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# General Solution of DP-conditions for Simple Wave Type Solutions of the One-Dimensional Gas Dynamics Equations

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**Abstract:** The manuscript is devoted to the one-dimensional gas dynamics equations. For an isentropic flow these equations are reduced to the equations written in the Riemann invariants. The system written in the Riemann invariants is hyperbolic and homogeneous. It allows obtaining simple waves, which are also called Riemann waves. For nonisentropic flows there are no Riemann invariants. The question is: what solutions could substitute the Riemann waves. By the method of differential constraints such types of solutions are found here. For these classes of solutions one can integrate the gas dynamics equations: finite formulas with one parameter are obtained. These solutions have similar properties with simple Riemann waves. For example, they describe a nonisentropic rarefaction wave. The rarefaction waves play the main role in many applications such as the problem of pulling a piston, decay of arbitrary discontinuity and others.

**Key words:** Differential constraints, Compatibility conditions, Riemann (simple) waves, Gas dynamics equations.
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## 1 Introduction

The method of differential constraints is one of the methods for constructing particular exact solutions of partial differential equations. The idea of the method was proposed by N.N.Yanenko [1]. A survey of the method can be found in the book [2]. The method is based on the following idea.

Consider a system of differential equations

$$S_i(x,u,p)=0, \quad (i=1,2,\ldots,s). \tag{1}$$

Here $x=(x_1,x_2,\ldots,x_n)$ are the independent variables, $u=(u^1,u^2,\ldots,u^m)$ are the dependent variables, $p=(p^j_\alpha)$ is the set of the derivatives $p^j_\alpha=\dfrac{\partial^{|\alpha|}u^j}{\partial x^\alpha}$, $(j=1,2,\ldots,m;\ |\alpha|\le q)$, $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$, $|\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_n$. Assume that a solution of system (1) satisfies the additional system of differential equations

$$\Phi_k(x,u,p)=0, \quad (k=1,2,\ldots,q). \tag{2}$$

The differential equations (2) are called differential constraints. A solution of system (1) satisfying (2) is called the solution characterized by the differential constraints (2).

The obtained system (1), (2) is an overdetermined system. The method of differential constraints requires for the overdetermined system (1), (2) to be compatible. The form of the differential constraints (the functions $\Phi_k$) and a part of equations of the given system (the functions $S_i$) may not be known a priori.

The application of the method of differential constraints involves two stages. The first stage is to find the set of differential constraints (2) under which the overdetermined system is compatible. On this stage in the process of compatibility analysis (reducing the system to involutive) the overdetermined system (1), (2) can be supplemented by new equations. The second stage of the method is to construct solutions of the involutive overdetermined system. Because the solution has to satisfy the differential constraints (additional equations), then it allows easier constructing particular solution of the given system (1).

The requirement of compatibility of system (1), (2) is very general. Therefore the method of differential constraints includes (almost) all known methods for constructing exact solutions of partial differential equations: group–invariant solutions, nonclassical and weak symmetries, partially invariant solutions, separation of variables, as well as many others.

Increasing the number of requirements on the differential constraints narrows the generality of the method and makes it more suitable for finding exact particular solutions. In [3] it is suggested to require involutiveness of the overdetermined system (1), (2). With this refinement the method of differential constraints becomes a practical tool for obtaining exact particular solutions. In this case the classification of differential constraints and solutions characterized by them is carried out with respect to the functional arbitrariness of solutions of the overdetermined system (1), (2) and order of highest derivatives, included in the differential constraints (2). Involutive conditions are called $DP$-conditions.

The manuscript is devoted to study one class of solutions of one-dimensional polytropic gas dynamics equations. Classification of all differential constraints of first order, with which the one-dimensional gas dynamics equations are involutive, were given in [4][1]. It has been known only one class of solutions characterized by two differential constraints with sound characteristic: Riemann (simple) waves. Here the general solution of $DP$-conditions is given: it is proven that except Riemann waves (for arbitrary polytropic exponent) there is only one class of solutions for which $DP$-conditions are satisfied. These solutions are called in the manuscript by generalized simple waves. The main features of this class of solutions are the following. They describe a nonisentropic rarefaction waves. The construction of them is reduced to integration a system of ordinary differential equations along characteristics.

[1]It should be also noted that in [5] method of differential constraints was applied to one-dimensional gas dynamics equations written in Lagrangian coordinates. Here this method is applied to Eulerian representation of these equations.

Example of generalized simple waves for two-dimensional plane gas dynamics equations was constructed in [6]. The solutions considered there generalize Prandtl-Myer flows. For a hyperbolic quasilinear systems with two dependent variables generalized simple waves were also studied in [7].

The manuscript is organized as follows. The first part introduces some knowledge about the method of differential constraints which are necessary for applications. The second part is devoted to the study of generalized simple waves for one-dimensional gas dynamics equations.

# 2 Method of differential constraints

Let us consider the quasilinear system of partial differential equations

$$\frac{\partial u}{\partial t} + Q\frac{\partial u}{\partial x} - f = 0. \tag{3}$$

Here $Q = Q(x,t,u)$ is a $m \times m$ matrix, $f = f(x,t,u)$ is a vector, $E_r$ is a $r \times r$ unity matrix. One is looking for solutions characterized by first order differential constraints[2]

$$\Phi_k(x,t,u,u_x) = 0, \quad (k = 1,2,\ldots,q). \tag{4}$$

It is assumed that the differential constraints satisfy the natural requirement

$$rank\left(\frac{\partial \Phi_k}{\partial u_x}\right) = q.$$

## 2.1 Involutive conditions

Without loss of generality one can rewrite the system of differential equations and the differential constraints in the more suitable form

$$S \equiv Lu_t + ALu_x - Lf = 0, \tag{5}$$

$$\Phi = B_1 Lu_x + \Psi = 0. \tag{6}$$

Here $L = L(x,t,u)$ is a nonsingular $m \times m$ matrix, $A = LQL^{-1}$, the function $\Psi = \Psi(x,t,u,y)$ depends on $x,t,u$ and $y = B_2 Lu_x$, $B_1$ and $B_2$ are rectangular $q \times m$ and $(q-1) \times m$ matrices with the elements

$$(B_1)_{ij} = \delta_{ij}, \ (1 \le i \le q, \ 1 \le j \le m),$$
$$(B_2)_{kj} = \delta_{q+k,j}, \ (1 \le k \le m-q, \ 1 \le j \le m),$$

$\delta_{ij}$ is the Kronecker's symbol. The matrices $B_1$ and $B_2$ have the following properties:

$$B_1B_1' = E_q, \ B_2B_2' = E_{m-q}, \ B_1'B_1 + B_2'B_2 = E_m,$$
$$B_1B_2' = 0, \ B_2B_1' = 0.$$

[2]The study of differential constraints of higher order of the system $(S)$ can be reduced to the study of differential constraints of first order for the prolonged system.

Note that if the matrix $A$ is a diagonal matrix, then the matrix $B_jAB_j$ is diagonal and $B_iAB_j = 0$ $(i, j = 1, 2; i \neq j)$. For a hyperbolic system (3) the matrix $A$ can be chosen diagonal.

For the overdetermined system (5), (6) in [8] it is proven the following.

**Theorem 1** *Overdetermined system (5), (6) is involutive if and only if*

$$(D_t\Phi + ZAB_1'D_x\Phi - ZD_xS)_{|(S\Phi)} = 0, \tag{7}$$

$$ZA - ZAB_1'Z = 0, \tag{8}$$

*where* $Z = B_1 + \Psi_y B_2$ *and* $(S\Phi)$ *means the manifold*

$$(S\Phi) \equiv \{(x, u, p)|S(x, u, p) = 0, \Phi(x, u, p) = 0\}.$$

Equations (8) mean that the symbol of the overdetermined system is involutive. In applications equations (8) are checked first, although they are contained in (7). Equations (8) mean that there are no new equations after prolongation the system. Equations (7), (8) are called $DP$-conditions.

It should be noted that equations (8) are equivalent to

$$B_1AB_2' - \Psi_y B_2AB_1'\Psi_y + \Psi_y B_2AB_2' - B_1AB_1'\Psi_y = 0.$$

If the matrix $A$ is a diagonal matrix with the diagonal entries $\lambda_i$ $(i = 1, 2, \ldots, m)$, then $B_1AB_2' = 0$, $B_2AB_1' = 0$, the matrices $B_1AB_1'$, $B_2AB_2'$ are diagonal and equations (8) become

$$(\lambda_i - \lambda_j)(\Psi_i)_{y_j} = 0, \ (i = 1, 2, ..., q; \ j = 1, 2, ..., m - q).$$

This means that $\Psi_i$ can only depend on $y_j$ such that $(\lambda_i - \lambda_j) = 0$. In particular, in the case of strictly hyperbolic systems $(\lambda_i - \lambda_j) \neq 0$ $(i \neq j)$, and equations (8) are reduced to [9]

$$\Psi_y = 0.$$

The last equations mean that for strictly hyperbolic systems the differential constraints are quasilinear.

If system (5), (6) is analytic, then its involutiveness provides an uniqueness and existence of the Cauchy problem. There are more weak requirements on the smoothness of system (5), (6) that are sufficient for the uniqueness and existence of the Cauchy problem. First proof for systems of the class $C^2$ was done in [10]. For systems of the class $C^1$ the existence theorem was done in [8].

Assume that

$$L \in C^1(D), \ A \in C^1(D), \ f \in C^1(D), \ \Psi \in C^1(D) \tag{9}$$

in open domain $D \subset R^m \times R^2$.

**Theorem 2** *Let system (5) be a hyperbolic system with (9), and let equations (7), (8) be satisfied. Then there exists an unique solution $u(x,t) \in C^1$ of the Cauchy problem for system (5), (6) with the initial data $u(x,0) = \varphi(x) \in C^1$ satisfying the differential constraints (6) at $t = 0$.*

There are also valid similar statements for other types of systems [8].

## 2.2 Generalized simple waves

One class of solutions, which generalizes class of simple waves is studied here. Assume that a system of quasilinear differential equations $(S)$ admits $q = m - 1$ quasilinear differential constraints

$$\Phi = B_1 L u_x + \Psi_y B_2 L u_x + \phi = 0,$$

where $\phi = \phi(u,x,t)$ and $\Psi_y = \Psi_y(u,x,t)$ is a $(m-1)\times m$ matrix, $\lambda = B_2 A B_2'$, $y = B_2 L u_x$. Also assume that $B_2 A B_1' = 0$ and

$$L \in C^1(D),\ A \in C^1(D),\ f \in C^1(D),\ \Psi_y \in C^1(D),\ \phi \in C^1(D) \tag{10}$$

A solution satisfying these differential constraints is called a generalized simple wave[3].

For this class of solutions the $DP$-conditions (7), (8) become the following

$$\lambda \Psi_y = -B_1 A (B_2' - B_1' \Psi_y), \tag{11}$$
$$\Omega_1 y^2 + \Omega_2 y + \Omega_3 = 0,$$

where $\Omega_1, \Omega_2, \Omega_3$ are vector-functions, which depend on $L, \Psi_y, A, \phi$ and their derivatives [12]. Note that (11) can be rewritten as

$$A(B_2' - B_1' \Psi_y) = \lambda (B_2' - B_1' \Psi_y). \tag{12}$$

Note also that in the strength of involutive conditions (11) the first function $\Omega_1 \equiv 0$. Because $\Omega_2$ and $\Omega_3$ do not depend on $y$, then the conditions of involutiveness require

$$\Omega_2 = 0,\ \Omega_3 = 0. \tag{13}$$

In (11) there are $2q$ equations.

By virtue of the differential constraints and condition (12) for these solutions one can define all derivatives along the characteristic $\frac{dx}{dt} = \lambda$:

$$\begin{array}{c} B_1 L \frac{du}{dt} = B_1 (A - \lambda E_m) B_1' \phi + B_1 L f, \\ B_2 L \frac{du}{dt} = B_2 L f,\ \frac{dx}{dt} = \lambda. \end{array} \tag{14}$$

Let $u_0(\xi) \in C^1$ satisfies the differential constraints

$$(B_1 + \Psi_y(u_0(\xi),\xi,0) B_2) L(u_0(\xi),\xi,0) u_0'(\xi) + \phi(u_0(\xi),\xi,0) = 0.$$

[3]There is a generalization of such class solutions for systems with more than two independent variables [11].

There exists the unique solution $(v(\xi,t),x(\xi,t))$ of the Cauchy problem of the system of ordinary differential equations (14) with the initial data at $t=0$:

$$v=u_0(\xi),\ x=a.$$

The dependence $x=x(\xi,t)$ can be solved with respect to $\xi=\xi(x,t)$ in some neighborhood of the point $(x_0,0)$. We show that $u(x,t)=v(\xi(x,t),t)$ is a solution of the overdetermined system $(S\Phi)$. Exchanging the variables $(x,t)$ onto $(\xi,t)$ we have

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t}+\lambda\frac{\partial}{\partial x},\ \frac{\partial}{\partial \xi}=x_\xi\frac{\partial}{\partial x},$$

where $x_\xi=\frac{\partial x}{\partial \xi}$. The left hand side of the differential constraints in the new independent variables is

$$B_1Lu_x+\Psi_yB_2Lu_x+\phi=\frac{1}{x_\xi}(B_1Lv_\xi+\Psi_yB_2Lv_\xi+x_\xi\phi).$$

Let $H=(B_1+\Psi_yB_2)Lv_\xi+x_\xi\phi$. By using the conditions of involutiveness (13), we obtain that $H=H(\xi,t)$ satisfies the linear differential equations

$$\frac{dH}{dt}=GH,$$

where $G=G(\xi,t)$ is some matrix, $\frac{d}{dt}$ is the partial derivative $\frac{\partial}{\partial t}$ in the variables $(\xi,t)$. Because the initial values $H(\xi,0)=0$ and by virtue of the uniqueness of solution of the Cauchy problem of system of ordinary differential equations

$$H(\xi,t)=0.$$

It means that the differential constraints are satisfied. Rewriting equations (14) in the coordinates $(x,t)$ one finds that

$$B_1L(u_t+\lambda u_x)=B_1(A-\lambda E_m)B_1'\phi+B_1Lf,$$
$$B_2L(u_t+\lambda u_x)=B_2Lf.$$

Substitution $\phi=-(B_1+\Psi_yB_2)Lu_x$ into the last equations gives

$$B_1(Lu_t+ALu_x)=B_1Lf,$$
$$B_2(Lu_t+ALu_x)=B_2Lf.$$

Here it is used that $\lambda=B_2AB_2'$ and conditions (12).

Therefore, for constructing a generalized simple wave one needs to satisfy the differential constraints on some curve $x_0(t)$ which is not the characteristic $x_0'\neq\lambda$, then the solution can be found by integrating the system of ordinary differential equations (14).

By the same way one can construct a solution of a problem with the initial data on a characteristic curve of the overdetermined system $(S\Phi)$ and with a singularity of the rarefaction wave type at the point $(0,0)$ [12]. In fact there exists an unique solution of the system $(S)$ in some domain $V \in R^2$ that satisfies the following conditions.

1. On the characteristic curve $\Pi : x = x_0(t)$ the value $u(x_0(t), t) = u_\lambda(t)$ satisfy (14).

2. The point $(0,0) \in \Pi \subset V$ is singular: the solution is multiply defined at this point. The value $u = u_0(a)$ of the solution at this point depends on the parameter $a$, $(u_0(0) = u_\lambda(0))$ and it defines the curve in the space $R^m$ satisfying the equations

$$(B_1 + \Psi_y(u_0(a), 0, 0)B_2)L(u_0(a), 0, 0)u_0'(a) = 0, \tag{15}$$

$$\frac{\partial \lambda}{\partial u}(u_0(a), 0, 0)u_0'(a) < 0, \ (0 \le a \le a_0). \tag{16}$$

Here the parameter $a$ plays role of the variable $\xi$ at the point of singularity $(0, 0)$.

The solution of this problem generalizes the well-known rarefaction wave in gas dynamics.

# 3 One-dimensional gas dynamics equations

An unsteady one-dimensional flow of a gas is described by the equations

$$\begin{cases} u_t + uu_x + \rho^{-1}p_x = 0, \\ \rho_t + u\rho_x + \rho u_x = 0, \\ p_t + up_x + A(\rho, p)u_x = 0. \end{cases} \tag{17}$$

Here $\rho$ is the density, $\tau = 1/\rho$ is the specific volume, $u$ is the velocity, $p$ is the pressure, $\varepsilon$ is the internal energy, $T$ is the temperature, $\eta$ is the entropy, and $c$ is a sound speed ($c^2 = A/\rho$). For a polytropic gas $A = \gamma p$, $\gamma > 1$, and the thermodynamical parameters are related by the following formulas

$$T = p(R\rho)^{-1}, \ \varepsilon = (\gamma - 1)^{-1}p/\rho, \ \eta = g(p\rho^{-\gamma}),$$

where $g$ is some function.

System (17) can be rewritten in a matrix form

$$S \equiv L\mathbf{u}_t + \Lambda L\mathbf{u}_x = 0,$$

with

$$\mathbf{u} = \begin{pmatrix} u \\ \rho \\ p \end{pmatrix}, \ L = \begin{pmatrix} 0 & -A & \rho \\ \rho c & 0 & 1 \\ -\rho c & 0 & 1 \end{pmatrix}, \ \Lambda = \begin{pmatrix} u & 0 & 0 \\ 0 & u + c & 0 \\ 0 & 0 & u - c \end{pmatrix}.$$

Since system (17) is strictly hyperbolic the differential constraints of first order for it must be quasilinear. The well-known Riemann waves (or simple waves) are obtained by assuming that $u = u(\rho)$, $p = p(\rho)$. The entropy in the Riemann waves is constant. It can be shown that the Riemann waves belong to the class of solutions which is characterized by the following differential constraints

$$p_x - \rho\alpha^2\rho_x = 0, \quad \rho u_x + \alpha\rho_x = 0,$$

where $\alpha = \pm c$. Note that the matrix $B_2 = (0,1,0)^*$ for $\alpha = c$ and $B_2 = (0,0,1)^*$ for $\alpha = -c$.

Let us study more general class of solutions which is characterized by the differential constraints

$$p_x - \alpha^2\rho_x = \psi, \quad u_x + \rho^{-1}\alpha u_x = \phi, \tag{18}$$

where $\psi = \psi(t,x,u,\rho,p)$ and $\phi = \phi(t,x,u,\rho,p)$. The $DP$-conditions (12), (13) for this class of solutions are

$$-\psi_p\gamma p - \psi_\rho\rho + \psi_u\alpha + \psi(\gamma+1) = 0 \tag{19}$$

$$\psi_t + u\psi_x - \psi_u(\psi\rho^{-1} + \alpha\phi) = 0 \tag{20}$$

$$-4\phi_p\alpha\gamma p\rho - 4\phi_\rho\alpha\rho^2 + 4\phi_u\gamma p - 3\psi\gamma + \phi\alpha\rho(3-\gamma) = 0 \tag{21}$$

$$\begin{gathered}-\psi_p\psi - \psi_u\phi - \psi_x - \phi_p\psi\alpha\rho + \phi_p\phi\gamma p\rho + \phi_\rho\phi\rho^2 - \\ \phi_t\rho + \phi_u\psi - \phi_u\phi\alpha\rho - \phi_x\rho(\alpha+u) - \phi^2\rho = 0\end{gathered} \tag{22}$$

Note that if $\psi = 0$, then $\phi$ can be not equal to zero only for $\gamma = 3$ and $\gamma = 5/3$ (one-atomic gas). But in the case $\psi = 0$ the one-dimensional gas dynamics equations are transformed to the Darboux equation [4]. If $\gamma = 3$ or $\gamma = 5/3$, then the general solution of the Darboux equation is expressed through the D'Alambert solution [13]. Therefore further the case of $\psi \neq 0$ is studied.

The general solution of equation (19) is

$$\psi = \rho^{\gamma+1}\Psi\left(t,x,\xi,\eta\right).$$

where $\xi = u + \frac{2\alpha}{\gamma-1}$, $\eta = p\rho^{-\gamma}$. After that one can solve equation (21). The general solution of equation (21) is

$$\phi = \rho^{(3-\gamma)/4}\Phi\left(t,x,\xi,\eta\right) - \frac{3\alpha\rho^{\gamma+1}}{p(3\gamma-1)}\Psi\left(t,x,\xi,\eta\right).$$

After substituting the representations of $\psi$ and $\phi$ into (20) one has

$$\Psi_t + (\xi - \frac{2\alpha}{\gamma-1})\Psi_x + \left(\rho^\gamma\frac{1}{3\gamma-1}\Psi - \alpha\rho^{(3-\gamma)/4}\Phi\right)\Psi_\xi = 0. \tag{23}$$

Splitting this equation[5] with respect to $\rho$ one obtains

$$\Psi_t = 0, \ \Psi_x = 0, \ \Psi_\xi = 0.$$

[4] See, for example, [13].

[5] For splitting it is essential that $\gamma > 1$.

After substituting the representations of $\psi$ and $\phi$ into (22) one has[6]

$$a_1\rho_1^5 + a_2\rho_1^{4+3\gamma} + a_3\rho_1^{3+6\gamma} + a_4\rho_1^{2+5\gamma} + a_5\rho_1^{2+\gamma} + a_6\rho_1^{3\gamma} = 0, \tag{24}$$

where $\rho_1 = \rho^{1/4}$, the coefficients $a_i$ depend on $(t, x, \xi, \eta)$. Analysis of the linear functions (powers of $\rho_1$) gives that for $\gamma > 1$, the degrees $4+3\gamma, 3+6\gamma$ and $2+5\gamma$ have different values and they differ from the degrees 5, $2+\gamma$, and $3\gamma$. Thus, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$. These equations give

$$\eta\Psi_\eta = \frac{3\gamma}{3\gamma-1}\Psi, \; \eta\Phi_\eta = \frac{3(\gamma-3)}{3\gamma-1}\Phi, \; \Phi_\xi = 0. \tag{25}$$

In the strength of $\Phi_\xi = 0$, equation (24) can be split with respect to $\xi$:

$$\Phi_x = 0. \tag{26}$$

The general solution of (25), (26) is

$$\Psi = k\eta^\beta, \; \Phi = h(t)\eta^q, \; \left(\beta = \frac{3\gamma}{3\gamma-1}, \; q = \frac{3(\gamma-3)}{3\gamma-1}\right),$$

where $k$ is constant. Equation (24) is reduced to the equation

$$\rho_1^{3-\gamma}(\gamma+1)\eta^q h^2 + 4h' = 0.$$

If $\gamma \neq 3$, then from this equation one obtains $h = 0$. If $\gamma = 3$, then $h = (t + k_1)^{-1}$, ($k_1 = const$).

**Theorem 3** The general solution of the $DP$-conditions for differential constraints (18) (for nonisentropic flows $\psi \neq 0$ and arbitrary polytropic exponent) is

$$\psi = k\rho^{\beta_1}p^\beta, \; \phi = -\frac{3\gamma}{(3\gamma-1)\alpha\rho}\psi,$$
$$\left(\beta_1 = 1 - \frac{\gamma}{(3\gamma-1)}, \; \beta = 1 + \frac{1}{(3\gamma-1)}, \quad k \neq 0\right). \tag{27}$$

A generalized simple wave ($k \neq 0$) along the characteristics

$$\frac{dx}{dt} = u - \alpha \tag{28}$$

satisfies the system of ordinary differential equations (equations (14))

$$\frac{dp}{d\rho} = \frac{p}{3\rho}, \; \frac{du}{d\rho} = \frac{\alpha}{3\gamma\rho}, \; \frac{d\rho}{dt} = -3\gamma k p^{\beta_1}\rho^{\beta_2+1}. \tag{29}$$

[6] All calculations are done in REDUCE [14]

These equations can be integrated. The general solution of (29) is

$$p = c_1\rho^{1/3},\ u = -\gamma_1^{-1}\sqrt{c_1}\rho^{-1/3} + c_3,\ \rho^{1/3} = \left(\gamma c_1^{\beta_1} kt + c_2\right)^{-1}, \tag{30}$$

where $\alpha = \gamma_1\sqrt{c_1}\rho^{-1/3}$ and $\gamma_1 = \pm\sqrt{\gamma}$. Hence,

$$u - \alpha = (\gamma - 1)\gamma_1^{-1}\sqrt{c_1}\left(\gamma c_1^{\beta_1} kt + c_2\right) + c_3,$$

and

$$x = \frac{(\gamma-1)}{2}\gamma_1^{-1}\sqrt{c_1}\left(\gamma c_1^{\beta_1} kt^2 + 2c_2 t\right) + c_3 t + \xi.$$

Here $c_1$, $c_2$, $c_3$ and $\xi$ are arbitrary constants of integration. These constants are defined, for example, by the initial values at $t = 0$ :

$$u_o(\xi) = u(0,\xi),\ \rho_o(\xi) = \rho(0,\xi),\ p_o(\xi) = p(0,\xi) \tag{31}$$

$$c_1(\xi) = p_o/\rho_o^{1/3},\ c_3 = u_o + \gamma_1^{-1}\sqrt{c_1}\rho_o^{-1/3},\ c_2 = \rho_o^{-1/3}, \tag{32}$$

$$x = \frac{(\gamma-1)}{2}\gamma_1^{-1}p_o^{1/2}\rho_o^{-1/6}t\left(\gamma p_o^{\beta_1/2}\rho_o^{-\beta_1/6}kt + 2\rho_o^{-1/3}t\right)$$
$$+ (u_o + \gamma_1^{-1}p_o^{1/2}\rho_o^{-1/2})t + \xi$$

The functions $u_o(\xi)$, $\rho_o(\xi)$, $p_o(\xi)$ have to satisfy the differential constraints (18) with the functions (27). It is proven in [8] that if the initial values at $t = 0$ (31) satisfy the differential constraints (18), then the solution of the gas dynamics equations satisfies this differential constraints at time $t > 0$. Note that in the initial values one can choose one arbitrary function, the other functions are defined by the system of ordinary differential equations. Let us take, for example, $\rho_o = const$. In this case equations (18) can be integrated as follows

$$p_o = \left(\xi\gamma_1\rho_o^{\beta_2+1/2}k + k_1\right)^{(1-3\gamma)},\ u_o = \frac{6\gamma_1}{(1-3\gamma)}\rho_o^{-1/2}p^{1/2} + k_2,$$

where $k_1$ and $k_2$ are arbitrary constants.

It should be also noted that by using the generalized simple waves one can obtain nonisentropic rarefaction waves. This solution is constructed by integrating (28), (29) with singular initial values, which satisfy the equations (equations (15), (16))

$$p_\xi - \rho\alpha^2\rho_\xi = 0,\quad \rho u_\xi + \alpha\rho_\xi = 0.$$

These conditions in gas dynamics are called $(p,u)$-diagram. Note that the $(p,u)$-diagram for nonisentropic case is the same as for isentropic rarefaction waves.

# 4 Conclusion

Nonisentropic solutions of one-dimensional gas dynamics equations having one arbitrary function in defining were constructed. This class of solutions is defined by two differential constraints of first order. They can be considered as a generalization of Riemann waves for nonisentropic case.

# References

[1] Yanenko, N.N.,*Compatibility theory and methods of integration of systems of nonlinear partial differential equations*, Proceedings of the Fourth All–Union Mathematics Congress, Nauka, Leningrad, 1964, 613.

[2] Sidorov, A.F., Shapeev, V.P., and Yanenko, N.N., *n gas dynamics*, Nauka, Novosibirsk, 1984.

[3] Shapeev, V.P., A problem of continuous adjoint of DP-solutions of one-dimensional inelastic continuum media, *Chisl.metody mech.splosh.sredy* **5(4)**(1974), 116–125.

[4] Raspopov, V.E., Shapeev, V.P. and Yanenko, N.N. The application of the method of differntial constraints to one–dimensional gas dynamics equations, *Izvestiya V.U.Z. Matematica* **11(150)**(1974), 69–74.

[5] Raspopov, V.E., Shapeev, V.P. and Yanenko, N.N. The method of differential constraints for one–dimensional gas dynamics equations, *Chisl.metody mech.splosh.sredy*, **8(2)**(1977), 100–105.

[6] Meleshko, S.V., Shapeev, V.P. 'An application of the differential constraints method for the two-dimensional equations of gas dynamics', *Prikl. matem. and mech.*, **63(6)**(1999), 909-916. (English transl. in *J.Appl.Maths Mechs*, **63(6)**, 885-891)

[7] Manganaro, N., Meleshko, S.V. Reduction Procedure and Generalized Simple Waves for Systems Written in the Riemann Variables, *Nonlinear Dynamics (An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems)*, **30(1)**(2002), 87-102.

[8] Meleshko, S.V. DP-conditions and the adjoin problem of different DP-solutions, *Chisl.metody mech.splosh. sredy* **11(6)**(1980), 96-109.

[9] Gzigzin, A.E. and Shapeev, V.P., 'To the problem about continuous adjoining of particular solutions of systems of partial differential equations', *Chisl.metody mech.splosh.sredy* **6(7)**(1975), 44–52.

[10] Gzigzin, A.E., On integrability of some nonanalytic involutive systems, *Docladу AS USSR*, **238(1)**(1978), 15–18.

[11] Meleshko, S.V., One class of solutions for the systems of quasilinear differential equations with many independent variables, *Chisl.metody mech.splosh.sredy*, **12(4)**(1981), 87-100.

[12] Meleshko, S.V., *A solution of wave problem for dynamics of nonelastic continuous media with method of differential constraints*, Ph.D. thesis, Institute of Hydrodynamics. Novosibirsk, 1981.

[13] Rozhdestvenskii, B.L. and Yanenko, N.N., *Systems of quasilinear equations and their applications to gas dynamics*, 2nd ed., Nauka, Moscow, 1978. (English translation published by Amer. Math. Soc., Providence, RI, 1983).

[14] Hearn, A.C. 'REDUCE Users Manual, ver. 3.3'. The Rand Corporation CP 78. Santa Monica, 1987.

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