Thai Journal of Mathematics
Volume 22 Number 3 (2024)
Pages 509–518

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

Inner Products on Intervals in $\mathbb{R}^n$

Daishi Kuroiwa* **and Tomoya Mori**

Department of Mathematical Sciences, Shimane University, Japan
e-mail : kuroiwa@riko.shimane-u.ac.jp (D. Kuroiwa)

Abstract In this paper, we defined an inner product on the collection of intervals in $\mathbb{R}^n$ within a vector space framework, demonstrating its consistency with the properties of standard inner products. We further established that the collection of intervals in $\mathbb{R}^n$ forms a Hilbert space under the proposed inner product. Additionally, we explored applications of this framework to interval linear programming problems and interval support vector machines, highlighting the practical relevance and usefulness of the theoretical results.

MSC: 46C05; 90C25; 90C46

Keywords: intervals in $\mathbb{R}^n$; embedding to a vector space; inner products; interval linear programming problem; interval support vector machine

Submission date: 08.11.2024 / Acceptance date: 10.12.2024

1. Introduction

In the real world, uncertainty is inherent in various situations. For instance, in production planning, prices fluctuate over time. One way to represent uncertainty in real numbers is through intervals. By defining upper and lower bounds on value changes within a given mathematical programming problem, we can consider both optimistic and pessimistic solutions. However, such a collection of intervals neither forms a vector space in the traditional sense nor allows for a well-defined inner product, for exapmle, see [1].

It is well-known that convex sets can be embedded into a vector space, see [2]. While this embedding addresses the vector space structure, to the best of our knowledge, no existing work explores the definition of an inner product for intervals in such spaces. The purpose of this paper is to define an inner product for a collection of intervals in $\mathbb{R}^n$, analyze its properties, and provide practical examples. The remainder of this paper is organized as follows. In Section 2, we introduce the collection of intervals in $\mathbb{R}^n$ and define the sum and non-negative scalar product on the collection. Section 3 presents the foundational results of this paper. We define a vector space in which the intervals in $\mathbb{R}^n$ are embedded, provide inner products on the vector space, and discuss the dimension of the inner product spaces. As a result, the inner product spaces are Hilbert spaces. In Section 4, we explore applications to interval linear programming and interval support vector machines. Finally, Section 5 concludes the paper with a summary of our results.

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*Corresponding author.

Published by The Mathematical Association of Thailand.

2. Intervals in $\mathbb{R}^n$

For any $a = (a_1, a_2, \ldots, a_n)^T, b = (b_1, b_2, \ldots, b_n)^T \in \mathbb{R}^n$, define partial order $\leqq$ on $\mathbb{R}^n$ by $a \leqq b$ if $a_i \leqq b_i$ for all $i \in \{1, \cdots, n\}$, and define the interval $[a, b]$ as follows:

$$[a, b] := \{x \in \mathbb{R}^n \mid a \leqq x \leqq b\} = \prod_{i=1}^{n} [a_i, b_i]$$

whenever $a \leqq b$. Let $\mathcal{M}$ be the family of all intervals in $\mathbb{R}^n$ as follows:

$$\mathcal{M} = \{[a, b] \mid a \leqq b\}.$$

For any $a \in \mathbb{R}^n$, $[a, a]$ means singleton $\{a\}$ and we treat the element a as the singleton interval $[a, a]$ if necessary. On $\mathcal{M}$, the sum and non-negative scalar product are defined as follows:

$$[a, b] + [c, d] = [a + c, b + d] \quad \text{and} \quad t[a, b] = [ta, tb]$$

for any $t \geqq 0$. Remark that we do avoid defining $[a, b] - [c, d]$ and $t[a, b]$ for any $t < 0$ in order to prevent misunderstanding regarding the additive inverse. Indeed the additive inverse of $[a, b]$ does not exist whenever $a \neq b$. In order to clarify the structure involving the additive inverse, we construct a vector space $\mathcal{N}$ in which $\mathcal{M}$ is embedded in the next section, by using the same way by Rådström, see [2]. The main purpose of the paper is to provide an inner product in the vector space $\mathcal{N}$.

3. A Hilbert Space in Which Intervals in $\mathbb{R}^n$ Are Embedded

3.1. A Vector Space in Which Intervals in $\mathbb{R}^n$ Are Embedded

At first we define a binary relation $\sim$ on $\mathcal{M}^2$ as follows:

$$([a, b], [c, d]) \sim ([a', b'], [c', d']) \iff \begin{cases} a - c = a' - c' \\ b - d = b' - d', \end{cases}$$

for any $([a, b], [c, d])$ and $([a', b'], [c', d']) \in \mathcal{M}^2$. We can see this binary relation is reflexive, symmetric and transitive, so it is an equivalence relation. For any $([a, b], [c, d]) \in \mathcal{M}^2$,

$$[a, b] \ominus [c, d] := \{([a', b'], [c', d']) \in \mathcal{M}^2 \mid ([a, b], [c, d]) \sim ([a', b'], [c', d'])\}$$

denotes the equivalence class to which $([a, b], [c, d])$ belongs.

$$\mathcal{N} = \mathcal{M}^2/\sim = \{[a, b] \ominus [c, d] \mid [a, b], [c, d] \in \mathcal{M}\}$$

denotes the quotient set of $\mathcal{M}^2$ by $\sim$. On $\mathcal{N}$, the sum and scalar product are defined as follows:

$$[a, b] \ominus [c, d] + [a', b'] \ominus [c', d'] := [a + a', b + b'] \ominus [c + c', d + d']$$

$$\lambda([a, b] \ominus [c, d]) := \begin{cases} [\lambda a, \lambda b] \ominus [\lambda c, \lambda d] & \text{if } \lambda \geqq 0, \\ [-\lambda c, -\lambda d] \ominus [-\lambda a, -\lambda b] & \text{if } \lambda < 0. \end{cases}$$

Then we can check that $\mathcal{N}$ is a vector space with this sum and scalar product. Here the null vector has the form $[a, b] \ominus [a, b]$ containing $[0, 0] \ominus [0, 0]$, and the additive inverse of $[a, b] \ominus [c, d] \in \mathcal{N}$ is $[c, d] \ominus [a, b]$. In order to embed $\mathcal{M}$ to $\mathcal{N}$, we identify an interval $[a, b]$ of $\mathcal{M}$ with $[a, b] \ominus [0, 0]$. These arguments are similar to Rådström, see [2].

3.2. Inner Products on $\mathbb{R}^{2n}$

Before to discuss inner products on $\mathcal{N}$, we observe inner products on $\mathbb{R}^{2n}$, by using the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, that is, $\langle x, y \rangle = x^T y$ for any $x, y \in \mathbb{R}^n$. Let P be a square $2n \times 2n$ invertible matrix over $\mathbb{R}$ which forms $P = \begin{pmatrix} S & T \\ U & V \end{pmatrix}$ where S, T, U, and V are square $n \times n$ matrices. Define the following function from $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ to $\mathbb{R}$ by

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle_P := \langle Sa + Tb, Sa' + Tb' \rangle + \langle Ua + Vb, Ua' + Vb' \rangle$$

for any $a, b, a', b' \in \mathbb{R}^n$.

Proposition 3.1. *The function* $\left\langle \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\rangle_P$ *is an inner product on* $\mathbb{R}^{2n}$.

Proof. For any $a, b \in \mathbb{R}^n$, observe the following value

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle_P = \langle Sa + Tb, Sa + Tb \rangle + \langle Ua + Vb, Ua + Vb \rangle .$$

It is clear that the value is non-negative from the positive semi-definiteness of the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. Next, if the value is zero, then $Sa + Tb = 0$ and $Ua + Vb = 0$, that is,

$$\begin{pmatrix} S & T \\ U & V \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since P is invertible, it follows that $a = b = 0$.

Also, it is relatively easy to confirm that the linearity in the second argument and the symmetry hold by using the matrix product properties $S(a' + a'') = Sa' + Sa''$, $S(\lambda a') = \lambda Sa'$, and the inner product property of $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. Therefore, the function is an inner product on $\mathbb{R}^{2n}$. This completes the proof. ∎

Remark that the converse of Proposition 3.1 holds as follows:

Proposition 3.2. *Any inner product on* $\mathbb{R}^{2n}$ *can be expressed in the form for some square* $2n \times 2n$ *invertible matrix* P *over* $\mathbb{R}$.

Proof. Let $\left\langle \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\rangle$ be an inner product on $\mathbb{R}^{2n}$, and define a square $2n \times 2n$ matrix by

$$Q = (\langle e_i, e_j \rangle)_{i,j \in \{1,2,\ldots,2n\}},$$

where $\{e_1, \ldots, e_{2n}\}$ is the standard basis of $\mathbb{R}^{2n}$. Then we can check that Q is a symmetric positive definite matrix and for all $a, b \in \mathbb{R}^n$,

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle = (a^T, b^T) Q \begin{pmatrix} a' \\ b' \end{pmatrix}.$$

Also there exists the square root R of Q, that is $R^2 = Q$. Let $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $n \times n$ matrix. Then we have

$$\begin{aligned}
\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle &= (a^T, b^T) Q \begin{pmatrix} a' \\ b' \end{pmatrix} \\
&= \left(R \begin{pmatrix} a \\ b \end{pmatrix} \right)^T R \begin{pmatrix} a' \\ b' \end{pmatrix} \\
&= \begin{pmatrix} Aa + Bb \\ Ca + Db \end{pmatrix}^T \begin{pmatrix} Aa' + Bb' \\ Ca' + Db' \end{pmatrix} \\
&= \langle Aa + Bb, Aa' + Bb' \rangle + \langle Ca + Db, Ca' + Db' \rangle \\
&= \left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle_R .
\end{aligned}$$

Since R is a symmetric positive definite matrix, R is also invertible. This completes the proof. ∎

We observe the inner product when S, T, U, and V are diagonal matrices, where $S = \operatorname{diag}(s)$, $T = \operatorname{diag}(t)$, $U = \operatorname{diag}(u)$, $V = \operatorname{diag}(v)$, $s = (s_1, \ldots, s_n)^T$, $t = (t_1, \ldots, t_n)^T$, $u = (u_1, \ldots, u_n)^T$, and $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle_P &= \langle \operatorname{diag}(s)a + \operatorname{diag}(t)b, \operatorname{diag}(s)a' + \operatorname{diag}(t)b' \rangle \\
&\quad + \langle \operatorname{diag}(u)a + \operatorname{diag}(v)b, \operatorname{diag}(u)a' + \operatorname{diag}(v)b' \rangle \\
&= \sum_{i=1}^{n} \left((s_i a_i + t_i b_i)(s_i a'_i + t_i b'_i) + (u_i a_i + v_i b_i)(u_i a'_i + v_i b'_i) \right).
\end{aligned}$$

Proposition 3.3. *If $s_i v_i - t_i u_i \neq 0$ for all $i = 1, 2, \ldots, n$, then $\left\langle \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\rangle_P$ is an inner product on $\mathbb{R}^{2n}$.*

Proof. It is a direct consequence from Proposition 3.1 because

$$\det P = \det \left(\begin{array}{ccc|ccc} s_1 & & O & t_1 & & O \\ & \ddots & & & \ddots & \\ O & & s_n & O & & t_n \\ \hline u_1 & & O & v_1 & & O \\ & \ddots & & & \ddots & \\ O & & u_n & O & & v_n \end{array} \right) = \prod_{i=1}^{n} (s_i v_i - t_i u_i)$$

and it is not zero from the assumption. ∎

Example 3.4. If $s_i > 0$, $v_i > 0$, and $t_i = u_i = 0$, then $s_i v_i - t_i u_i = s_i v_i > 0$ and $\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle_P = \sum_{i=1}^{n} (s_i a_i a'_i + v_i b_i b'_i)$ is an inner product on $\mathbb{R}^{2n}$ and called a weighted inner product on $\mathbb{R}^{2n}$. Especially if $s_i = v_i = 1$ and $t_i = u_i = 0$, then $\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle_P =$

$\langle a, a' \rangle + \langle b, b' \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. If $s_i = v_i = 1$, $u_i = -1$, and $t_i = 0$, then $s_i v_i - t_i u_i = 1$ and $\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle_P = \langle a, a' \rangle + \langle b - a, b' - a' \rangle$ is an inner product on $\mathbb{R}^{2n}$ which has another structure from the standard inner product.

3.3. An Inner Product on $\mathcal{N}$

Now we define an inner product on $\mathcal{N}$. Let $\left\langle \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\rangle$ be an inner product in $\mathbb{R}^{2n}$. Define a function from $\mathcal{N}^2$ to $\mathbb{R}$ by

$$\langle [a, b] \ominus [c, d], [a', b'] \ominus [c', d'] \rangle := \left\langle \begin{pmatrix} a - c \\ b - d \end{pmatrix}, \begin{pmatrix} a' - c' \\ b' - d' \end{pmatrix} \right\rangle,$$

for any $[a, b] \ominus [c, d], [a', b'] \ominus [c', d'] \in \mathcal{N}$.

Theorem 3.5. *The function defined above is well-defined on $\mathcal{N}^2$ and it is an inner product on $\mathcal{N}$.*

Proof. At first, we show that this function is well-defined. Let $[a, b] \ominus [c, d], [a', b'] \ominus [c', d'], [a'', b''] \ominus [c'', d''], [a''', b'''] \ominus [c''', d'''] \in \mathcal{N}$, and assume that $[a, b] \ominus [c, d] = [a'', b''] \ominus [c'', d'']$ and $[a', b'] \ominus [c', d'] = [a''', b'''] \ominus [c''', d''']$.

Since $([a, b], [c, d]) \sim ([a'', b''], [c'', d''])$ and $([a', b'], [c', d']) \sim ([a''', b'''], [c''', d'''])$, we have

$$\begin{cases} a - c = a'' - c'' \\ b - d = b'' - d'' \end{cases} \quad \text{and} \quad \begin{cases} a' - c' = a''' - c''' \\ b' - d' = b''' - d'''. \end{cases}$$

We observe that

$$\begin{aligned} \langle [a, b] \ominus [c, d], [a', b'] \ominus [c', d'] \rangle &= \left\langle \begin{pmatrix} a - c \\ b - d \end{pmatrix}, \begin{pmatrix} a' - c' \\ b' - d' \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a'' - c'' \\ b'' - d'' \end{pmatrix}, \begin{pmatrix} a''' - c''' \\ b''' - d''' \end{pmatrix} \right\rangle \\ &= \langle [a'', b''] \ominus [c'', d''], [a''', b'''] \ominus [c''', d'''] \rangle, \end{aligned}$$

and then the function is well-defined.

(i) We show the positive definiteness. For any $[a, b] \ominus [c, d] \in \mathcal{N}$,

$$\langle [a, b] \ominus [c, d], [a, b] \ominus [c, d] \rangle = \left\langle \begin{pmatrix} a - c \\ b - d \end{pmatrix}, \begin{pmatrix} a - c \\ b - d \end{pmatrix} \right\rangle \geqq 0,$$

and $\langle [a, b] \ominus [c, d], [a, b] \ominus [c, d] \rangle = 0$ if and only if $\begin{pmatrix} a - c \\ b - d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, that is, $a = c$ and $b = d$, or equivalently $[a, b] \ominus [c, d] = [a, b] \ominus [a, b]$, which is the null vector of $\mathcal{N}$.

(ii) We show the symmetry as follows:

$$\begin{aligned} \langle [a, b] \ominus [c, d], [a', b'] \ominus [c', d'] \rangle &= \left\langle \begin{pmatrix} a - c \\ b - d \end{pmatrix}, \begin{pmatrix} a' - c' \\ b' - d' \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a' - c' \\ b' - d' \end{pmatrix}, \begin{pmatrix} a - c \\ b - d \end{pmatrix} \right\rangle \\ &= \langle [a', b'] \ominus [c', d'], [a, b] \ominus [c, d] \rangle. \end{aligned}$$

(iii) The additivity in the second argument is shown as follows:

$$\begin{aligned}
&\langle [a,b]\ominus[c,d]\,,([a',b']\ominus[c',d'])+([a'',b'']\ominus[c'',d''])\rangle\\
&=\langle [a,b]\ominus[c,d]\,,[a'+a'',b'+b'']\ominus[c'+c'',d'+d'']\rangle\\
&=\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} a'+a''-c'-c''\\ b'+b''-d'-d''\end{pmatrix}\right\rangle\\
&=\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} a'-c'\\ b'-d'\end{pmatrix}\right\rangle+\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} a''-c''\\ b''-d''\end{pmatrix}\right\rangle\\
&=\langle [a,b]\ominus[c,d]\,,[a',b']\ominus[c',d']\rangle+\langle [a,b]\ominus[c,d]\,,[a'',b'']\ominus[c'',d'']\rangle\,.
\end{aligned}$$

Next, we show the homogeneity in the second argument. When $\lambda \geqq 0$, we have

$$\begin{aligned}
\langle [a,b]\ominus[c,d]\,,\lambda([a',b']\ominus[c',d'])\rangle &=\langle [a,b]\ominus[c,d]\,,[\lambda a',\lambda b']\ominus[\lambda c',\lambda d']\rangle\\
&=\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} \lambda a'-\lambda c'\\ \lambda b'-\lambda d'\end{pmatrix}\right\rangle\\
&=\lambda\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} a'-c'\\ b'-d'\end{pmatrix}\right\rangle\\
&=\lambda\,\langle [a,b]\ominus[c,d]\,,[a',b']\ominus[c',d']\rangle\,,
\end{aligned}$$

and when $\lambda < 0$, we have

$$\begin{aligned}
\langle [a,b]\ominus[c,d]\,,\lambda([a',b']\ominus[c',d'])\rangle &=\langle [a,b]\ominus[c,d]\,,[-\lambda c',-\lambda d']\ominus[-\lambda a',-\lambda b']\rangle\\
&=\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} -\lambda c'+\lambda a'\\ -\lambda d'+\lambda b'\end{pmatrix}\right\rangle\\
&=\lambda\left\langle \begin{pmatrix} a-c\\ b-d\end{pmatrix},\begin{pmatrix} a'-c'\\ b'-d'\end{pmatrix}\right\rangle\\
&=\lambda\,\langle [a,b]\ominus[c,d]\,,[a',b']\ominus[c',d']\rangle\,.
\end{aligned}$$

From (i) to (iii), the function is an inner product on $\mathcal{N}$. This completes the proof. ■

Remark that this new inner product has consistency with the standard inner product because $\langle [a,a]\ominus[0,0],[b,b]\ominus[0,0]\rangle_P = \langle a,b\rangle$ for all $a,b\in\mathbb{R}^n$, where $s_i = v_i = \frac{1}{\sqrt{2}}$ and $t_i = u_i = 0$.

3.4. The Dimension of $\mathcal{N}$

From the previous theorem, $\mathcal{N}$ is established as a pre-Hilbert space equipped with a defined sum, scalar product, and inner product. We now proceed to show that $\mathcal{N}$ is finite-dimensional, and therefore can be regarded as a Hilbert space.

Theorem 3.6. *The dimension of $\mathcal{N}$ is $2n$ and $\mathcal{N}$ is a Hilbert space for the defined sum, scalar product, and inner product.*

Proof. At first we define a map φ from $\mathcal{N}$ to $\mathbb{R}^{2n}$ as follows:

$$\varphi([a,b]\ominus[c,d]) := \begin{pmatrix} a-c\\ b-d\end{pmatrix}.$$

If $[a,b]\ominus[c,d] = [a',b']\ominus[c',d']$, then $a-c = a'-c'$ and $b-d = b'-d'$. This shows that the map φ is well-defined. This map is also linear. Indeed, the additivity is shown as follows: for any $[a, b] \ominus [c, d]$, $[a', b'] \ominus [c', d'] \in \mathcal{N}$,

$$\begin{aligned}\varphi([a,b]\ominus[c,d]+[a',b']\ominus[c',d']) &= \varphi([a+a',b+b']\ominus[c+c',d+d']) \\ &= \begin{pmatrix}(a+a')-(c+c')\\(b+b')-(d+d')\end{pmatrix} \\ &= \begin{pmatrix}a-c\\b-d\end{pmatrix}+\begin{pmatrix}a'-c'\\b'-d'\end{pmatrix} \\ &= \varphi([a,b]\ominus[c,d])+\varphi([a',b']\ominus[c',d']).\end{aligned}$$

The homogeneity is shown as follows: for any $[a, b] \ominus [c, d] \in \mathcal{N}$ and $\lambda \in \mathbb{R}$, if $\lambda \geqq 0$, then

$$\begin{aligned}\varphi(\lambda([a,b]\ominus[c,d])) &= \varphi([\lambda a,\lambda b]\ominus[\lambda c,\lambda d]) \\ &= \begin{pmatrix}\lambda a-\lambda c\\\lambda b-\lambda d\end{pmatrix} \\ &= \lambda\begin{pmatrix}a-c\\b-d\end{pmatrix} \\ &= \lambda\varphi([a,b]\ominus[c,d]),\end{aligned}$$

and if $\lambda < 0$, then

$$\begin{aligned}\varphi(\lambda([a,b]\ominus[c,d])) &= \varphi([-\lambda c,-\lambda d]\ominus[-\lambda a,-\lambda b]) \\ &= \begin{pmatrix}-\lambda c-(-\lambda a)\\-\lambda d-(-\lambda b)\end{pmatrix} \\ &= \lambda\begin{pmatrix}a-c\\b-d\end{pmatrix} \\ &= \lambda\varphi([a,b]\ominus[c,d]).\end{aligned}$$

The injectivity is shown as follows: for any $[a, b] \ominus [c, d]$, $[a', b'] \ominus [c', d'] \in \mathcal{N}$, if $\varphi([a, b] \ominus [c, d]) = \varphi([a', b'] \ominus [c', d'])$, then

$$\begin{pmatrix}a-c\\b-d\end{pmatrix}=\begin{pmatrix}a'-c'\\b'-d'\end{pmatrix},$$

that is, $a-c = a'-c'$ and $b-d = b'-d'$, and this shows that $[a, b]\ominus[c, d] = [a', b']\ominus[c', d']$. Finally, the surjectivity is shown as follows: for any $y, z \in \mathbb{R}^n$, put

$$a_i = y_i, \quad b_i = \max\{y_i, z_i\}, \quad c_i = 0, \quad \text{and} \quad d_i = \max\{y_i - z_i, 0\},$$

then

$$a_i \leqq b_i, \quad c_i \leqq d_i, \quad a_i - c_i = y_i, \quad \text{and} \quad b_i - d_i = z_i,$$

for each $i \in \{1, 2, \ldots, n\}$. The last equality is shown since $\max\{y_i, z_i\} = z_i + \max\{y_i - z_i, 0\}$. Therefore, $[a, b] \ominus [c, d] \in \mathcal{N}$ and

$$\varphi([a,b]\ominus[c,d])=\begin{pmatrix}a-c\\b-d\end{pmatrix}=\begin{pmatrix}y\\z\end{pmatrix}.$$

Consequently, we have that the dimension of $\mathcal{N}$ is $2n$. Therefore $\mathcal{N}$ is also a Hilbert space because every pre-Hilbert spaces have finite dimension are known to be Hilbert space. ■

4. Applications

4.1. Application to Interval Linear Programming Problems

Consider the following interval programming problem:

$$(\mathrm{P}_0)\begin{cases} \text{Maximize} & \langle c, x\rangle \\ \text{subject to} & \langle a_i, x\rangle \leq b_i, \quad i=1,2,\ldots,m \\ & x=[\underline{x},\overline{x}]\in\mathcal{M}, \end{cases}$$

where $c=[\underline{c},\overline{c}]\in\mathcal{M}$, $a_i=[\underline{a}_i,\overline{a}_i]\in\mathcal{M}$, $i=1,2,\ldots,m$, $b\in\mathbb{R}^m$. It is natural that the coefficients of mathematical programming problem may contain some errors, and intervals can represent the range in which these errors may move. Here, the above function $\langle\cdot,\cdot\rangle:\mathcal{M}^2\to\mathbb{R}$ is given by an arbitrary inner product $\left\langle \begin{pmatrix}\cdot\\\cdot\end{pmatrix}, \begin{pmatrix}\cdot\\\cdot\end{pmatrix}\right\rangle$ on $\mathbb{R}^{2n}$ as follows:

$$\langle c, x\rangle = \left\langle \begin{pmatrix}\underline{c}\\\overline{c}\end{pmatrix}, \begin{pmatrix}\underline{x}\\\overline{x}\end{pmatrix}\right\rangle.$$

Instead of the problem (P_0), we solve the next problem (P) on Hilbert space $\mathcal{N}$:

$$(\mathrm{P})\begin{cases} \text{Maximize} & \langle c\ominus 0, x\ominus x'\rangle \\ \text{subject to} & \langle a_i\ominus 0, x\ominus x'\rangle \leq b_i, \quad i=1,2,\ldots,m, \\ & x\ominus x'\in\mathcal{N}, \end{cases}$$

which is a kind of relaxation of (P_0), because $\langle c\ominus 0, x\ominus x'\rangle = \left\langle \begin{pmatrix}\underline{c}\\\overline{c}\end{pmatrix}, \begin{pmatrix}\underline{x}-\underline{x}'\\\overline{x}-\overline{x}'\end{pmatrix}\right\rangle$ and it equals $\langle c, x\rangle$ when $x'=[0,0]$. Therefore, the optimal value of (P_0), $\mathrm{val}(\mathrm{P}_0)$, is less than or equal to the optimal value of (P), val(P). Also, if $x\ominus x'$ is a solution of (P) and $\underline{x}-\underline{x}'\leq\overline{x}-\overline{x}'$, then $[\underline{x}-\underline{x}', \overline{x}-\overline{x}']$ is a solution of (P_0). Next, problem (P) can be characterized by duality. Since (P) is a linear programming problem with finite affine constraints on $\mathcal{N}$, then FM, the Farkas Minkowski property, holds and Theorem 4.1 in [3] can be applied to this problem. Therefore, the objective value of (P) is given by its dual form as follows:

$$\begin{aligned}
\mathrm{val}(\mathrm{P}) &= \max\{\langle c\ominus 0, x\ominus x'\rangle \mid \langle a_i\ominus 0, x\ominus x'\rangle\leq b_i,\ x\ominus x'\in\mathcal{N}\} \\
&= -\inf\{\langle -c\ominus 0, x\ominus x'\rangle \mid \langle a_i\ominus 0, x\ominus x'\rangle\leq b_i,\ x\ominus x'\in\mathcal{N}\} \\
&= -\max_{y_i\geq 0}\inf_{x\ominus x'\in\mathcal{N}}\left\{\langle -c\ominus 0, x\ominus x'\rangle + \sum_{i=1}^m y_i(\langle a_i\ominus 0, x\ominus x'\rangle - b_i)\right\} \\
&= -\max_{y_i\geq 0}\inf_{x\ominus x'\in\mathcal{N}}\left\{\left\langle \left(-c+\sum_{i=1}^m y_i a_i\right)\ominus 0, x\ominus x'\right\rangle - \sum_{i=1}^m y_i b_i\right\}.
\end{aligned}$$

We can see that $-c+\sum_{i=1}^m y_i a_i\neq 0$ for any $y_i\geq 0$ if and only if $\mathrm{val}(P)=+\infty$. Otherwise

$$\mathrm{val}(P) = -\max_{\substack{y_i\geq 0\\ \sum_{i=1}^m y_i a_i = c}} -\sum_{i=1}^m y_i b_i = \min_{\substack{y_i\geq 0\\ \sum_{i=1}^m y_i a_i = c}} \sum_{i=1}^m y_i b_i = \mathrm{val}(D),$$

where val(D) is the optimal value of the following programming problem:

$$(\mathrm{D})\begin{cases}\text{Minimize} & \langle b, y\rangle \\ \text{subject to} & \sum_{i=1}^{m} y_i a_i = c \\ & y_i \geq 0, \quad i = 1, 2, \ldots, m,\end{cases}$$

or equivalently

$$(\mathrm{D})\begin{cases}\text{Minimize} & \langle b, y\rangle \\ \text{subject to} & \sum_{i=1}^{m} y_i \underline{a_i} = \underline{c} \\ & \sum_{i=1}^{m} y_i \overline{a_i} = \overline{c} \\ & y_i \geq 0, \quad i = 1, 2, \ldots, m.\end{cases}$$

The latter form of problem (D) becomes a typical linear optimization problem.

4.2. Application to Interval Support Vector Machines

Consider the following interval data points for classification:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_l, y_l),$$

where $x_i \in \mathcal{M}$ and $y_i \in \{1, -1\}$, $i = 1, 2, \ldots, l$. It is a natural situation that data contains some errors. Assume that the lower limit of the i-th data is $\underline{x}_i$ and the upper limit is $\overline{x}_i$. To consider the classification of the data, we solve the one of the following data sets on $\mathcal{N}$:

$$(x_1 \ominus 0, y_1), (x_2 \ominus 0, y_2), \ldots, (x_l \ominus 0, y_l).$$

Since $\mathcal{N}$ is a Hilbert space, the soft-margin problem for this classification is easily given as follows:

$$\begin{aligned}\text{Minimize} \quad & \|w \ominus w'\|^2 + C\sum_{i=1}^{l} \xi_i \\ \text{subject to} \quad & y_i(\langle w \ominus w', x_i \ominus 0\rangle - b) \geq 1 - \xi_i, \quad \xi_i \geq 0, \quad \forall i = 1, 2, \ldots, l, \\ & w \ominus w' \in \mathcal{N},\ b \in \mathbb{R},\ \xi \in \mathbb{R}^l,\end{aligned}$$

where $\langle w \ominus w', x \ominus x'\rangle := \left\langle \begin{pmatrix} \underline{w} - \underline{w'} \\ \overline{w} - \overline{w'} \end{pmatrix}, \begin{pmatrix} \underline{x} - \underline{x'} \\ \overline{x} - \overline{x'} \end{pmatrix} \right\rangle$, and $\left\langle \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\rangle$ is an arbitrary inner product on $\mathbb{R}^{2n}$. Define $L : \mathcal{N} \times \mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}$ by

$$\begin{aligned}L(w \ominus w', b, \xi, u, v) &= \|w \ominus w'\|^2 + C\sum_{i=1}^{l} \xi_i \\ &\quad + \sum_{i=1}^{l} u_i(1 - \xi_i - y_i(\langle w \ominus w', x_i \ominus 0\rangle - b)) - \sum_{i=1}^{l} v_i \xi_i.\end{aligned}$$

If $(w \ominus w', b, \xi) \in \mathcal{N} \times \mathbb{R} \times \mathbb{R}^l$ is a solution of the soft-margin problem, by using the Lagrange duality theorem, Theorem 5.1 in [3], there exist $u, v \in \mathbb{R}^l$ such that $u_i \geq 0$, $u_i(1 - \xi_i - y_i(\langle w \ominus w', x_i \ominus 0\rangle - b)) = 0$, $v_i \geq 0$, $v_i \xi_i = 0$, $i = 1, 2, \ldots, l$, and

$$(0 \ominus 0, 0, 0) \in \partial L(\cdot, \cdot, \cdot, u, v)(w \ominus w', b, \xi).$$

Then we have

$$0 \ominus 0 = 2w \ominus w' - \sum_{i=1}^{l} u_i y_i x_i \ominus 0, \quad 0 = \sum_{i=1}^{l} u_i y_i, \quad \text{and} \quad 0 = (C, \ldots, C)^T - u - v,$$

that is,

$$w \ominus w' = \frac{1}{2} \sum_{i=1}^{l} u_i y_i x_i \ominus 0, \quad w' = 0, \quad 0 = \sum_{i=1}^{l} u_i y_i, \quad \text{and} \quad 0 = (C, \ldots, C)^T - u - v.$$

In this case,

$$\begin{aligned} L(w \ominus w', b, \xi, u, v) &= -\frac{1}{4} \| \sum_{i=1}^{l} u_i y_i x_i \ominus 0 \|^2 + \sum_{i=1}^{l} u_i \\ &= -\frac{1}{4} \sum_{i=1}^{l} \sum_{j=1}^{l} u_i u_j y_i y_j \left\langle \begin{pmatrix} \underline{x}_i \\ \overline{x}_i \end{pmatrix}, \begin{pmatrix} \underline{x}_j \\ \overline{x}_j \end{pmatrix} \right\rangle + \sum_{i=1}^{l} u_i, \end{aligned}$$

and the dual problem is the following quadratic maximization problem:

$$\begin{aligned} &\text{Maximize} \quad -\frac{1}{4} \sum_{i=1}^{l} \sum_{j=1}^{l} u_i u_j y_i y_j \left\langle \begin{pmatrix} \underline{x}_i \\ \overline{x}_i \end{pmatrix}, \begin{pmatrix} \underline{x}_j \\ \overline{x}_j \end{pmatrix} \right\rangle + \sum_{i=1}^{l} u_i \\ &\text{subject to} \quad \sum_{i=1}^{l} u_i y_i = 0, \text{ and } 0 \leq u_i \leq C, \quad \forall i = 1, 2, \ldots, l. \end{aligned}$$

This problem is the typical dual form of the soft-margin problem for SVM.

5. Conclusions

In this study, we introduced a novel inner product structure for intervals within a vector space setting. We showed that this inner product preserves the key properties of standard inner products, which allowed us to establish that the set of intervals forms a Hilbert space. This new framework not only enriches the theoretical understanding of intervals but also provides a solid foundation for practical applications. We applied this structure to interval linear programming and interval support vector machines, illustrating that our interval Hilbert space approach makes these models easier to analyze and interpret, especially in situations with uncertainties represented as intervals. This approach enables inner product calculations directly within the space of intervals, offering new insights for problems where uncertainties play a significant role.

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