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Common Fixed Point Theorems for Six Weakly Compatible Mappings in D^*-Metric Spaces

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Abstract : In this paper, we give some new definitions of D^*-metric spaces and we prove a common fixed point theorem for six mappings under the condition of weakly compatible mappings in complete D^*-metric spaces. We get some improved versions of several fixed point theorems in complete D^*-metric spaces.

Keywords : D^*-metric contractive mapping; Complete D^*-metric space; Common fixed point theorem.
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1 Introduction and Preliminaries

In 1922, the Polish mathematician, Banach, proved a theorem which ensures under appropriate conditions, the existence and uniqueness of a fixed point.His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering.Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [11], Jungck introduced more generalized commuting mappings, called *compatible* mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems(see, e.g.,([1, 2, 3, 4, 6, 8, 9, 12, 13, 14, 16]). One such generalization is generalized metric space or D-metric space initiated by Dhage [5] in 1992. He proved some results on fixed points for a self-map satisfying a contraction for complete and bounded D-metric spaces. Rhoades [11] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma [15] introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition. Dhage [5] introduced the notion of generalized metric or D-metric spaces and claimed that D-metric convergence defines a Hausdorff topology and that D-metric is sequentially continuous in all the three variables.Many authors have taken these claims for granted and used them in proving fixed point theorems in D-metric spaces. Unfortunately, almost all theorems in D-metric spaces are not valid .In this paper, we introduce D^*-metric which is a probable modification of the definition of D-metric introduced by Dhage [5] and prove some basic properties in D^*-metric spaces. We also prove a theorem for six maps in D^*-metric spaces .

In what follows (X, D^*) will denote a D^*-metric space, $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1.1. *Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$.*

(1) $D^*(x, y, z) \geq 0$,

(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$,(symmetry) where p is a permutation function,

(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Immediate examples of such a function are

(a) $D^*(x, y, z) = \max\{D^*(x, y), D^*(y, z), D^*(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}^+$ then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise .} \end{cases}$$

Remark 1.2. *In a D^*-metric space, we prove that $D^*(x, x, y) = D^*(x, y, y)$. For*

(i) $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and similarly

(ii) $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$.

Hence by (i),(ii) we get $D^*(x, x, y) = D^*(x, y, y)$.

Let (X, D^*) be a D^*-metric space. For $r > 0$ define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}$$

Example 1.3. *Let $X = \mathbb{R}$. Denote $D^*(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x,y,z \in \mathbb{R}$. Thus*

$$\begin{aligned} B_{D^*}(1,2) &= \{y \in \mathbb{R} : D^*(1,y,y) < 2\} \\ &= \{y \in \mathbb{R} : |y-1| + |y-1| < 2\} \\ &= \{y \in \mathbb{R} : |y-1| < 1\} \\ &= (0,2) \end{aligned}$$

Definition 1.4. *Let (X, D^*) be a D^*-metric space and $A \subset X$.*

(1) If for every $x \in A$ there exist $r > 0$ such that $B_{D^*}(x,r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*-bounded if there exists $r > 0$ such that $D^*(x,y,y) < r$ for all $x,y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if

$$D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0 \quad as \quad n \to \infty.$$

That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow D^*(x, x, x_n) < \epsilon \quad (*)$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \Longrightarrow D^*(x, x_n, x_m) < \epsilon \quad (**)$$

Indeed, if have $(*)$,then

$$D^*(x_n, x_m, x) = D^*(x_n, x, x_m) \leq D^*(x_n, x, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon$$

Conversely, set $m = n$ in $(**)$ we have $D^*(x_n, x_n, x) < \epsilon$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D^*-metric space (X, D^*) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $r > 0$ such that $B_{D^*}(x,r) \subset A$. Then τ is a topology on X (induced by the D^*-metric).

Lemma 1.5. *Let (X, D^*) be a D^*-metric space. If $r > 0$, then the ball $B_{D^*}(x,r)$ with center $x \in X$ and radius r is the open ball.*

Proof. Let $z \in B_{D^*}(x,r)$ so that $D^*(x,z,z) < r$. If set $D^*(x,z,z) = \delta$ and $r' = r - \delta$ then we prove that $B_{D^*}(z,r') \subseteq B_{D^*}(x,r)$. Let $y \in B_{D^*}(z,r')$, by triangular inequality we have $D^*(x,y,y) = D^*(y,y,x) \leq D^*(y,y,z) + D^*(z,x,x) < r' + \delta = r$. Hence $B_{D^*}(z,r') \subseteq B_{D^*}(x,r)$. That is ball $B_{D^*}(x,r)$ is open ball. □

Lemma 1.6. *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist
$n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D^*(x, x, x_n) < \frac{\epsilon}{2}$
and
$n_2 \in \mathbb{N}$ such that for every $n \geq n_2 \Longrightarrow D^*(y, y, x_n) < \frac{\epsilon}{2}$.
If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$$D^*(x, x, y) \leq D^*(x, x, x_n) + D^*(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Hence $D^*(x, x, y) = 0$ is a contradiction. So, $x = y$. □

Lemma 1.7. *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X is converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$ for each $\epsilon > 0$ there exists
$n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D^*(x_n, x_n, x) < \frac{\epsilon}{2}$
and
$n_2 \in \mathbb{N}$ such that for every $m \geq n_2 \Longrightarrow D^*(x, x_m, x_m) < \frac{\epsilon}{2}$.
If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by triangular inequality we have
$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Definition 1.8. *Let (X, D^*) be a D^*- metric space. D^* is said to be continuous function on $X^3 \times (0, \infty)$ if*

$$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z),$$

whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$ i.e.

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$$

Lemma 1.9. *Let (X, D^*) be a D^*- metric space. Then D^* is continuous function on X^3.*

Proof. Since the sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$ i.e.

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$$

for each $\epsilon > 0$ there exists
$n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D^*(x, x, x_n) < \frac{\epsilon}{3}$
$n_2 \in \mathbb{N}$ such that for every $n \geq n_2 \Longrightarrow D^*(y, y, y_n) < \frac{\epsilon}{3}$,
and similarly there exist $n_3 \in \mathbb{N}$ such that for every $n \geq n_3 \Longrightarrow D^*(z, z, z_n) < \frac{\epsilon}{3}$.

If set $n_0 = \max\{n_1, n_2, n_3\}$, then for every $n \geq n_0$ by triangular inequality we have

$$\begin{aligned} D^*(x_n, y_n, z_n) &\leq D^*(x_n, y_n, z) + D^*(z, z_n, z_n) \\ &\leq D^*(x_n, z, y) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &\leq D^*(z, y, x) + D^*(x, x_n, x_n) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &< D^*(x, y, z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= D^*(x, y, z) + \epsilon \end{aligned}$$

Hence we have

$$D^*(x_n, y_n, z_n) - D^*(x, y, z) < \epsilon$$

$$\begin{aligned} D^*(x, y, z) &\leq D^*(x, y, z_n) + D^*(z_n, z, z) \\ &\leq D^*(x, z_n, y_n) + D^*(y_n, y, y) + D^*(z_n, z, z) \\ &\leq D^*(z_n, y_n, x_n) + D^*(x_n, x, x) + D^*(y_n, y, y) + D^*(z_n, z, z) \\ &< D^*(x_n, y_n, z_n) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= D^*(x_n, y_n, z_n) + \epsilon \end{aligned}$$

That is,

$$D^*(x, y, z) - D^*(x_n, y_n, z_n) < \epsilon$$

Therefore we have $|D^*(x_n, y_n, z_n) - D^*(x, y, z)| < \epsilon$, that is

$$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$$

□

Definition 1.10. *Let (X, D^*) is a D^*- metric space, then D^* is said to be of the $first$ type if for every $x, y \in X$ we have*

$$D^*(x, x, y) \leq D^*(x, y, z)$$

for every $z \in X$.

In 1998, Jungck and Rhoades [8] introduced the following concept of weak compatibility.

Definition 1.11. *Let A and S be mappings from a D^*-metric space (X, D^*) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.*

2 Main Results

A class of implicit relation

Throughout this section (X, D^*) denotes a D^*-metric space and Φ denotes a family of mappings such that for each $\phi \in \Phi$, $\phi : (\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+$, is continuous and increasing in each co-ordinate variable. Also $\gamma(t) = \phi(t,t,t,t) < t$ for every $t \in \mathbb{R}^+$.

Example 2.1. *Let* $\phi : (\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+$ *is define by*

$$\phi(t_1, t_2, t_3, t_4) = \frac{1}{5}(t_1 + t_2 + t_3 + t_4)$$

Our main result, for a complete D^*-metric space X, reads follows:

Theorem 2.2. *Let A, B, C, S, T and R be self-mappings of a complete D^*- metric space (X, D^*) where D^* is first type with :*

(i)$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, $C(X) \subseteq R(X)$ *and* $A(X)$ *or* $B(X)$ *or* $C(X)$ *is a closed subset of* X,

(ii) $D^*(Ax, By, Cz) \le q\phi(D^*(Rx, Ty, Sz), D^*(Rx, Ty, By), D^*(Ty, Sz, Cz), D^*(Sz, Rx, Ax))$, *for every* $x, y, z \in X$, *some* $0 < q < 1$ *and* $\phi \in \Phi$,

(iii) *the pair* (A, R) , (B, T) *and* (S, C) *are weak compatible.*
Then A, B, C, S, T and R have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point . By (i), there exists $x_1, x_2, x_3 \in X$ such that

$$Ax_0 = Tx_1 = y_0, \quad Bx_1 = Sx_2 = y_1 \quad \text{and} \quad Cx_2 = Rx_3 = y_2.$$

Inductively, construct sequence $\{y_n\}$ in X such that $y_{3n} = Ax_{3n} = Tx_{3n+1}$, $y_{3n+1} = Bx_{3n+1} = Sx_{3n+2}$ and $y_{3n+2} = Cx_{3n+2} = Rx_{3n+3}$, for $n = 0, 1, 2, \cdots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $D^*_m = D^*(y_m, y_{m+1}, y_{m+2})$. Then, we have

$$\begin{aligned} D^*_{3n} &= D^*(y_{3n}, y_{3n+1}, y_{3n+2}) = D^*(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}) \\ &\le q\phi\begin{pmatrix} D^*(Rx_{3n}, Tx_{3n+1}, Sx_{3n+2}), D^*(Rx_{3n}, Tx_{3n+1}, Bx_{3n+1}) \\ D^*(Tx_{3n+1}, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Rx_{3n}, Ax_{3n}) \end{pmatrix} \\ &= q\phi\begin{pmatrix} D^*(y_{3n-1}, y_{3n}, y_{3n+1}), D^*(y_{3n-1}, y_{3n}, y_{3n+1}) \\ D^*(y_{3n}, y_{3n+1}, y_{3n+2}), D^*(y_{3n+1}, y_{3n-1}, y_{3n}) \end{pmatrix} \\ &= q\phi(D^*_{3n-1}, D^*_{3n-1}, D^*_{3n}, D^*_{3n-1}). \end{aligned}$$

We prove that $D^*_{3n} \le D^*_{3n-1}$, for every $n \in \mathbb{N}$. If $D^*_{3n} > D^*_{3n-1}$ for some $n \in \mathbb{N}$, by above inequality we have $D^*_{3n} < qD^*_{3n}$, is a contradiction. Now, if $m = 3n + 1$, then

$$\begin{aligned}
D^*_{3n+1} &= D^*(y_{3n+1}, y_{3n+2}, y_{3n+3}) \\
&= D^*(y_{3n+3}, y_{3n+1}, y_{3n+2}) \\
&= D^*(Ax_{3n+3}, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq q\phi\left(\begin{array}{l} D^*(Rx_{3n+3}, Tx_{3n+1}, Sx_{3n+2}), D^*(Rx_{3n+3}, Tx_{3n+1}, Bx_{3n+1}) \\ D^*(Tx_{3n+1}, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Rx_{3n+3}, Ax_{3n+3}) \end{array}\right) \\
&= q\phi\left(\begin{array}{l} D^*(y_{3n+2}, y_{3n}, y_{3n+1}), D^*(y_{3n+2}, y_{3n}, y_{3n+1}) \\ D^*(y_{3n}, y_{3n+1}, y_{3n+2}), D^*(y_{3n+1}, y_{3n+2}, y_{3n+3}) \end{array}\right) \\
&= q\phi(D^*_{3n}, D^*_{3n}, D^*_{3n}, D^*_{3n+1}).
\end{aligned}$$

Similarly, if $D^*_{3n+1} > D^*_{3n}$ for some $n \in \mathbb{N}$ we have $D^*_{3n+1} < qD^*_{3n+1}$ is a contradiction. If $m = 3n+2$,then

$$\begin{aligned}
D^*_{3n+2} &= D^*(y_{3n+2}, y_{3n+3}, y_{3n+4}) \\
&= D^*(y_{3n+3}, y_{3n+4}, y_{3n+2}) \\
&= D^*(Ax_{3n+3}, Bx_{3n+4}, Cx_{3n+2}) \\
&\leq q\phi\left(\begin{array}{l} D^*(Rx_{3n+3}, Tx_{3n+4}, Sx_{3n+2}), D^*(Rx_{3n+3}, Tx_{3n+4}, Bx_{3n+4}) \\ D^*(Tx_{3n+4}, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Rx_{3n+3}, Ax_{3n+3}) \end{array}\right) \\
&= q\phi\left(\begin{array}{l} D^*(y_{3n+2}, y_{3n+3}, y_{3n+1}), D^*(y_{3n+2}, y_{3n+3}, y_{3n+4}) \\ D^*(y_{3n+3}, y_{3n+1}, y_{3n+2}), D^*(y_{3n+1}, y_{3n+2}, y_{3n+3}) \end{array}\right) \\
&= q\phi(D^*_{3n+1}, D^*_{3n+2}, D^*_{3n+1}, D^*_{3n+1}).
\end{aligned}$$

Similarly, if $D^*_{3n+2} > D^*_{3n+1}$ for some $n \in \mathbb{N}$ we have $D^*_{3n+2} < qD^*_{3n+2}$ is a contradiction.

Hence for every $n \in \mathbb{N}$ we have $D^*_n \leq qD^*_{n-1}$. That is

$$D^*_n = D^*(y_n, y_{n+1}, y_{n+2}) \leq qD^*(y_{n-1}, y_n, y_{n+1}) \leq \cdots \leq q^n D^*(y_0, y_1, y_2).$$

Since D^* is a first type , hence we have

$$D^*(y_n, y_n, y_{n+1}) \leq q^n D^*(y_0, y_1, y_2).$$

Therefore

$$D^*(y_n, y_n, y_m) \leq D^*(y_n, y_n, y_{n+1}) + D^*(y_{n+1}, y_{n+1}, y_{n+2}) + \cdots + D^*(y_{m-1}, y_{m-1}, y_m).$$

Hence

$$\begin{aligned}
D^*y_n, y_n, y_m) &\leq q^n D^*(y_0, y_1, y_2) + q^{n+1} D^*(y_0, y_1, y_2) + \cdots + q^{m-1} D^*(y_0, y_1, y_2) \\
&= (q^n + q^{n+1} + \cdots + q^{m-1}) D^*(y_0, y_1, y_2) \\
&\leq D^*(y_0, y_1, y_2) \frac{q^n}{1-q} \longrightarrow 0.
\end{aligned}$$

Thus the sequence $\{y_n\}$ is Cauchy and by the completeness of X, $\{y_n\}$ converges to y in X. That is,$\lim_{n\to\infty} y_n = y$

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} Bx_{3n+1} = \lim_{n\to\infty} Cx_{3n+2}\\ &= \lim_{n\to\infty} Tx_{3n+1} = \lim_{n\to\infty} Rx_{3n+3} = \lim_{n\to\infty} Sx_{3n+2} = y\end{aligned}$$

Let $C(X)$ be a closed subset of X , hence there exist $u \in X$ such that $Ru = y$. We prove that $Au = y$. For
$D^*(Au, Bx_{3n+1}, Cx_{3n+2})$

$$\leq q\phi \begin{pmatrix} D^*(Ru, Tx_{3n+1}, Sx_{3n+2}), D^*(Ru, Tx_{3n+1}, Bx_{3n+1}) \\ D^*(Tx_{3n+1}, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Ru, Au) \end{pmatrix}$$

On making $n \longrightarrow \infty$ we get

$$D^*(Au, y, y) \leq q\phi \begin{pmatrix} D^*(Ru, y, y), D^*(Ru, y, y) \\ D^*(y, y, y), D^*(y, Ru, Au) \end{pmatrix}.$$

If $D^*(y, y, Au) > 0$, then we have $D^*(Au, y, y) < qD^*(y, y, Au)$ is a contradiction. Thus $Au = y$. By the weak compatibility of the pair (R, A) we have $ARu = RAu$. Hence $Ay = Ry$. We prove that $Ay = y$, if $Ay \neq y$, then
$D^*(Ay, Bx_{3n+1}, Cx_{3n+2})$

$$\leq q\phi \begin{pmatrix} D^*(Ry, Tx_{3n+1}, Sx_{3n+2}), D^*(Ry, Tx_{3n+1}, Bx_{3n+1}) \\ D^*(Tx_{3n+1}, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Ry, Ay) \end{pmatrix}$$

As $n \longrightarrow \infty$ we have

$$\begin{aligned} D^*(Ay, y, y) &\leq q\phi \begin{pmatrix} D^*(Ry, y, y), D^*(Ry, y, y) \\ D^*(y, y, y), D^*(y, Ry, Ay) \end{pmatrix} \\ &\leq qD^*(Ay, y, y) \end{aligned}$$

a contradiction. Therefore,$Ry = Ay = y$, that is, y is a common fixed of R, A . Since $y = Ay \in A(X) \subseteq R(X)$, hence there exists $v \in X$ such that $Tv = y$. We prove that $Bv = y$. For
$D^*(y, Bv, Cx_{3n+2}) = D^*(Ay, Bv, Cx_{3n+2})$

$$\leq q\phi \begin{pmatrix} D^*(Ry, Tv, Sx_{3n+2}), D^*(Ry, Tv, Bv) \\ D^*(Tv, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Ry, Ay) \end{pmatrix}$$

On making $n \longrightarrow \infty$ we get

$$D^*(y, Bv, y) \leq q\phi \begin{pmatrix} D^*(y, y, y), D^*(y, y, Bv) \\ D^*(y, y, y), D^*(y, y, y) \end{pmatrix} \leq qD^*(y, y, Bv)$$

Thus $Bv = y$. By the weak compatibility of the pair (B, T) we have $TBv = BTv$. Hence $By = Ty$. We prove that $By = y$, if $By \neq y$, then

$$D^*(Ay, By, Cx_{3n+2}) \leq q\phi \begin{pmatrix} D^*(Ry, Ty, Sx_{3n+2}), D^*(Ry, Ty, By) \\ D^*(Ty, Sx_{3n+2}, Cx_{3n+2}), D^*(Sx_{3n+2}, Ry, Ay) \end{pmatrix}$$

As $n \longrightarrow \infty$ we have

$$D^*(y, By, y) \leq q\phi \left(\begin{array}{l} D^*(Ry, Ty, y), D^*(Ry, By, By) \\ D^*(By, y, y), D^*(y, y, y) \end{array} \right) \leq qD^*(y, By, y)$$

a contradiction. Therefore,$By = Ty = y$, that is, y is a common fixed of B, T . Similarly, since $y = By \in B(X) \subseteq S(X)$,hence there exists $w \in X$ such that $Sw = y$. We prove that $Cw = y$. For
$D^*(y, y, Cw) = D^*(Ay, By, Cw)$

$$\leq q\phi \left(\begin{array}{l} D^*(Ry, Ty, w), D^*(Ry, Ty, By) \\ D^*(Ty, Sw, Cw), D^*(Sw, Ry, Ay) \end{array} \right) \leq qD^*(y, y, Cw)$$

Thus $Cw = y$. By weak compatible the pair (C, S) we have $CSw = SCw$, hence $Cy = Sy$. We prove that $Cy = y$, if $Cy \neq y$, then
$D^*(y, y, Cy) = D^*(Ay, By, Cy)$

$$\leq q\phi \left(\begin{array}{l} D^*(Ry, Ty, Sy), D^*(Ry, Ty, By) \\ D^*(Ty, Sy, Cy), D^*(Sy, Ry, Ay) \end{array} \right) \leq qD^*(y, y, Cy)$$

a contradiction. Therefore,$Cy = Sy = y$, that is, y is a common fixed of C, S.Thus

$$Ay = Sy = Ty = By = Cy = Ry = y$$

To prove uniqueness, let v be another common fixed point of T, A, B, C, R, S.
If $D^*(y, y, v) > 0$, hence
$D^*(y, y, v) = D^*(Ay, By, Cv))$

$$\leq q\phi \left(\begin{array}{l} D^*(Ry, Ty, Sv), D^*(Ry, Ty, By) \\ D^*(Ty, Sv, Cv), D^*(Sv, Ry, Ay) \end{array} \right) \leq qD^*(y, y, v)$$

a contradiction. Therefore, $y = v$ is the unique common fixed point of self-maps T, A, B, C, R, S. □

Corollary 2.3. *Let S, T, R and $\{A_\alpha\}_{\alpha \in I}$, $\{B_\beta\}_{\beta \in J}$ and $\{C_\gamma\}_{\gamma \in K}$ be the set of all self-mappings of a complete D^*-metric space (X, D^*), where D^* is first type satisfying:*

(i)*there exists $\alpha_0 \in I$, $\beta_0 \in J$ and $\gamma_0 \in K$ such that $A_{\alpha_0}(X) \subseteq T(X)$, $B_{\beta_0}(X) \subseteq S(X)$ and $C_{\gamma_0}(X) \subseteq R(X)$,*

(ii)*$A_{\alpha_0}(X)$ or $B_{\beta_0}(X)$ or $C_{\gamma_0}(X)$ is a closed subset of X,*

(iii) *$D^*(A_\alpha x, B_\beta y, C_\gamma z) \leq q\phi(D^*(Rx, Ty, Sz), D^*(Rx, Ty, B_\beta y),$ $D^*(Ty, Sz, C_\gamma z), D^*(Sz, Rx, A_\alpha x))$, for every $x, y, z \in X$, some $0 < q < 1$ and $\phi \in \Phi$, and every $\alpha \in I, \beta \in J$,$\gamma \in K$,*

(iv) *the pair (A_{α_0}, R) or (B_{β_0}, T) or (C_{γ_0}, S) are weak compatible.*
Then $A, B, C, S, T$and R have a unique common fixed point in X.

Proof. By Theorem 2.2 R, S, T and A_{α_0} , B_{β_0} and C_{γ_0} for some $\alpha_0 \in I, \beta_0 \in J, \gamma_0 \in K$ have a unique common fixed point in X. That is there exist a unique $a \in X$ such that $R(a) = S(a) = T(a) = A_{\alpha_0}(a) = B_{\beta_0}(a) = C_{\gamma_0}(a) = a$. Let there exist $\lambda \in J$ such that $\lambda \neq \beta_0$ and $D^*(a, B_\lambda a, a) > 0$ then we have

$$\begin{aligned} D^*(a, B_\lambda a, a) &= D^*(A_{\alpha_0}a, B_\lambda a, C_{\gamma_0}a) \\ &\leq q\phi \left(\begin{array}{l} D^*(Ra, Ta, Sa), D^*(Ra, Ta, B_\lambda a) \\ D^*(Ta, Sa, C_{\gamma_0}a), D^*(Sa, Ra, A_{\alpha_0}a) \end{array} \right) \\ &< qD^*(a, a, B_\lambda a) \end{aligned}$$

is a contradiction. Hence for every $\lambda \in J$ we have $B_\lambda(a) = a$. Similarly for every $\delta \in I$ and $\eta \in K$ we get $A_\delta(a) = C_\eta(a) = a$. Therefore for every $\delta \in I, \lambda \in J$ and $\eta \in K$ we have $A_\delta(a) = B_\lambda(a) = C_\eta(a) = R(a) = S(a) = T(a) = a$. □

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