differ. Equ. 81 (1989) 1–49.

[4] S. Asghar, T. Hayat, P.D. Ariel, Unsteady Couette flows in a second grade fluid with variable material properties, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 154–159.

[5] T. Aziz, F.M. Mah, A note on the solutions of some nonlinear equations arising in third-grade fluid flows: An exact approach, Sci. World J. 2014 (2014) Article ID 109128.

[6] G. Barenblat, I. Zheltov, I. Kochiva, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, J. Appl. Math. Mech. 24 (1960) 1286–1303.

[7] T.B. Benjamin, J.L. Bona, J.J. Mahony, Models equation of long waves in nonlinear dispersive systems, Philos. Trans. R. Soc. Lond. Ser. A 272 (1220) (1972) 47–78.

[8] J.L. Bona, V.A. Dougalis, An initial and boundary value problem for a model equation for propagation of long waves, J. Math. Anal. Appl. 75 (1980) 503–522.

[9] A. Bouziani, Solvability of nonlinear pseudoparabolic equation with a nonlocal boundary condition, Nonlinear Anal. 55 (2003) 883–904.

[10] A. Bouziani, Initial-boundary value problem for a class of pseudoparabolic equations with integral boundary conditions, J. Math. Anal. Appl. 291 (2004) 371–386.

[11] Y. Cao, J. Yin, C. Wang, Cauchy problems of semilinear pseudoparabolic equations, J. Differ. Equ. 246 (2009) 4568–4590.

[12] Y. Cao, J.X. Lin, Y.H. Li, One-dimensional viscous diffusion of higher order with gradient dependent potentials and sources, Acta. Math. Sin. 246 (12) (2018) 4568–4590.

[13] D.Q. Dai, Y. Huang, A moment problem for one-dimensional nonlinear pseudoparabolic equation, J. Math. Anal. Appl. 328 (2007) 1057–1067.

[14] T. Hayat, M. Khan, M. Ayub, Some analytical solutions for second grade fluid flows for cylindrical geometries, Math. Comput. Modelling 43 (2006) 16–29.

[15] T. Hayat, F. Shahzad, M. Ayub, Analytical solution for the steady flow of the third grade fluid in a porous half space, Appl. Math. Model. 31 (2007) 2424–2432.

[16] A. Mahmood, N.A. Khan, C. Fetecau, M. Jamil, Q. Rubbab, Exact analytic solutions for the flow of second grade fluid between two longitudinally oscillating cylinders, Journal of Prime Research in Mathematics 5 (2009) 192–204.

[17] L.A. Medeiros, M.M. Miranda, Weak solutions for a nonlinear dispersive equation, J. Math. Anal. Appl. 59 (1977) 432–441.

[18] V. Padron, Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation, Trans. Amer. Math. Soc. 356 (2004) 2739–2756.

[19] M. Sajid, T. Hayat, Series solution for steady flow of a third grade fluid through porous space, Transp. Porous Media 71 (2008) 173–183.

[20] Y.D. Shang, B.L. Guo, On the problem of the existence of global solution for a class of nonlinear convolutional intergro-differential equation of pseudoparabolic type, Acta Math. Appl. Sin. 26 (2003) 512–524.

[21] L. Zhang, Decay of solution of generalized Benjamin-Bona-Mahony-Burgers equations in n-space dimensions, Nonlinear Anal. TMA. 25 (1995) 1343–1369.

[22] X. Zhu, F. Li, Y. Li, Global solutions and blow-up solutions to a class pseudoparabolic equations with nonlocal term, Appl. Math. Comput. 329 (2018) 38–51.

[23] N.T. Long, A.P.N. Dinh, T.N. Diem, Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator, J. Math. Anal. Appl. 267 (2002) 116–134.

[24] N.T. Long, L.T.P. Ngoc, On a nonlinear Kirchhoff-Carrier wave equation in the unit membrane: The quadratic convergence and asymptotic expansion of solutions, Demonstratio Math. 40 (2007) 365–392.

[25] L.X. Truong, L.T.P. Ngoc, N.T. Long, High-order iterative schemes for a nonlinear Kirchhoff-Carrier wave equation associated with the mixed homogeneous conditions, Nonlinear Anal. TMA. 71 (1–2) (2009) 467–484.

[26] L.T.P. Ngoc, L.H.K. Son, N.T. Long, A N-order iterative scheme for a nonlinear Carrier wave in an annular with Robin-Dirichlet conditions, Nonlinear Funct. Anal. Appl. 22 (2017) 147–169.

[27] R.E. Showalter, Hilbert Space Methods for Partial Differential Equations, Electronic J. Differ. Equ. Monograph 01, 1994.

[28] V. Lakshmikantham, S. Leela, Differential and Integral Inequalities, Vol. 1., Academic Press, New York, 1969.

[29] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non-linéaires, Dunod; Gauthier-Villars, Paris, 1969.