Thai Journal of Mathematics
Volume 22 Number 3 (2024)
Pages 533–543
http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On the Quaternionic Normal Curves in the Euclidean Space

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Abstract In this paper, we define the quaternionic normal curves in Euclidean 3-space E^3 and Euclidean 4-space E^4. We obtain some characterizations of quaternionic normal curves in terms of their curvature functions. Moreover, we give necessary and sufficient condition for a quaternionic curves to be quaternionic normal curves in E^3 and E^4 respectively.

MSC: 53A04; 11R52
Keywords: normal curves; real quaternion; quaternionic curve; position vector

Submission date: 13.01.2020 / Acceptance date: 08.11.2023

1. Introduction

In mathematics, the quaternions are extensions of the complex numbers. They were first described by Irish mathematician Sir William Rowan Hamilton in 1843 and were applied to mechanics in three-dimensional space. Hamilton defined a quaternion as the quotient of two directed lines in a three-dimensional space or equivalently as the quotient of two vectors. Quaternions can also be represented as the sum of a scalar and a vector.

In the Euclidean space E^3, it is well known that normal curves, i.e., curves with position vector always lying in their normal plane, are spherical curves [1]. Analogously, timelike normal curves in Minkowski 3-space E_1^3 are defined as the curves whose normal planes always contain a fixed point. Therefore, the position vector of such curves (with respect to some chosen origin), always lies in its normal plane [2]. In particular, timelike normal curves lie in pseudosphere in E_1^3. Recently, İlarslan [3], has been studied some characterizations of spacelike normal curves in the Minkowski 3-space E_1^3. Also İlarslan and Nesovic [4] have been investigated spacelike and timelike normal curves in E_1^4.

In [5], the Serret-Frenet formulae for a quaternionic curves in E^3 and E^4 are given by Baharathi and Nagaraj. In analogy with the Euclidean case, Serret-Frenet formulae for a quaternionic curves in semi-Euclidean space E_2^4 is defined in [6]. Quaternionic inclined curves and harmonic curvatures for the quaternionic curves are given in [7]. Moreover,

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Published by The Mathematical Association of Thailand.

characterization of quaternionic B_2-slant helices in Euclidean space E^4 given in [8] and quaternionic Mannheim curves are studied in semi Eucliden space E^4 in [9].

In this paper, we define the quaternionic normal curves in E^3 and E^4. and obtain some characterizations of quaternionic normal curves in terms of their curvature functions. Moreover, we give the necessary and sufficient condition for quaternionic curves to be quaternionic normal curves in E^3 and E^4 respectively.

2. Preliminaries

In this section a brief summary of the theory of quaternions in the Euclidean space and normal curves are presented.

The space of quaternions Q are isomorphic to R^4, four-dimensional vector space over the real numbers. There are three operations in Q : addition, scalar multiplication, and quaternion multiplication defined by the sum of two elements of Q is defined to be their sum as elements of R^4. Similarly the product of an element of Q by a real number is defined to be the same as the product in R^4.

A real quaternion q is an expression of the form $q = ae_1 + be_2 + ce_3 + de_4$, where a, b, c and d are real numbers, and e_1, e_2, e_3 are quaternionic units which satisfy the non-commutative multiplication rules,

$$\begin{aligned} \text{i) } e_i \times e_i &= -e_4, \qquad (e_4 = 1,\ 1 \leq i \leq 3) \\ \text{ii) } e_i \times e_j &= e_k = -e_j \times e_i, \quad (1 \leq i, j \leq 3), \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space E^4. A real quaternion can be written as a linear combination of scalar part $S_q = d$ and vectorial part $V_q = ae_1 + be_2 + ce_3$. The product of two quaternions can be expanded as

$$p \times q = S_p S_q - < V_p, V_q > + S_p V_q + S_q V_p + V_q \wedge V_q$$

for every $p, q \in Q$, where $<,>$ and $\wedge$ are inner product and cross product on E^3, respectively. The conjugate of the quaternion q is denoted by $\overline{q}$ and defined as

$$\overline{q} = S_q - V_q = de_4 - ae_1 - be_2 - ce_3,$$

and is called by "Hamiltonian conjugation of q". The h-inner product of two quaternions is defined by

$$h(p, q) = \frac{1}{2}(p \times \overline{q} + q \times \overline{p}),$$

where h is the symmetric, non-degenerate, real valued and bilinear form. Thus, a norm can be defined in Q, that is

$$\|q\|^2 = h(q, q) = a^2 + b^2 + c^2 + d^2.$$

Let m_1, m_2 be fixed points in Q and $r_1, r_2 > 0$ be constants. The $2-$sphere defined by

$$S^2(m_1, r_1) = \left\{ u \in Q \,\middle|\, h(u - m_1, u - m_1) = r_1^2 \right\};$$

and the $3-$sphere is defined by

$$S^3(m_2, r_2) = \left\{ v \in Q \,\middle|\, h(v - m_2, v - m_2) = r_2^2 \right\}.$$

where m_i is the center and r_i is the radius, for $i = 1, 2$.

The three-dimensional Euclidean space E^3 is identified with the space of spatial quaternion $\{\gamma \in Q \,|\gamma + \overline{\gamma} = 0\}$ in an obvious manner.

Theorem 2.1. *Let $I = [0, 1]$ be an interval in the real line R and $s \in I$ be the parameter along the smooth curve*

$$\gamma : I \subset R \longrightarrow Q, \qquad \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \quad (1 \leq i \leq 3),$$

where the tangent $\gamma'(s) = t$ has unit length $\|t(s)\| = 1$ for all s. This unitarity condition implies;

$$t' \times \overline{t} + t \times \overline{t'} = 0.$$

The last equation implies that t' is orthogonal to t and $t' \times \overline{t}$ is a spatial quaternion. Let $\{t(s), n_1(s), n_2(s)\}$ be the Frenet trihedron in the point $\gamma(s)$ of the quaternionic curve γ. Then Frenet equations are

$$\begin{aligned} t'(s) &= k(s) n_1(s) \\ n_1'(s) &= -k(s) t(s) + r(s) n_2(s) \\ n_2'(s) &= -r(s) n_1(s) \end{aligned} \tag{2.1}$$

where t is the unit tangent, n_1 is the unit principal normal, n_2 is the unit binormal vector fields, k is the principal curvature and r is the torsion of the quaternionic curve γ, [5].

In this section, the four-dimensional Euclidean space E^4 is identified with the space of unit quaternion.

Theorem 2.2. *Let*

$$\beta : I \subset R \longrightarrow Q, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1$$

be a smooth curve (β) in E^4 defined over the interval I. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum_{i=1}^{4} \gamma_i'(s) e_i$ has unit magnitude. Let $\{T, N_1, N_2, N_3\}$ be the Frenet apparatus of the differentiable Euclidean space curve in E^4. Then the Frenet equations are

$$\begin{aligned} T'(s) &= K N_1(s) \\ N_1'(s) &= -K T(s) + k N_2(s) \\ N_2'(s) &= -k N_1(s) + (r - K) N_3(s) \\ N_3'(s) &= -(r - K) N_2, \end{aligned} \tag{2.2}$$

where $N_1 = t \times T$, $N_2 = n_1 \times T$, $N_3 = n_2 \times T$ and $K = \|T'(s)\|$, [5].

It is obtained the Frenet formulae in [5] and the apparatus for the curve β by making use of the Frenet formulae for a curve γ in E^3. Moreover, there are relationships between curvatures of the curves β and γ. These relations can be explained that the torsion of β is the principal curvature of the curve γ. Also, the bitorsion of β is $(r - K)$, where r is the torsion of γ and K is the principal curvature of β. These relations are only determined for quaternions, [5].

For further quaternions concepts see [10].

Now, we recall some basic notions about normal curves. In E^3, it is well-known that to each unit speed curve $\alpha : I \subset R \rightarrow E^3$ with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields t, n and b called the tangent, the principal normal and the binormal vector fields respectively. At each point $\alpha(s)$ of the curve α, the planes spanned by $\{t,n\}$, $\{t,b\}$ and $\{n,b\}$ are called as the osculating plane, the rectifying plane and the normal plane respectively. For simplicity, the curves $\alpha : I \subset R \to E^3$ for which the position vector α lie in their rectifying plane, are called *rectifying curves* , the curves for which the position vector lie in their normal plane, are called *normal curves.* and the curves for which the position vector α lie in their osculating plane, are called *osculating curves.* Therefore, the position vector with respect to a given origin, of a normal curve α in E^3, satisfies the equation

$$\alpha(s) = \lambda(s)t(s) + \mu(s)b(s),$$

where λ and μ are arbitrary differentiable functions in terms of the arc length parameter s.

Analogously, the normal curve in 4-space E^4 can be defined. Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in E^4, consisting of the tangent, principal normal, first binormal vector field, and second binormal vector field, respectively. Normal curve as a curve whose position vector always lies in the orthogonal complement $T^{\perp}$ of its tangent vector field T of the curve. Consequently, the position vector with respect to a given origin, of a normal curve α in E^4, satisfies the equation

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B_1(s) + \nu(s)B_2(s),$$

where λ, μ and ν are arbitrary differentiable functions in terms of the arc length parameter s.

3. Some Characterization of Spatial Quaternionic Normal Curves

In a similar manner to [3], we can define the spatial quaternionic normal curves as follows. The position vector of the spatial quaternionic normal curve satisfies the following equation

$$\gamma(s) = \lambda(s)n_1(s) + \mu(s)n_2(s)$$

where λ and μ are arbitrary differentiable functions. The following theorems provide some simple characterizations of spatial quaternionic normal curves.

Theorem 3.1. *Let $\gamma = \gamma(s)$ be a unit speed spatial quaternionic normal curve in E^3 with curvatures $k(s) > 0$, $r(s) \neq 0$ for each $s \in I \subset R$. Then the following statements hold:*

(i) *The curvatures $k(s)$ and $r(s)$ satisfy the following*

$$\frac{1}{k(s)} = c_1 \cos\left(\int r(s)ds\right) + c_2 \sin\left(\int r(s)ds\right), \quad c_1, c_2 \in R;$$

(ii) *The principal normal and binormal component of the position vector of the quaternionic curve are given by*

$$\begin{aligned} h(\gamma(s), n_1) &= -c_1 \cos\left(\int r(s)ds\right) - c_2 \sin\left(\int r(s)ds\right), \\ h(\gamma(s), n_2) &= c_1 \sin\left(\int r(s)ds\right) - c_2 \cos\left(\int r(s)ds\right), \quad c_1, c_2 \in R. \end{aligned}$$

respectively. Furthermore, if $\gamma(s)$ is a unit speed spatial quaternionic curve in E^3 with the curvatures $k(s) > 0$, $r(s) \neq 0$ for each $s \in I \subset R$ and one of the statements (i) *and* (ii) *holds, then* γ *is a spatial quaternionic normal curve or congruent to a spatial quaternionic normal curve.*

Proof. (i) Let $\gamma(s)$ be a unit speed spatial quaternionic curve in E^3, where s is arclength parameter. Then by definition we have

$$\gamma(s) = \lambda(s)n_1(s) + \mu(s)n_2(s). \tag{3.1}$$

By taking the derivative of (3.1) with respect to s and applying the Frenet formulas, we get

$$-k\lambda = 1\ , \quad \lambda' - r\mu = 0\ , \quad r\lambda + \mu' = 0\ . \tag{3.2}$$

From the first and second equation in (3.2), we get

$$\lambda(s) = -\frac{1}{k(s)}\ , \quad \mu(s) = -\frac{1}{r(s)}\left(\frac{1}{k(s)}\right)'\ . \tag{3.3}$$

Thus

$$\gamma(s) = -\frac{1}{k(s)}n_1(s) - \frac{1}{r(s)}\left(\frac{1}{k(s)}\right)' n_2(s). \tag{3.4}$$

Further, from the third equation in (3.2) and using (3.3), we find the following differential equation

$$\left[\frac{1}{r}\left(\frac{1}{k}\right)'\right]' + \frac{r}{k} = 0. \tag{3.5}$$

This equation, can be written as

$$[p(s)y'(s)]' + \frac{y(s)}{p(s)} = 0, \text{for } y(s) = \frac{1}{k}, p(s) = \frac{1}{r}.$$

If we change variables in the above equation as $t = \int \frac{1}{p(s)}ds$, then we get

$$\frac{d^2y}{dt^2} + y = 0.$$

The solution of the previous differential equation is

$$y = c_1\cos(t) + c_2\sin(t),$$

where $c_1, c_2 \in R$. Therefore,

$$\frac{1}{k(s)} = c_1\cos\left(\int r(s)ds\right) + c_2\sin\left(\int r(s)ds\right). \tag{3.6}$$

(ii) By appliying (3.6) into (3.3) and (3.4), we get

$$\begin{aligned}\lambda &= -\left[c_1\cos\left(\int r(s)ds\right) + c_2\sin\left(\int r(s)ds\right)\right],\\ \mu &= \left[c_1\sin\left(\int r(s)ds\right) - c_2\cos\left(\int r(s)ds\right)\right],\end{aligned}$$

and

$$\begin{aligned}\gamma(s) &= -\left[c_1\cos\left(\int r(s)ds\right) + c_2\sin\left(\int r(s)ds\right)\right]n_1(s) \\ &\quad + \left[c_1\sin\left(\int r(s)ds\right) - c_2\cos\left(\int r(s)ds\right)\right]n_2(s).\end{aligned} \tag{3.7}$$

Therefore, from (3.7) we obtain

$$h(\gamma,\gamma) = c_1^2 + c_2^2 \tag{3.8}$$

$$h(\gamma(s), n_1) = -c_1 \cos\left(\int r(s)ds\right) - c_2 \sin\left(\int r(s)ds\right) \tag{3.9}$$

$$h(\gamma(s), n_2) = c_1 \sin\left(\int r(s)ds\right) - c_2 \cos\left(\int r(s)ds\right), \tag{3.10}$$

where $c_1, c_2 \in R$.

Now, suppose that statement (i) holds. Then we have

$$\frac{1}{k(s)} = c_1 \cos\left(\int r(s)ds\right) + c_2 \sin\left(\int r(s)ds\right). \tag{3.11}$$

By taking the derivative of (3.11) with respect to s, we get

$$\left[\frac{1}{r}\left(\frac{1}{k}\right)'\right]' = -\frac{r}{k}. \tag{3.12}$$

By using (3.12) and Frenet equations (2.1), we obtain

$$\frac{d}{ds}\left[\gamma(s) + \frac{1}{k}n_1 + \frac{1}{r}\left(\frac{1}{k}\right)' n_2\right] = 0.$$

Consequently, γ is congruent to a spatial quaternionic normal curve.

Next, assume that statement (ii) holds. Then the equation (3.8) is satisfied. Differentiating (3.8) with respect to s, we find $h(\gamma(s), t) = 0$, that is, γ spatial quaternionic normal curve. ∎

Theorem 3.2. *Let $\gamma = \gamma(s)$ be a unit speed spatial quaternionic normal curve in E^3 with curvatures $k(s) > 0$, $r(s) \neq 0$. Then γ curve lies on S^2 if and only if*

$$\frac{1}{k} = \pm\sqrt{b^2 - c^2} \cos\left(\int r(s)ds\right) + c \sin\left(\int r(s)ds\right), \quad c \in R,\ b \in R^+. \tag{3.13}$$

Proof. Let us first assume that the curve lies on S^2. Then $h(\gamma,\gamma) = b^2$, $b \in R^+$. By putting this into (3.8), we get $c_1 = \pm\sqrt{b^2 - c_2^2}$. By using the last equation and (3.6), we obtain that (3.13) holds.

Conversely, assume that (3.13) holds . Then, by differentiating with respect to s, we get $h(\gamma,\gamma) = b^2$, for $b \in R^+$. It implies that the curve lies on S^2. ∎

4. Some Characterization of Quaternionic Normal Curves in Q

Let $\beta = \beta(s)$ be a unit speed quaternionic normal curve, lying fully in Q. Then its position vector satisfies

$$\beta(s) = \lambda(s)N_1(s) + \mu(s)N_2(s) + \nu(s)N_3(s). \tag{4.1}$$

By taking the derivative of (4.1) with respect to s and using the Frenet equations (2.2), we get

$$T = -K\lambda T + (\lambda' - k\mu)N_1 + (k\lambda + \mu' - (r-K)\upsilon)N_2 + ((r-K)\mu + \upsilon')N_3$$

and therefore

$$-K\lambda = 1\ ,\quad \lambda' - k\mu = 0\ ,\quad k\lambda + \mu' - (r-K)\upsilon = 0\ ,\quad (r-K)\mu + \upsilon' = 0. \quad (4.2)$$

From the first three equations we find

$$\begin{aligned} \lambda(s) &= -\frac{1}{K(s)}\ ,\quad \mu(s) = -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)', \\ \upsilon(s) &= -\frac{1}{(r(s)-K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]. \end{aligned} \quad (4.3)$$

Applying the relation (4.3) into (4.1), we get that the position vector of the quaternionic normal curve β is:

$$\begin{aligned} \beta(s) &= -\frac{1}{K(s)}N_1 - \frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' N_2 \\ &\quad -\frac{1}{(r(s)-K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] N_3. \end{aligned} \quad (4.4)$$

Then we have the following theorem.

Theorem 4.1. *Let $\beta(s)$ be a unit speed quaternionic curve, lying fully in Q. Then $\beta(s)$ is congruent to a quaternionic normal curve if and only if*

$$-\frac{(r(s)-K(s))}{k(s)}\left(\frac{1}{K(s)}\right)' = \left[\frac{1}{(r(s)-K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]\right]'. \quad (4.5)$$

Proof. Let $\beta(s)$ be congruent to a quaternionic normal curve. Then relations (4.2) and (4.3) imply that (4.5) holds.

Conversely, assume that relation (4.5) holds. Let the vector $m \in Q$ be given by

$$\begin{aligned} m(s) &= \beta(s) + \frac{1}{K(s)}N_1 + \frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' N_2 \\ &\quad +\frac{1}{(r(s)-K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right] N_3. \end{aligned} \quad (4.6)$$

Differentiating (4.6) with respect to s and by applying (2.2), we get

$$\begin{aligned} m'(s) &= \frac{(r(s)-K(s))}{k(s)}\left(\frac{1}{K(s)}\right)' N_3 \\ &\quad +\left(\frac{1}{(r(s)-K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]\right)' N_3. \end{aligned}$$

From the relation (4.5), it follows that m is a constant vector, which means that β is congruent to a quaternionic normal curve. ∎

Theorem 4.2. *Let $\beta(s)$ be a unit speed quaternionic curve, lying fully in Q. If β is a quaternionic normal curve, then the following statements hold:*

(i) *The coefficient of the first normal and the second normal component of the position vector* β *are*

$$\begin{aligned} h(\beta, N_1) &= -\frac{1}{K(s)}, \\ h(\beta, N_2) &= -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)', \end{aligned}$$

respectively.

(ii) *The coefficient of the second normal and the third normal component of the position vector* β *are*

$$\begin{aligned} h(\beta, N_2) &= -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)', \\ h(\beta, N_3) &= -\frac{1}{(r(s)-K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]. \end{aligned}$$

respectively.

Conversely, if $\beta(s)$ *is a unit speed quaternionic curve, lying fully in* Q*, and one of statements* (i) *or* (ii) *holds, then* β *is a quaternionic normal curve.*

Proof. If $\beta(s)$ is a quaternionic normal curve, it is easy to check that relation (4.4) implies statements (i) and (ii).

Conversely, if statement (i) holds, by differentiating the equation $h(\beta, N_1) = -\frac{1}{K(s)}$ with respect to s and by applying (2.2), we find $h(\beta, T) = 0$ which means that β is a quaternionic normal curve. If statement (ii) holds, we conclude that β is a quaternionic normal curve in a similar way. ■

In the next theorem, we obtain interesting geometric characterization of quaternionic normal curves.

Theorem 4.3. *Let* $\beta(s)$ *be a unit speed quaternionic curve, lying fully in* Q*. Then* β *is congruent to a quaternionic normal curve if and only if* β *lies on* S^3 *in* Q*.*

Proof. Assume that β is congruent to a quaternionic normal curve. Then, by straightforward calculations and by Theorem 4.1, we get

$$\begin{gathered} 2\tfrac{1}{K}\left(\tfrac{1}{K}\right)' + 2\tfrac{1}{k}\left(\tfrac{1}{K}\right)'\left(\tfrac{1}{k}\left(\tfrac{1}{K}\right)'\right)' \\ +2\tfrac{1}{(r-K)}\left[\tfrac{k}{K} + \left(\tfrac{1}{k}\left(\tfrac{1}{K}\right)'\right)'\right]\left(\tfrac{1}{(r-K)}\left[\tfrac{k}{K} + \left(\tfrac{1}{k}\left(\tfrac{1}{K}\right)'\right)'\right]\right)' = 0. \end{gathered}$$

On the other hand, the previous equation is the differential of the equation

$$\left(\frac{1}{K}\right)^2 + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)^2 + \left(\frac{1}{(r-K)}\left[\frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)^2 = c, \quad c \in R. \tag{4.7}$$

By using (4.6), it is easy to check that

$$h(\beta - m, \beta - m) = \left(\frac{1}{K}\right)^2 + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)^2 + \left(\frac{1}{(r-K)}\left[\frac{k}{K} + \left(\frac{1}{k}\left(\frac{1}{K}\right)'\right)'\right]\right)^2,$$

which on gives $h(\beta-m,\beta-m)=c$, by combining with (4.7). Consequently, β lies on S^3 in Q.

Conversely, if β lies on S^3 a hypersphere in Q, then $h(\beta-m,\beta-m)=c$, $c\in R$, where $m\in Q$ is a constant vector. By taking the derivative of the previous equation with respect to s, we obtain $h(\beta-m,T)=0$ which proves the theorem. ■

Recall that an arbitrary curve β in Q is called a W-curve (or a helix), if it has constant curvature functions. The following theorem gives the characterization of quaternionic W-curve in Q, in terms of quaternionic normal curves.

Theorem 4.4. *Every unit speed quaternionic W-curve, lying fully in Q, is congruent to a quaternionic normal curve.*

Proof. By assumption we have $K(s)=c_1$, $k(s)=c_2$, $(r-K)(s)=c_3$, where $c_1,c_2,c_3\in R-\{0\}$. Since the curvature functions obviously satisfy relation (4.5), β is congruent to a normal curve by Theorem 4.1. ■

Lemma 4.5. *A unit speed quaternionic $\beta(s)$, lying fully in Q, is congruent to a quaternionic normal curve if and only if there exists a differentiable function $f(s)$ such that*

$$\begin{aligned} f(s)(r(s)-K(s)) &= \frac{k(s)}{K(s)}+\left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)', \\ f'(s) &= -\frac{(r(s)-K(s))}{k(s)}\left(\frac{1}{K(s)}\right)'. \end{aligned} \tag{4.8}$$

By using the similar methods as in [4], as well as Lemma 4.5, we obtain the following theorem which gives the necessary and the sufficient conditions for a quaternionic curvs in Q to be quaternionic normal curve.

Theorem 4.6. *Let $\beta(s)$ be a unit speed quaternionic curve in Q. Then β is congruent to a quaternionic normal curve if and only if there exist constants a_0, $b_0\in R$ such that*

$$\begin{aligned} -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)' &= \left(a_0+\int\frac{k(s)}{K(s)}\cos\theta(s)ds\right)\cos\theta(s) \\ &\quad+\left(b_0+\int\frac{k(s)}{K(s)}\sin\theta(s)ds\right)\sin\theta(s), \end{aligned} \tag{4.9}$$

where $\theta(s)=\int_0^s(r(s)-K(s))\,ds$.

Proof. If $\beta(s)$ is congruent to a quaternionic normal curve, according to Lemma 4.5 there exists a differentiable function $f(s)$ such that relation (4.8) holds. Let us define differentiable functions $\theta(s)$, $a(s)$ and $b(s)$ by

$$\begin{aligned} \theta(s) &= \int_0^s(r(s)-K(s))\,ds, \\ a(s) &= -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\cos\theta(s)+f(s)\sin\theta(s)-\int\frac{k(s)}{K(s)}\cos\theta(s)\,ds, \\ b(s) &= -\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\sin\theta(s)-f(s)\cos\theta(s)-\int\frac{k(s)}{K(s)}\sin\theta(s)\,ds. \end{aligned} \tag{4.10}$$

By using (4.8), we find $\theta'(s) = (r(s) - K(s))$, $a'(s) = 0$, $b'(s) = 0$ and thus

$$a(s) = a_0 \ , \ \ b(s) = b_0 \ , \ \ a_0, b_0 \in R. \tag{4.11}$$

By multiplying the second and the third equations in (4.10), with $\cos\theta(s)$ and $\sin\theta(s)$, respectively adding the obtained equations and from (4.11), we conclude that relation (4.9) holds.

Conversely, assume that there exist constants a_0, $b_0 \in R$ such that the relation (4.9) holds. By taking the derivative of (4.9) with respect to s, we find

$$-\frac{k(s)}{K(s)} - \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)' = (r(s) - K(s))\begin{bmatrix} -\left(a_0 + \int \frac{k(s)}{K(s)}\cos\theta(s)ds\right)\sin\theta(s) \\ +\left(b_0 + \int \frac{k(s)}{K(s)}\sin\theta(s)ds\right)\cos\theta(s)\end{bmatrix}. \tag{4.12}$$

Let us define the differentiable function $f(s)$ by

$$f(s) = \frac{1}{(r(s) - K(s))}\left[\frac{k(s)}{K(s)} + \left(\frac{1}{k(s)}\left(\frac{1}{K(s)}\right)'\right)'\right]. \tag{4.13}$$

Next, relations (4.12) and (4.13) imply

$$f(s) = \left(a_0 + \int \frac{k(s)}{K(s)}\cos\theta(s)ds\right)\sin\theta(s) - \left(b_0 + \int \frac{k(s)}{K(s)}\sin\theta(s)ds\right)\cos\theta(s).$$

By using this relation and (4.9), we obtain $f'(s) = -\frac{(r(s)-K(s))}{k(s)}\left(\frac{1}{K(s)}\right)'$. Finally, Lemma 4.5 implies that β is congruent to a quaternionic normal curve. ∎

Acknowledgements

We would like to thank the anonymous referees and the editor for providing their useful suggestions which greatly improved the presentation of the paper.

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