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Convergence of Iterative Process for Generalized I-Asymptotically Quasi-Nonexpansive Mappings

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Abstract : In this paper, we establish iterative process for convergence to common fixed point of generalized I-asymptotically quasi-nonexpansive mappings in Banach spaces. The results obtained in this paper improve and extend the corresponding results in the existing literature.

Keywords : I-asymptotically quasi-nonexpansive mapping, generalized I-asymptotically quasi-nonexpansive mapping, Ishikawa iterative schemes, convergence theorems.
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1 Introduction

Let K be a nonempty subset of uniformly convex Banach space X. Let T be a self-mapping of K. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

A mapping $T : K \longrightarrow K$ is called nonexpansive provided

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$ and $n \geq 1$. T is called asymptotically nonexpansive mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda_n)\|x - y\|$$

for all $x, y \in K$ and $n \geq 1$.

T is called quasi-nonexpansive mapping provided

$$\|T^n x - p\| \leq \|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

T is called asymptotically quasi-nonexpansive mapping if there exist a sequence $\{\lambda_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda_n)\|x - p\|$$

for all $x \in K$ and $p \in F(T)$ and $n \geq 1$.

Remark 1.1. *From above definitions, it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive. But the converse does not hold.*

Let $T, I : K \longrightarrow K$. Then T is called I-nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\|$$

for all $x, y \in K$.

T is called I- asymptotically nonexpansive on K if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda'_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \lambda'_n)\|I^n x - I^n y\|$$

for all $x, y \in K$ and $n = 1, 2, ...$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$ the following inequality holds:

$$\|T^n x - T^n y\| \leq L\|I^n x - I^n y\|$$

and I is uniformly Γ-Lipschitzian if there exists a constant $\Gamma > 0$ such that for all $x, y \in K$ the following inequality holds:

$$\|I^n x - I^n y\| \leq \Gamma\|x - y\|$$

T is called I- asymptotically quasi-nonexpansive on K if there exists a sequence $\{\lambda'_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} \lambda'_n = 0$ such that

$$\|T^n x - p\| \leq (1 + \lambda'_n)\|I^n x - p\|$$

for all $x \in K$ and $p \in F(T) \cap F(I)$ and $n = 1, 2, ...$

Remark 1.2. *From the above definitions it follows that if $F(T) \cap F(I)$ is nonempty, a I-nonexpansive mapping must be I-quasi-nonexpansive, and linear I-quasi-nonexpansive mappings are I-nonexpansive mappings. But it is easily seen that there exist nonlinear continuous I-quasi-nonexpansive mappings which are not I-nonexpansive.*

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied Ghosh and Debnath [1], Goebel and Kirk [2], Liu [6, 7], Petryshyn and Williamson[9] in the settings of Hilbert spaces and uniformly convex Banach spaces. The class of asymptotically nonexpansive maps which an important generalization of the class nonexpansive maps was introduced by Goebel and Kirk [2]. They proved that every asymptotically nonexpansive self-mapping of a nonempty closed convex bounded subset of a uniformly convex Banach space has a fixed point. Also in [2], they extended this result to broader class of uniformly Lipschitzian mappings.

Since 1972, iterative techniques for convergence to fixed points of nonexpansive mappings and their generalizations in setting of Hilbert spaces or Banach spaces have been studied by many authors. For example, in 1973, Petryshyn and Williamson [9] proved a necessary and sufficient condition for a Mann iterative sequence to convergence to fixed points for quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [1] extended Petryshyn and Williamson's results and gave some necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings. Subsequently, in 2001 and 2002, Liu Qihou [6, 7] extended the results of Ghosh and Debnath to the more general asymptotically quasi-nonexpansive mapping and gave some necessary and sufficient conditions for Ishikawa iterative sequence and Ishikawa iterative sequence with errors to converge to fixed points for asymptotically quasi-nonexpansive mappings in Banach spaces and uniformly Banach spaces. Recently, in 2006, Lan [5] introduced a new class of iterative processes with errors for approximating the common fixed point of two generalized asymptotically quasi-nonexpansive mappings and gave some strong convergence results for Ishikawa iterative sequence to fixed point for this class of mappings.

More recently, Rhoades and Temir [10] and Yao and Wang [15] introduced a class of I-nonexpansive mapping. Rhoades and Temir [10] proved weak convergence of iterative sequence for I-nonexpansive mapping to common fixed point. Yao and Wang [15] proved strong convergence of iterative sequence for I-quasi-nonexpansive mapping to common fixed point. In [14], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved.

Recall some definitions and notations.

2 Preliminaries and Notations

Let X be a normed linear space, T be self-mapping on X. Let $\{x_n\}$ be sequence of the Ishikawa iterative scheme [4] associated with T, $x_0 \in X$,

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases} \tag{2.1}$$

for every $n \in \mathbb{N}$, where $0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1$.

Let $S, T : K \longrightarrow K$ be two mappings. In 2006, Lan [5] introduced the following iterative scheme with errors. The sequence x_n in K defined by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n + \psi_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n + \varphi_n \end{cases} \tag{2.2}$$

for every $n \in \mathbb{N}$, where $0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1$ and $\{\varphi_n\}$, $\{\psi_n\}$ are two sequences in K.

Define the Ishikawa iterative process of the generalized I-asymptotically quasi-nonexpansive mappings in uniformly convex Banach space X as follows

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T^n x_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n I^n y_n \end{cases} \tag{2.3}$$

for every $n \in \mathbb{N}$, where $0 \leq \{\alpha_n\}, \{\beta_n\} \leq 1$.

Definition 2.1. *[12] Let X be a real normed linear space and K a nonempty subset of X. A mapping $T : K \longrightarrow K$ is called generalized asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exist sequences of real numbers $\{u_n\}$, $\{\varphi_n\}$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} \varphi_n$ such that*

$$\|T^n x - p\| \leq \|x - p\| + u_n\|x - p\| + \varphi_n$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

If, in Definition 2.1, $\varphi_n = 0$ for all $n \geq 1$ then T becomes asymptotically quasi-nonexpansive mapping and hence the class of generalized asymptotically quasi-nonexpansive mappings includes the class of asymptotically quasi-nonexpansive mappings.

Recall that a Banach space X is said to satisfy Opial's condition [8] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known from [8] that all l_r spaces for $1 < r < \infty$ have this property. However, the L_r space do not have unless $r = 2$.

Lemma 2.2. *[13] Let $\{a_n\}$, $\{b_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: $\forall n \geq 1$, $a_{n+1} \leq (1+\sigma_n)a_n + b_n$, where $\sum_{n=0}^{\infty} \sigma_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.3. *[11] Let K be a nonempty closed bounded convex subset of a Banach space X and $\{\alpha_n\}$ a sequence $[\epsilon, 1-\epsilon]$, for some $\epsilon \in (0,1)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in K such that*

$$\limsup_{n\to\infty} \|x_n\| \leq c,$$

$$\limsup_{n\to\infty} \|y_n\| \le c$$

and

$$\limsup_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = c$$

holds for some $c \ge 0$. *Then*

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Definition 2.4. *The mappings* $T, I : K \to K$ *are said to satisfying condition (A) if there is a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0, f(r) > 0$, *for all* $r \in [0,\infty)$ *such that* $\frac{1}{2}(\|x - Tx\| + \|x - Ix\|) \ge f(d(x,F))$ *for all* $x \in K$, *where* $d(x,F) = inf\{\|x - p\| : p \in F = F(T) \cap F(I)\}$.

In this paper, we consider T and I self-mappings of K, where T is a generalized I-asymptotically quasi-nonexpansive mapping and $I : K \to K$ be an asymptotically quasi-nonexpansive mapping.

We establish the weak and strong convergence of the sequence of Ishikawa iterates to a common fixed point of T and I. The purpose of this paper is to study iterative process for convergence to common fixed point of generalized I-asymptotically quasi-nonexpansive mappings and prove some sufficient and necessary conditions for Ishikawa iterative sequences of generalized I-asymptotically quasi-nonexpansive mappings to converge to common fixed point.

3 Weak and strong convergence generalized I-asymptotically quasi-nonexpansive mappings

Definition 3.1. *Let* X *be a Banach space and* K *a nonempty subset of* X. T *is called generalized* I*-asymptotically quasi-nonexpansive mapping if* $F = F(T) \cap F(I) \ne \emptyset$ *and there exist sequences of real numbers* $\{u_n\}$, $\{\varphi_n\}$ *with* $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} \varphi_n$ *such that*

$$\|T^n x - p\| \le \|I^n x - p\| + u_n\|I^n x - p\| + \varphi_n$$

for all $x \in K$, $p \in F$ *and* $n \ge 1$.

If, in Definition 3.1, $\varphi_n = 0$ for all $n \ge 1$ then T becomes I-asymptotically quasi-nonexpansive mapping.

Lemma 3.2. *Let* X *be an uniformly convex Banach space,* K *be a nonempty closed convex subset of* X. T *is generalized asymptotically I-quasi-nonexpansive mappings on* K *with* $\{u_n\}$, $\{\varphi_n\} \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} u_n < \infty$ *and* $\sum_{n=1}^{\infty} \varphi_n < \infty$, I *is asymptotically quasi-nonexpansive mappings on* K *with* $\{v_n\} \subset [0,\infty)$ *such that*

$\sum_{n=1}^{\infty} v_n < \infty$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in [0,1]. Let* $\{x_n\}$ *be the sequence defined in* (2.3) *with* $F = F(T) \cap F(I) \neq \emptyset$. *Then* $\lim_{n\to\infty} \|x_n - p\|$ *exists for common fixed point* p *of* T *and* I.

Proof. For any $p \in F(T) \cap F(I)$

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n I^n y_n + (1-\alpha_n)x_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|I^n y_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n(1+v_n)\|y_n - p\|\end{aligned}$$

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n \|T^n x_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1+u_n)\|I^n x_n - p\| + \varphi_n] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1+u_n)(1+v_n)\|x_n - p\| + \varphi_n] \\ &= \|x_n - p\|((1-\beta_n) + \beta_n(1+u_n)(1+v_n)) + \beta_n\varphi_n \\ &= \|x_n - p\|(1 + \beta_n u_n + \beta_n v_n + \beta_n u_n v_n) + \beta_n \varphi_n\end{aligned}$$

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n(1+v_n)\|y_n - p\|. \\ &\leq (1-\alpha_n)\|x_n - p\| \\ &\quad +\alpha_n(1+v_n)[(1 + \beta_n u_n + \beta_n v_n + \beta_n u_n v_n)\|x_n - p\| + \beta_n\varphi_n] \\ &= \|x_n - p\|[(1-\alpha_n) \\ &\quad +(\alpha_n + \alpha_n v_n)(1 + \beta_n u_n + \beta_n v_n + \beta_n u_n v_n)] + \beta_n\alpha_n(1+v_n)\varphi_n \\ &= \|x_n - p\|[1 - \alpha_n + \alpha_n + \alpha_n\beta_n u_n + \alpha_n\beta_n v_n + \alpha_n\beta_n u_n v_n \\ &\quad +\alpha_n v_n + \alpha_n\beta_n u_n v_n + \alpha_n\beta_n v_n^2 + \alpha_n\beta_n u_n v_n^2] + \beta_n\alpha_n(1+v_n)\varphi_n \\ &\leq \|x_n - p\|[1 + \alpha_n\beta_n(u_n + v_n) + 2\alpha_n\beta_n u_n v_n + \alpha_n v_n + \alpha_n\beta_n v_n^2 \\ &\quad +\alpha_n\beta_n u_n v_n^2] + \beta_n\alpha_n(1+v_n)\varphi_n.\end{aligned}$$

Thus we obtain

$$\|x_{n+1} - p\| \leq (1+\gamma_n)\|x_n - p\| + \Psi_n$$

where

$$\gamma_n = \alpha_n\beta_n(u_n + v_n) + 2\alpha_n\beta_n u_n v_n + \alpha_n v_n + \alpha_n\beta_n v_n^2 + \alpha_n\beta_n u_n v_n^2$$

with $\sum_{n=1}^{\infty} \gamma_n < \infty$.

$\Psi_n = \beta_n\alpha_n(1+v_n)\varphi_n$ with $\sum_{n=1}^{\infty} \Psi_n < \infty$.

By Lemma 2.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T) \cap F(I)$.

□

Lemma 3.3. *Let X be an uniformly convex Banach space, K be a nonempty closed convex subset of X. Let T be uniformly L-Lipschitzian, generalized asymptotically I-quasi-nonexpansive mappings on K with respect to $\{\varphi_n\}$ and I be uniformly Γ-Lipschitzian, asymptotically quasi-nonexpansive mappings on K such that $F(T) \cap F(I) \neq \emptyset$ in K. Suppose that for any given $x \in K$, the sequence $\{x_n\}$ is generated by (2.3). If $F = F(T) \cap F(I) \neq \emptyset$, then*

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \lim_{n\to\infty} \|Ix_n - x_n\| = 0.$$

Proof. By Lemma 3.2 for any $p \in F(T) \cap F(I)$, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = k$. If $k = 0$ by continuity of T and I, then the proof is completed.

Now suppose $k > 0$.

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n T^n x_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n \|T^n x_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n(1+u_n)\|I^n x_n - p\| + \beta_n \varphi_n \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n(1+u_n)(1+v_n)\|x_n - p\| + \beta_n \varphi_n \\
&\leq \|x_n - p\|(1 + \beta_n u_n + \beta_n v_n + \beta_n u_n v_n) + \beta_n \varphi_n.
\end{aligned}$$

Taking lim sup on both sides in the above inequality,

$$\limsup_{n\to\infty} \|y_n - p\| \leq k. \tag{3.1}$$

Since I is asymptotically quasi-nonexpansive mappings on K, we can get that, $\|I^n y_n - p\| \leq (1+v_n)\|y_n - p\|$, which on taking $\limsup_{n\to\infty}$ and using (3.1), gives

$$\limsup_{n\to\infty} \|I^n y_n - p\| \leq k.$$

Further,

$$\lim_{n\to\infty} \|x_{n+1} - p\| = k$$

means that

$$\lim_{n\to\infty} \|\alpha_n I^n y_n + (1-\alpha_n)x_n - p\| = k$$

$$\lim_{n\to\infty} (1-\alpha_n)\|x_n - p\| + \alpha_n \|I^n y_n - p\| = k.$$

It follows from Lemma 2.3

$$\lim_{n\to\infty} \|I^n y_n - x_n\| = 0. \tag{3.2}$$

Now,

$$\begin{aligned}
\|x_n - p\| &\leq \|x_n - I^n y_n\| + \|I^n y_n - p\| \\
&\leq \|x_n - I^n y_n\| + (1+v_n)\|y_n - p\|
\end{aligned}$$

which on taking $\lim_{n\to\infty}$ implies

$$\begin{aligned} k &= \lim_{n\to\infty} \|x_n - p\| \\ &\le \limsup_{n\to\infty}(\|x_n - I^n y_n\| + (1+v_n)\|y_n - p\|) \\ &= \limsup_{n\to\infty} \|y_n - p\| \le k. \end{aligned}$$

Then we obtain,

$$\limsup_{n\to\infty} \|y_n - p\| = k.$$

Next,

$$\begin{aligned} \|T^n x_n - p\| &\le (1+u_n)\|I^n x_n - p\| + \varphi_n \\ &\le (1+u_n)(1+v_n)\|x_n - p\| + \varphi_n. \end{aligned}$$

Taking $\lim_{n\to\infty}$ on both sides in the above inequality,

$$\begin{aligned} \lim_{n\to\infty} \|T^n x_n - p\| &\le \lim_{n\to\infty}[(1+u_n)(1+v_n))\|x_n - p\| + \varphi_n] \\ &\le \lim_{n\to\infty} \|x_n - p\| \le k. \end{aligned}$$

Further,

$$\lim_{n\to\infty} \|\alpha_n(T^n x_n - p) + (1-\alpha_n)(x_n - p)\| = \lim_{n\to\infty} \|y_n - p\| = k.$$

By Lemma 2.3, we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.3}$$

We have also,

$$\begin{aligned} \|I^n x_n - x_n\| &\le \|I^n x_n - I^n y_n\| + \|I^n y_n - x_n\| \\ &\le \Gamma\|x_n - y_n\| + \|I^n y_n - x_n\| \\ &\le \Gamma\|x_n - [(1-\beta_n)x_n + \beta_n T^n x_n\| + \|I^n y_n - x_n\| \\ &\le \Gamma\|\beta_n(T^n x_n - x_n)\| + \|I^n y_n - x_n\| \\ &\le \Gamma\beta_n\|(T^n x_n - x_n)\| + \|I^n y_n - x_n\|. \end{aligned}$$

Thus from (3.2) and (3.3), we obtain

$$\lim_{n\to\infty} \|I^n x_n - x_n\| = 0. \tag{3.4}$$

$$\begin{aligned} \|x_{n+1} - I x_{n+1}\| &\le \|x_{n+1} - I^{n+1} x_{n+1}\| + \|I^{n+1} x_{n+1} - I x_{n+1}\| \\ &\le \|x_{n+1} - I^{n+1} x_{n+1}\| + \Gamma\|I^n x_{n+1} - x_{n+1}\|. \end{aligned}$$

Taking lim sup on both sides in the above inequality and from (3.4), we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - Ix_{n+1}\| \leq 0.$$

That is,

$$\lim_{n\to\infty} \|x_n - Ix_n\| = 0. \tag{3.5}$$

Next,

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|I^n x_{n+1} - x_{n+1}\|.\end{aligned}$$

Also, taking lim sup on both sides in the above inequality and from (3.3), (3.4), we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - Tx_{n+1}\| \leq 0.$$

That is,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.6}$$

Then the proof is completed.

□

Theorem 3.4. *Let X be uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be the same as Lemma 3.2. If $F = F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T and I.*

Proof. Let $p \in F = F(T) \cap F(I)$. Then, as in Lemma 3.2, it follows $\lim_{n\to\infty} \|x_n - p\|$ exists and so for $n \geq 1$, $\{x_n\}$ is bounded on K. Then by the reflexivity of X and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ weakly. If $F(T) \cap F(I)$ is a singleton, then the proof is complete. For $p \in F(T) \cap F(I)$, T is generalized I-asymptotically quasi-nonexpansive on K and I is asymptotically quasi-nonexpansive on K. The proof is completed if $\{x_n\}$ converges weakly to a common fixed point of T and I, i.e., it suffices to show that the weak limit set of the sequence $\{x_n\}$ consists of exactly one point. We assume that $F(T) \cap F(I)$ is not singleton. Suppose $p, q \in w(\{x_n\})$, where $w(\{x_n\})$ denotes the weak limit set of $\{x_n\}$. Let $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ which converge weakly to p and q, respectively. By Lemma 3.3 and Lemma 2.2 guarantees that $Ip = p$ and $Tp = p$. In the same way $Iq = q$ and $Tq = q$.

Next we prove the uniqeness.Assume that $p \neq q$ and $\{x_{n_k}\} \rightharpoonup p$, $\{x_{n_j}\} \rightharpoonup q$. By Opial's condition, we conclude that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \lim_{k\to\infty} \|x_{n_k} - p\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\
&= \lim_{n\to\infty} \|x_n - q\| = \lim_{j\to\infty} \|x_{n_j} - q\| \\
&< \lim_{j\to\infty} \|x_{n_j} - p\| = \lim_{n\to\infty} \|x_n - p\|.
\end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T) \cap F(I)$.

□

Theorem 3.5. *Let X be Banach space, K be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be the same as Lemma 3.2. If $T, I : K \longrightarrow K$ satisfy condition (A) and T and I are continuous mapping and $F = F(T) \cap F(I) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T and I.*

Proof. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F = F(T) \cap F(I)$. Furthermore, in the proof of Lemma 3.2, we obtain

$$\|x_{n+1} - p\| \leq (1 + \gamma_n)\|x_n - p\| + \Psi_n \tag{3.7}$$

where

$$\gamma_n = \alpha_n\beta_n(u_n + v_n) + 2\alpha_n\beta_n u_n v_n + \alpha_n v_n + \alpha_n\beta_n v_n^2 + \alpha_n\beta_n u_n v_n^2$$

with $\sum_{n=1}^{\infty} \gamma_n < \infty$.

$\Psi_n = \beta_n\alpha_n(1 + v_n)\varphi_n$ with $\sum_{n=1}^{\infty} \Psi_n < \infty$. By Lemma 2.2, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T) \cap F(I)$.
By (3.7), we get

$$d(x_{n+1}, F) \leq (1 + \gamma_n)d(x_n, F) + \Psi_n.$$

Then by Lemma 2.2, $\lim_{n\to\infty} d(x_n, F)$ exists and the condition (A) guarantees that

$$\lim_{n\to\infty} f(d(x_n, F)) = 0 \tag{3.8}$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. In fact, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $1+x \leq e^x$

for all $x > 0$, from (3.7) and (3.8) for any $p \in F = F(T) \cap F(I)$, we obtain

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + \gamma_{n+m-1})\|x_{n+m-1} - p\| + \Psi_{n+m-1} \\
&\leq \exp(\gamma_{n+m-1} + \gamma_{n+m-2})\|x_{n+m-2} - p\| \\
&\quad + \exp(\gamma_{n+m-1})(\Psi_{n+m-1} + \Psi_{n+m-2}) \\
&\vdots \\
&\leq \exp(\sum_{\imath=n}^{n+m-1} \gamma_i)\|x_n - p\| + \exp(\sum_{\imath=n}^{n+m-1} \gamma_i) \sum_{\imath=n}^{n+m-1} \Psi_n \\
&\leq M\|x_n - p\| + M \sum_{\imath=n}^{n+m-1} \Psi_n
\end{aligned}$$

for all natural numbers m,n where $M = \exp(\sum_{\imath=1}^{\infty} \gamma_i) < \infty$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} \Psi_n < \infty$, for any given $\epsilon > 0$, there exists a positive integer N_0 such that for all $n \geq N_0$, $d(x_n, F) < \frac{\epsilon}{6M}$ and $\sum_{\imath=n}^{\infty} \Psi_i < \frac{\epsilon}{3M}$. There exists $p_0 \in F = F(T) \cap F(I)$ such that $\|x_{n_0} - p_0\| < \frac{\epsilon}{6M}$.
Hence, for all $n \geq N_0$ and $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_0\| + \|x_n - p_0\| \\
&\leq M\|x_{n_0} - p_0\| + M(\sum_{\imath=n_0}^{n+m-1} \Psi_i) + M\|x_{n_0} - p_0\| + M(\sum_{\imath=n_0}^{n-1} \Psi_i) \\
&\leq 2M\|x_{n_0} - p_0\| + M(\sum_{\imath=n_0}^{n+m-1} \Psi_i) + M(\sum_{\imath=n_0}^{n-1} \Psi_i) \\
&\leq 2M\frac{\epsilon}{6M} + M\frac{\epsilon}{3M} + M\frac{\epsilon}{3M} = \epsilon
\end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence in X.
Thus, the completeness of X implies that $\{x_n\}$ is convergent. Assume that $\{x_n\}$ converges to a point p.

Then $p \in K$, because K is closed subset of X. Therefore the set $F = F(T) \cap F(I)$ is closed. $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(p, F) = 0$

Thus $p \in F$. This completes the proof. □

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