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Some Results in Fuzzy Metric-Like Spaces Using MA-Simulation Function

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Abstract In this manuscript, we use MA-simulation function which is already introduced, has further been utilizes to establish new fixed point results in fuzzy metric-like spaces. Moreover, it is shown that the proven result is quite a unified one to generllize several existing results. Furthermore, some new results are established to show the utility of our results. Some illustrative examples are also given which exhibit the usability of our results. Finally, we provide an application of our main result.

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1. Introduction

Menger, 1951, introduced the pioneer concept of a statistical metric [1]. Based on the concept of a statistical metric, in 1975, Kramosil and Michalek introduced the notion of a fuzzy metric in [2]. Here, we call it a KM-fuzzy metric. A KM-fuzzy metric is, in a certain sense, equivalent to a statistical metric, but there are essential differences in their definitions and interpretations. In 1994, George and Veeramani [3], see also [4], slightly modified the original concept of a KM-fuzzy metric, we call this modification by a GV-fuzzy metric. This modification allows many natural examples of fuzzy metrics, in particular, fuzzy metrics constructed from metrics. GV-fuzzy metrics appear to be more appropriate also for the study of induced topological structures. Along with the principal interest of many researchers in the theoretical aspects of the theory of fuzzy

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metrics in particular, the topological and sequential properties of fuzzy metric spaces, their completeness, fixed points of mappings, etc.fuzzy metrics have also aroused interest among specialists working in various applied areas of mathematics.

2. Preliminaries

Definition 2.1. [5] A binary operation $* : [0,1]\times[0,1]\to[0,1]$ is said to be continuous t-norm if it satisfies the following conditions:

(I) $*$ is commutative and associative;
(II) $*$ is continuous;
(III) $a*1=a$ for all $a\in[0,1]$;
(IV) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

For classical examples of continuous t-norm, we recall t-norms T_l, T_p and T_m defined as $T_l(a,b)=\max(a+b-1,0)$, $T_p(a,b)=ab$ and $T_m(a,b)=\min(a,b)$, $\forall a,b\in[0,1]$ respectively.

A fuzzy metric space in the sense of George and Veeramani [3] is defined as follows:

Definition 2.2. [3] The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $t,s>0$:

(GV-1) $M(x,y,t)>0$;
(GV-2) $M(x,y,t)=1\ \forall t>0$ iff $x=y$;
(GV-3) $M(x,y,t)=M(y,x,t)$;
(GV-4) $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$;
(GV-5) $M(x,y,.):(0,\infty)\to[0,1]$ is continuous.

It is worth pointing out that due to (GV-1) and (GV-2), $0<M(x,y,t)<1$ for all $t>0$ provided $x\ne y$, (cf. [6]). In what follows, fuzzy metric spaces in the sense of George and Veeramani will be called GV-fuzzy metric spaces. It is known that, for all $x,y\in X$, $M(x,y,.)$ is nondecreasing function. Several examples of fuzzy metric spaces can be found in George and Veeramani [3], Sapena [7], Gregori et al. [6] and Roldan et al. [8].

Remark 2.3. [9] The function $M(x,y,t)$ is often interpreted as the nearness between x and y with respect to t.

Remark 2.4. [10] For every $x,y\in X$, the mapping $M(x,y,.)$ is nondecreasing on $(0,\infty)$.

Definition 2.5. [11] The 3-tuple $(X,F,*)$ is a fuzzy metric like space if X is an arbitrary set $*$ is continuous norm and F is a fuzzy set in $X^2\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $t,s>0$:

(FML-1) $F(x,y,t)>0$;
(FML-2) if $F(x,y,t)=1\ \forall t>0$ then $x=y$;
(FML-3) $F(x,y,t)=F(y,x,t)$;
(FML-4) $F(x,z,t+s)\ge F(x,y,t)*F(y,z,s)$;
(FML-5) $F(x,y,.):(0,\infty)\to[0,1]$ is continuous.

Here F (endowed with $*$) is called a fuzzy metric-like on X.

Remark 2.6. A fuzzy metric-like space satisfies all of the conditions of a fuzzy metric space except that $F(x,x,t)$ may be less than 1 for all $t>0$ and for some (or may be for all) $x\in X$. Also, every fuzzy metric space is fuzzy metric-like space with unit self fuzzy distance, that is, with $F(x,x,t)=1$ for all $t>0$ and for all $x\in X$.

Note that, the axiom (GV-2) in Definition 2.2 gives the idea that when $x=y$ the degree of nearness of x and y is perfect, or simply 1, and then $M(x,x,t)=1$ for each $x\in X$ and for each $t>0$. While in fuzzy metric-like space, $F(x,x,t)$ may be less than 1, that is, the concept of fuzzy metric-like is applicable when the degree of nearness of x and y is not perfect for the case $x=y$.

Example 2.7. If $X=[0,1]$, then the triplet $(X,F,*_l)$ is a fuzzy metric-like space, where the fuzzy set F is defined by

$$F(x,y,t)=\begin{cases}1 & \text{if } x=y=0,\\ \dfrac{x+y}{2} & \text{otherwise}\end{cases}$$

for all $t>0$.

Using the following propositions, several examples of fuzzy metric-like spaces can be obtained.

Proposition 2.8. [11] *Let (X,σ) be any metric-like space (for the related definitions we refer to Harandi [12]. Then the triplet $(X,F,*_p)$ is a fuzzy metric-like space, where the fuzzy set F is given by*

$$F(x,y,t)=\frac{kt^n}{kt^n+m\sigma(x,y)}$$

for all $x,y\in X,\ t>0$, where $k\in\mathbb{R},\ m>0$ and $n\geq 1$.

Remark 2.9. [11] Proposition 2.8 shows that every metric-like space induces a fuzzy metric-like spaces. For $k=n=m=1$ the induced fuzzy metric-like space $(X,F_\sigma,*_p)$ is called the standard fuzzy metric-like space, where

$$F_\sigma(x,y,t)=\frac{t}{t+\sigma(x,y)}$$

for all $x,y\in X,\ t>0$.

Proposition 2.10. [11] *Let (X,σ) be any metric-like space. Then the triplet $(X,F,*_p)$ is a fuzzy metric-like space, where the fuzzy set F is defined by*

$$F(x,y,t)=e^{-\frac{\sigma(x,y)}{t^n}}$$

for all $x,y\in X,\ t>0$, where $n\geq 1$.

Example 2.11. Let $X=N$. Define $*$ by $a*b=ab$ and the fuzzy set F in $X^2\times(0,\infty)$ by

$$F(x,y,t)=\frac{1}{e^{\max\{x,y\}/t}}$$

for all $x,y\in X,t>0$. Then since $\sigma(x,y)=\max(x,y)$ for all $x,y\in X$ is a fuzzy metric-like on X (see [12]) therefore by Proposition 1.10 $(X,F,*)$ is a fuzzy metric-like space, but not a fuzzy metric space, as $F(x,x,t)=\frac{1}{e^{x/t}}\neq 1$ for all $x>0$ and $t>0$.

Example 2.12. [11] Let $X = [0,1]$. Define $*$ by $a * b = ab$ and the fuzzy set F in $X^2 \times (0,\infty)$ by

$$F(x,y,t) = \begin{cases} \frac{x}{y^3} & \text{if } x \leq y; \\ \frac{y}{x^3} & \text{if } y \leq x \end{cases},$$

for all $x, y \in X, t > 0$. Then $(X, F, *)$ is a fuzzy metric-like space.

We point out that the Propositions 2.8 and 2.10 are also hold even if we employ the minimum $t-$norm $*_m$ rather than product $t-$norm $*_p$ (see [11]).

Proposition 2.13. *Let (X,σ) be the bounded metric-like space, that is there exists $K > 0$ such that $\sigma(x,y) \leq K$ for all $x, y \in X$. Then the triplet $(X, F, *_l)$ is a fuzzy metric-like space, where the fuzzy set F is defined by*

$$F(x,y,t) = 1 - \frac{\sigma(x,y)}{K+t}$$

for all $x, y \in X,\ t > 0$.

Proof. The proofs of the properties (FML1)-(FML5) are obvious. For (FML4), let $x, y, z \in X$, $t > 0$, then since $\sigma(x,y) + \sigma(y,z) \geq \sigma(x,z)$, we have

$$1 - \frac{\sigma(x,y) + \sigma(y,z)}{K+t} \leq 1 - \frac{\sigma(x,z)}{K+t}.$$

It follows from the above inequality that

$$\max\left\{1 - \frac{\sigma(x,y) + \sigma(y,z)}{K+t}, 0\right\} \leq 1 - \frac{\sigma(x,z)}{K+t}.$$

which implies that (FML4) holds. ∎

Now we define convergent and Cauchy sequences in fuzzy metric-like spaces, and the completeness of fuzzy metric-like spaces.

Definition 2.14. [11] Let $(X, F, *)$ be a fuzzy metric-like space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ is said to be convergent to $x \in X$ and x is called a limit of $\{x_n\}$ if for all $t > 0$,

$$\lim_{n\to\infty} F(x_n, x, t) = F(x, x, t)$$

(ii) $\{x_n\}$ is said to be Cauchy if, for all $t > 0$ and each $p \geq 1$, the limit $\lim_{n\to\infty} F(x_{n+p}, x_n, t)$ exists.

(iii) $(X, F, *)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n\to\infty} F(x_n, x, t) = F(x, x, t) = \lim_{n\to\infty} F(x_{n+p}, x_n, t)$ for all $t > 0$ and each $p \geq 1$.

Remark 2.15. [11] In a fuzzy metric-like space, the limit of a convergent sequence need not be unique and a convergent sequence need not be a Cauchy sequence.

Lemma 2.16. [10, 13] *In a fuzzy metric-like space $(X, M, *)$, the mapping M is continuous on $X \times X \times (0,\infty)$.*

We need the following in the subsequent discussion.

Definition 2.17. [13] Let $(X, M, *)$ be a fuzzy metric-like space. A mapping $T : X \times X$ is said to be α-admissible if there exists a function $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ such that $\forall\ t > 0$

$$x, y \in X, \alpha(x, y, t) \geq 1 \implies \alpha(Tx, Ty, t) \geq 1.$$

Definition 2.18. [14] Let $(X, M, *)$ be a fuzzy metric-like space. An α-admissible mapping $T : X \times X$ is said to be triangular α-admissile if $\forall\ t > 0$

$$x, y, z \in X, \alpha(x, y, t) \geq 1 \text{ and } \alpha(y, z, t) \geq \implies \alpha(x, z, t) \geq 1.$$

Lemma 2.19. [14] *Let $(X, M, *)$ be a fuzzy metric-like space and $T : X \times X$ α-admissible mapping. Assume that there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0, t)$. Define a sequence $\{x_0\} \subseteq X$ by $x_n = Tx_{n-1}$, $\forall\ n \in \mathbb{N}$. Then we have*

$$\alpha(x_n, x_m, t) \geq 1,\ \forall\ m, n \in \mathbb{N},\ n < m.$$

3. Main Results

Inspired by Khojasteh et al. [15], we hereby introduce a new simulation function, namely, MA-simulation function. By utilizing this function, we define a new type of contraction, namely, α-admissible Ξ_{MA}-contraction, which will be utilized to establish a new result unifying several results of the existing literature besides deducing some new ones.

Definition 3.1. [16] A mapping $\xi : (0, 1] \times (0, 1] \to \mathbb{R}$ is said to be a MA-simulation function if it satisfies the following:

(ξ_1) $\xi(t, s) < \dfrac{1}{t} - \dfrac{1}{s},\ \forall\ s, t \in (0, 1)$;

(ξ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0, 1]$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0, 1)$ and $t_n < s_n,\ \forall\ n \in \mathbb{R}$ then

$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

We denote the set of all MA-simulation functions by Ξ_{MA}.

In the following lines, we furnish some examples of MA-simulation function.

Example 3.2. Let $\xi : (0, 1] \times (0, 1] \to \mathbb{R}$ be defined as

$$\xi(t, s) = k\Big(\frac{1}{t} - 1\Big) - \Big(\frac{1}{s} - 1\Big),$$

$\forall\ s, t \in (0, 1]$ and $k \in (0, 1)$.

Example 3.3. Let $\xi : (0, 1] \times (0, 1] \to \mathbb{R}$ be defined as

$$\xi(t, s) = \psi\Big(\frac{1}{t} - 1\Big) - \Big(\frac{1}{s} - 1\Big),$$

$\forall\ s, t \in (0, 1]$ where $\psi : [0, \infty) \to [0, \infty)$ is right continuous functions such that $\psi(r) < r,\ \forall\ r > 0$.

Example 3.4. Let $\xi : (0, 1] \times (0, 1] \to \mathbb{R}$ be defined as

$$\xi(t, s) = \Big(\frac{1}{t} - 1\Big) - \psi\Big(\frac{1}{t} - 1\Big) - \Big(\frac{1}{s} - 1\Big),$$

$\forall\ s, t \in (0, 1]$ where $\psi : [0, \infty) \to [0, \infty)$ is a given function such that $\psi(r) > 0$, for $r > 0$ and $\psi(0) = 0$.

Example 3.5. Let $\xi : (0,1] \times (0,1] \to \mathbb{R}$ be defined as

$$\xi(t,s) = s - \psi(t), \ \forall\ s,t \in (0,1]$$

where $\psi : (0,1] \to (0,1]$ is non-decreasing and left-continuous such that $\psi(r) > r, \ \forall\ r \in (0,1)$.

Example 3.6. Let $\xi : (0,1] \times (0,1] \to \mathbb{R}$ be defined as

$$\xi(t,s) = \Big(\frac{1}{t} - 1\Big)\psi\Big(\frac{1}{t} - 1\Big) - \Big(\frac{1}{s} - 1\Big),$$

$\forall\ s,t \in (0,1]$ where $\psi : [0,\infty) \to (0,1]$ is a given function such that $\lim_{r\to s^+}\psi(r) < 1, \ \forall\ s > 0$.

Example 3.7. Let $\xi : (0,1] \times (0,1] \to \mathbb{R}$ be defined as

$$\xi(t,s) = \Big(\frac{1}{t} - 1\Big) - \int_0^{\frac{1}{s}-1} \psi(s)ds,$$

$\forall\ s,t \in (0,1]$ where $\psi : [0,\infty) \to [0,\infty)$ is a given function such that $\int_0^r \psi(s)ds$ exists and $\int_0^r \psi(s)ds > r$ for each $r > 0$.

Now, we introduce the notion of α-admissible Ξ_{MA}-contraction.

Definition 3.8. Let $(X, M, *)$ be a fuzzy metric-like space and $\xi \in \Xi_{MA}$. A mapping $T : X \to X$ is said to be an α-admissible Ξ_{MA}-contraction if there exists a $\xi \in \Xi_{MA}$ such that $\forall\ t > 0$, it satisfies the following

$$x, y \in X, \ \alpha(x,y,t) \geq 1 \Rightarrow \xi\Big(M(x,y,t), M(Tx,Ty,t)\Big) \geq 0, \tag{3.1}$$

Now, we are equipped to present our main result.

Theorem 3.9. *Let $(X, M, *)$ be a complete fuzzy metric-like space and $T : X \to X$ an α-admissible Ξ_{MA}-contraction with respect to ξ. Assume that the following conditions are satisfied:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$;
(b) T is triangular α-admissible;
(c) T is continuous, or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$, $\forall\ n \in N, \ t > 0$ and $\{x_n\} \to x$, for some $x \in X$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x, t) \geq 1, \ \forall\ k \in \mathbb{N}$ and $t > 0$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Define a Picard sequence $\{x_n = T^n x_0\}$. Suppose there exists some $m_0 \in \mathbb{N}$ such that $T^{m_0}(x_0) = T^{m_0+1}x_0$, i.e., $x_{m_0} = x_{m_0+1}$, then x_{m_0} is a fixed point of T. Now, assume that $T_{n-1}x_0 \neq T_n x_0, \ \forall\ n \in \mathbb{N}$. Then, using Lemma 2.19, we have

$$\alpha(x_n, x_m, t) \geq 1, \ \forall\ m, n \in \mathbb{N}, \ n < m. \tag{3.2}$$

Thus, using (3.2) and (3.1), for $x = x_{n-1}$ and $y = x_n$, we obtain

$$0 \le \xi\Big(M(x_{n-1}, x_n, t), M(Tx_{n-1}, Tx_n, t)\Big) = \xi\Big(M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\Big) < \frac{1}{M(x_{n-1}, x_n, t)} - \frac{1}{M(x_n, x_{n+1}, t)},$$

which implies

$$M(x_{n-1}, x_n, t) < M(x_n, x_{n+1}, t).$$

Therefore, $\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence of positive real numbers in $(0, 1]$. Let $r(t) = \lim_{n\to\infty} M(x_n, x_{n+1}, t)$. We assert that $r(t) = 1$, $\forall\ t > 0$. Suppose on contrary that $r(t_0) < 1$, for some $t_0 > 0$. Then, as $\{t_n = M(x_{n-1}, x_n, t_0)\} \to r(t_0)$ and $\{s_n = M(x_n, x_{n+1}, t_0)\} \to r(t_0)$ so using (ξ_2), we obtain

$$0 \le \limsup_{n\to\infty} \xi\Big(M(x_{n-1}, x_n, t_0), M(x_n, x_{n+1}, t_0)\Big) < 0.$$

a contradiction. Thus $r(t) = 1$, $\forall\ t > 0$, we get $(\forall\ t > 0)$

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{3.3}$$

Next, we have to prove that $\{x_n\}$ is a Cauchy sequence. Suppose it is not so, then there exists $0 < \epsilon_0 < 1$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $k \le n(k) < m(k)$ and

$$M(x_{n(k)}, x_{m(k)}, t_0) \le 1 - \epsilon_0.$$

By Remark 2.3, we get

$$M\Big(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}\Big) \le 1 - \epsilon_0. \tag{3.4}$$

Now, suppose that $m(k)$ is the smallest integer corresponding to $n(k)$ satisfying (3.4). Then, we get

$$M\Big(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}\Big) \le 1 - \epsilon_0. \tag{3.5}$$

Now, using condition (FML-4), (3.4) and (3.5), we obtain

$$\begin{aligned} 1 - \epsilon_0 &\ge M(x_{n(k)}, x_{m(k)}, t_0) \\ &\ge M\Big(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}\Big) * M\Big(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{2}\Big) \\ &> (1 - \epsilon_0) * M\Big(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{2}\Big) \end{aligned}$$

Letting $k \to \infty$ and applying t-norm, it yields

$$1 - \epsilon_0 \ge M(x_{n(k)}, x_{m(k)}, t_0) \ge 1 - \epsilon_0$$

and hence

$$\lim_{n\to\infty} M(x_{n(k)}, x_{m(k)}, t_0) = 1 - \epsilon_0. \tag{3.6}$$

Also, again by (3.1) and (ξ_2), for $x = x_{n_k - 1}$, $y = x_{m_k - 1}$ and $t = t_0$, we get

$$0 \le \xi\Big(M(x_{n(k)-1}, x_{m(k)-1}, t_0), M(x_{n(k)}, x_{m(k)}, t_0)\Big) < \frac{1}{M(x_{n(k)-1}, x_{m(k)-1}, t_0)} - \frac{1}{M(x_{n(k)}, x_{m(k)}, t_0)},$$

so that

$$\begin{aligned} M(x_{n(k)}, x_{m(k)}, t_0) &> M(x_{n(k)-1}, x_{m(k)-1}, t_0) \\ &\geq M\Big(x_{n(k)-1}, x_{n(k)}, \frac{t_0}{2}\Big) * M\Big(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}\Big) \\ &> M\Big(x_{n(k)-1}, x_{n(k)}, \frac{t_0}{2}\Big) * (1-\epsilon_0) \end{aligned}$$

which on letting $k \to \infty$ and using t-norm yields

$$1-\epsilon_0 > \lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}, t_0) \geq 1-\epsilon_0.$$

Hence, we have

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}, t_0) = 1-\epsilon_0. \tag{3.7}$$

Henceforth, by (3.2), we have $\alpha(x_{n(k)-1}, x_{m(k)-1}, t_0) \geq 1$, thus taking $\{t_k = M(x_{n(k)-1}, x_{m(k)-1}, t_0)\}$ and $\{s_k = M(x_{n(k)}, x_{m(k)}, t_0)\}$ and applying (ξ_2), we obtain

$$0 \leq \limsup_{k\to\infty} \xi\Big(M(x_{n(k)-1}, x_{m(k)-1}, t_0), M(x_{n(k)}, x_{m(k)}, t_0)\Big) < 0,$$

a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in $(X, M, *)$. Now, by the completeness of X, there exists $x \in X$ such that $\{x_n\} \to x$. If T is continuous, then we have $\{Tx_n\} \to Tx$, which by uniqueness of limit implies that $Tx = x$. This completes the proof. ■

In the next theorem, we present the uniqueness of fixed point.

Theorem 3.10. *In addition to the hypothesis of Theorem 3.9, if the following condition is fulfilled:*

(d) for every $x, y \in Fix(T)$, there exists $w \in X$ such that $\alpha(x, w, t) \geq 1$ and $\alpha(y, w, t) \geq 1$, $\forall\ t > 0$,

then the fixed point of T is unique.

Proof. The existence part is followed by Theorem 3.9. For the uniqueness of fixed point, assume that x and x^* are two distinct fixed points of T. Then by condition (d), there exists a point $w \in X$ such that $\alpha(x, w, t) \geq 1$ and $\alpha(x^*, w, t) \geq 1$, $\forall\ t > 0$.

Construct a sequence $\{w_n\} \subseteq X$ by $w_0 = w$ and $w_{n+1} = Tw_n$, $\forall\ n \in \mathbb{N} \cup \{0\}$. By triangular α-admissibility, we have

$$\alpha(x, w_n, t) \geq 1 \text{ and } \alpha(x^*, w_n, t) \geq 1,\ \forall\ n \in \mathbb{N} \cup \{0\} \text{ and } t > 0 \tag{3.8}$$

Now, using (3.8) and applying (3.1) (for $x = x$ and $y = w_n$), we derive

$$M(x, w_{n+1}, t) > M(x, w_n, t),\ \forall\ n \in \mathbb{N} \cup \{0\} \text{ and } t > 0 \tag{3.9}$$

which shows that $\{M(x, w_n, t)\}$ is an increasing sequence of positive real numbers in $(0, 1]$. Let $L(t) = \lim_{n\to\infty} M(x, w_n, t)$. Our claim is that $L(t) = 1$, $\forall\ t > 0$. Assume on contrary that there exists some $t_0 > 0$ such that $L(t_0) < 1$. Thus, for $\{t_n = M(x, w_n, t_0)\}$ and $\{s_n = M(x, w_{n+1}, t_0)\}$, by (ξ_2) and applying (3.1), we obtain

$$0 \leq \lim_{n\to\infty} \xi\Big(M(x, w_n, t_0), M(x, w_{n+1}, t_0)\Big) < 0, \tag{3.10}$$

a contradiction. Hence, $L(t) = 1$, $\forall\ t > 0$. Thus, we have $\lim_{n\to\infty} M(x, w_n, t) = 1$, $\forall\ t > 0$, i.e., $\lim_{n\to\infty} w_n = x$. In the similar way, we can prove that $\lim_{n\to\infty} w_n = x^*$. By uniqueness of limit, we obtain $x = x^*$ and it completes the proof. ■

Now, we present the following example which exhibits the utility of Theorem 3.9.

Example 3.11. Let $X = [0, 1]$. Define t-norm $* : [0, 1]\times[0, 1] \to [0, 1]$ by $p*q = \min\{p, q\}$ and define fuzzy metric-like space F by

$$F(x, y, t) = \frac{t}{t + \sigma(x, y)},$$

where $\sigma(x, y) = x^2 + y^2$ is metric-like space.
Then $(X, F, .)$ is a complete fuzzy metric-like space. Define mappings $\alpha : X\times X\times(0, \infty) \to [0, \infty)$ and $T : X \to X$ by

$$\alpha(x, y, t) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{1}{2}], \\ 0 & \text{otherwise} \end{cases}$$

and

$$Tx = \begin{cases} \frac{ax}{1 + x} & \text{if } x \in [0, \frac{1}{2}], \\ x & \text{otherwise} \end{cases}$$

where $a \in (0, 1)$. Then, we have ($\forall\ x, y \in X$ and $t > 0$)

$$\frac{1}{F(x, y, t)} - 1 = \frac{t + \sigma(x, y)}{t} - 1 = \frac{\sigma(x, y)}{t} = \frac{x^2 + y^2}{t}$$

Also, for $x, y \in X$ such that $\alpha(z, y, t) \geq 1$, we have

$$\begin{aligned} \frac{1}{F(Tx, Ty, t)} - 1 &= \frac{t + \sigma(Tx, Ty)}{t} - 1 = \frac{\sigma(Tx, Ty)}{t} \\ &= \frac{(Tx)^2 + (Ty)^2}{t} = \frac{(\frac{ax}{1+x})^2 + (\frac{ay}{1+y})^2}{t} \\ &= \frac{\frac{a^2x^2}{(1+x)^2} + \frac{a^2y^2}{(1+y)^2}}{t}. \end{aligned}$$

Then, taking $\xi(t, s) = k(\frac{1}{t} - 1) - (\frac{1}{s} - 1)$, for any $k \in [a, 1)$, for each $a \in (0, 1)$, we obtain (for $x, y \in X$)

$$\begin{aligned} \alpha(x, y, t) \geq 1 \Rightarrow \xi\Big(F(x, y, t), F(Tx, Ty, t)\Big) \\ &= k\Big(\frac{x^2 + y^2}{t}\Big) - \Big(\frac{\frac{a^2x^2}{(1+x)^2} + \frac{a^2y^2}{(1+y)^2}}{t}\Big) \\ &= \frac{x^2}{t}\Big(k - \frac{a^2}{(1 + x)^2}\Big) + \frac{y^2}{t}\Big(k - \frac{a^2}{(1 + y)^2}\Big) \geq 0 \end{aligned}$$

for all $t > 0$. Hence, all the conditions of Theorem 3.1 are satisfied and the conclusion of the theorem holds, i.e., T has a unique fixed point (namely $x = 0$). But the result of Gregori and Sapena [6, 10] can not be applied. Indeed, for any $x, y \in (\frac{1}{2}, 1]$, there does not exist any $k \in (0, 1)$ such that (3.1) is satisfied.

Next, Theorem 3.9 can be improved as follows:

Theorem 3.12. *The conclusions of Theorems 3.9 and 3.10 hold if we replace α-admissible Ξ_{MA}-contraction by the following (retaining the other conditions are same):*

$$x, y \in X,\ \ \alpha(x,y,t) \geq 1 \Rightarrow \xi(F(x,y,t), F(T^n x, T^n y, t) \geq 0$$

for some $n \in \mathbb{N}$ and $\forall\ t > 0$.

Proof. By Theorem 3.9, T^n has a unique fixed point, $x \in X$ (say), i.e., $T^n x = x$. Also $T^n(Tx) = T(T^n x) = Tx$, i.e., Tx is the fixed point of T. But, by the uniqueness of fixed point of T, we have $Tx = x$. Since fixed point of T is also that of T^n, so T has a unique fixed point. ∎

The following example exhibits that Theorem 3.12 is a genuine extension of Theorem 3.9.

Example 3.13. Let $X = [-1, 1]$ and define a mapping F on $X \times X \times (0, \infty)$ as:

$$F(x,y,t) = \begin{cases} 1 & \text{if } x = y, \\ \min\{1 - \frac{|x|}{2}, 1 - \frac{|y|}{2}\} & \text{otherwise.} \end{cases}$$

The space $(X, F, *)$ is a complete fuzzy metric-like space with minimum t-norm. Define mappings $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ and $T : X \to X$ as:

$$\alpha(x,y,t) = 1,\ \forall\ x, y \in X \text{ and } t > 0$$

and

$$Tx = \begin{cases} 0 & \text{if } -1 < x < 0, \\ 1 & \text{if } 0 \leq x \leq 1, \end{cases}$$

respectively. We have

$$T^2 x = 1,\ \ x \in X$$

Then, with $\xi(t,s) = k(\frac{1}{t} - 1)(\frac{1}{s} - 1)$, $\forall\ t, s \in (0, 1]$ and any $k \in (0, 1)$, all the conditions of Theorem 3.12 are satisfied and T has a unique fixed point (namely $x = 1$). But Theorem 3.9 can not be applied here, since T is not an α-admissible Ξ_{MA}-contraction mapping. Indeed, for $x = \frac{-1}{2}$ and $y = \frac{1}{2}$, $F(Tx, Ty, t) < F(x, y, t)$, $\forall\ t > 0$, which is a contradiction.

4. Consequences

In this section, we deduce some corollaries as consequences of Theorem 3.12 starting with the following one.

Corollary 4.1. (Banach [17] type) *Let $(X, F, *)$ be a complete fuzzy metric-like space and $T : X \to X$ a mapping satisfying*

$$x, y \in X\ \ \alpha(x,y,t) \geq 1 \Rightarrow \frac{1}{F(Tx,Ty,t)} - 1 \leq k\Big(\frac{1}{F(x,y,t)} - 1\Big),$$

for all $t > 0$ and $k \in (0, 1)$. Then T has a unique fixed point.

Proof. The proof follows from Theorem 3.12 and Example 3.2. ∎

By taking $\alpha(x,y,t) = 1,\ \forall\ x, y \in X$ and $t > 0$, Corollary 4.1 reduces to the following result by Gregori and Sapena [6].

Corollary 4.2. *Let $(X, F, *)$ be a complete fuzzy metric-like space and $T : X \to X$ a mapping satisfying*

$$\frac{1}{F(Tx, Ty, t)} - 1 \leq k\Big(\frac{1}{F(x, y, t)} - 1\Big),$$

for all $x, y \in X$, $t > 0$ and $k \in (0, 1)$. Then T has a unique fixed point.

In the next corollary, we are going to present Boyd and Wong [18] type result in the setting of fuzzy metric-like spaces.

Corollary 4.3. (Boyd and Wong [18] type) *Let $(X, F, *)$ be a complete fuzzy metric-like space and $T : X \to X$ a mapping satisfying*

$$x, y \in X \;\; \alpha(x, y, t) \geq 1 \Rightarrow \frac{1}{F(Tx, Ty, t)} - 1 \leq \psi\Big(\frac{1}{F(x, y, t)} - 1\Big),$$

for all $t > 0$, where $\psi : [0, \infty) \to [0, \infty)$ is a given function such that $\psi(r) < r$, for all $r > 0$ and $\psi(0) = 0$. Then T has a unique fixed point.

Proof. In view of Theorem 3.9 and Example 3.3, the result follows. ■

Following is [19]-type fixed point result.

Corollary 4.4. (Abbas et al. [19] type) *Let $(X, F, *)$ be a complete fuzzy metric-like space and $T : X \to X$ a mapping satisfying*

$$x, y \in X \;\; \alpha(x, y, t) \geq 1 \Rightarrow \frac{1}{F(Tx, Ty, t)} - 1 \leq \Big(\frac{1}{F(x, y, t)} - 1\Big) - \psi\Big(\frac{1}{F(x, y, t)} - 1\Big),$$

for all $t > 0$, where $\psi : [0, \infty) \to [0, \infty)$ is a given function such that $\psi(r) > 0$, for all $r > 0$ and $\psi(0) = 0$. Then T has a unique fixed point.

Proof. Taking into account of Theorem 3.9 and Example 3.4, the result follows. ■

Next, we present the following results, which are known in some natural settings but seems new to the fuzzy setting.

Corollary 4.5. *Let $(X, F, *)$ be a complete fuzzy metric-like space and $T : X \to X$ a mapping satisfying*

$$x, y \in X \;\; \alpha(x, y, t) \geq 1 \Rightarrow F(Tx, Ty, t) \geq \psi(F(x, y, t)),$$

for all $t > 0$, where $\psi : (0, 1] \to (0, 1]$ is nondecreasing and left-continuous function such that $\psi(r) > r$, for all $r \in (0, 1)$. Then T has a unique fixed point.

Proof. The result follows from Theorem 3.9 and Example 3.5. ■

Corollary 4.6. *Let $(X, F, *)$ be a complete fuzzy metric-like space and $T : X \to X$ a mapping satisfying*

$$x, y \in X \;\; \alpha(x, y, t) \geq 1 \Rightarrow \frac{1}{F(Tx, Ty, t)} - 1 \leq \Big(\frac{1}{F(x, y, t)} - 1\Big).\psi\Big(\frac{1}{F(x, y, t)} - 1\Big),$$

for all $t > 0$, where $\psi : [0, \infty) \to [0, \infty)$ is a given function such that $\min_{r \to s+} \psi(r) > 0$, for all $s > 0$. Then T has a unique fixed point.

Proof. The result follows from Theorem 3.9 and Example 3.6. ■

Corollary 4.7. *Let* $(X, F, *)$ *be a complete fuzzy metric-like space and* $T : X \to X$ *a mapping satisfying*

$$x, y \in X \ \alpha(x, y, t) \geq 1 \Rightarrow \int_0^{\frac{1}{F(Tx,Ty,t)}-1} \psi(s)ds \leq \frac{1}{F(x, y, t)} - 1,$$

for all $t > 0$*, where* $\psi : [0, \infty) \to [0, \infty)$ *is a given function such that* $\int_0^r \psi(s)ds$ *exists and* $\int_0^r \psi(s)ds > 0$*, for each* $r > 0$*. Then* T *has a unique fixed point.*

Proof. In view of Theorem 3.9 and Example 3.7, this result follows. ∎

5. Application

In recent past, many authors used various sufficient conditions to find the existence and uniqueness of solutions of integral equations in varied settings. In this section, we consider a Fredholm non-linear integral equation and utilize our proved result in the setting of fuzzy metric-like spaces to find its unique solution. We see that by applying Theorem 3.9, this Fredholm non-linear integral equation has a unique solution under certain specific conditions and without these conditions, we cannot apply our results to find the unique solution.

To accomplish this, we consider the following:

$$x(t) = \int_a^b K(t, s)h(x(s))ds + g(t), \tag{5.1}$$

for all $t \in \Omega = [a, b](a, b \in \mathbb{R})$, $K \in C(\Omega \times \Omega, \mathbb{R})$ and $g, h \in C(\Omega, \mathbb{R})$.

Let Φ be the collection of all mappings $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ϕ_1) ϕ is non-decreasing;
(ϕ_2) $\phi(t) \leq t, \ \forall \ t \in [0, \infty)$.

Now, we are equipped to present our theorem in this section as follows:

Theorem 5.1. *Consider the integral equation* (5.1) *with* $K \in C(\Omega \times \Omega, \mathbb{R})$ *and* $g \in C(\Omega, \mathbb{R})$*. If the following conditions are satisfied:*

(A_1) there exists a positive number λ *and* $\phi \in \Phi$ *such that* $\forall \ x, y \in C(\Omega, \mathbb{R})$*, the following condition holds:*

$$h(x) - h(y) \leq \lambda\phi(x - y), \tag{5.2}$$

(A_2) $\lambda \sup_{t\in\Omega} \int_a^b |K(t, s)|ds \leq \frac{1}{2}$.

Then, the equation (5.1) *has a unique solution in* $C(\Omega, \mathbb{R})$*.*

Proof. Observe that $X = C(\Omega, \mathbb{R})$ is a complete metric space with respect to sup-metric

$$\sigma(x, y) = \sup_{t\in\Omega}(|x(t)| + |y(t)| + a).$$

Also, the space $(X, F, *)$ with

$$F(x, y, t) = \frac{t}{t + \sigma(x, y)}, \ \forall\ x, y \in X \text{ and } t > 0$$

is a complete fuzzy metric-like space with product t-norm.
Now we define a mapping $T : X \to X$ as:

$$Sx(t) = \int_a^b K(t, s)h(x(s))ds + g(t), \qquad \forall\ t \in \Omega. \tag{5.3}$$

Using (5.2) and (5.3), we have

$$\begin{aligned} Tx(t) - Ty(t) &= \int_a^b K(t, s)[h(x(s)) - h(y(s))]ds \\ &\leq \lambda \int_a^b K(t, s)\phi(x(s) - y(s))ds \end{aligned} \tag{5.4}$$

Using(ϕ_1), we have

$$\phi(x(s) - y(s) \leq \phi(\sup(|x(s)| + |y(s)| + a)) = \phi(\sigma(x, y)). \tag{5.5}$$

Applying (5.5) in (5.4), we obtain

$$\phi(x(s) - y(s) \leq \lambda \int_a^b K(t, s)\phi(\sigma(x, y))ds.$$

Taking supremum over $t \in \Omega$, using conditions (A_2) and (ϕ_2), we get

$$\begin{aligned} \sigma(Tx, Ty) &\leq \lambda\phi(\sigma(x, y)) \int_a^b |K(t, s)|ds \\ &\leq \frac{1}{2}\phi(\sigma(x, y)) \leq \frac{1}{2}(\sigma(x, y)). \end{aligned} \tag{5.6}$$

Now, we have

$$\frac{1}{F(Tx, Ty, t)} - 1 = \frac{\sigma(Tx, Ty)}{t} \leq \frac{\sigma(x, y)}{2t} = \frac{1}{2}\Big(\frac{1}{F(x, y, t)} - 1\Big).$$

By taking $\xi(t, s) = \frac{1}{2}\left(\frac{1}{t} - 1\right) - \left(\frac{1}{s} - 1\right)$. $\forall\ t, s \in (0, 1]$, all the conditions of Theorem 3.9 are satisfied with $\alpha(x, y, t) = 1$, $\forall\ x, y \in X$ and $t > 0$. Hence, by the conclusions of Theorems 3.9 and 3.10, (5.1) has a unique solution in $C(\Omega, \mathbb{R})$. ∎

6. Conclusion

In this paper, motivated by the work of Khojasteh et al. [15] and Karapinar [20], we introduce a new simulation function besides proposing the concept of a new contraction namely, α-admissible Ξ_{MA}-contraction and utilize the same to prove fixed point results ensuring the existence and uniqueness of fixed points. Furthermore, via some corollaries, we demonstrate that our main result is general enough to unify several results of the existing literature. Finally, by presenting an application, we exhibit the usability of our main result.

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