Thai Journal of Mathematics
Volume 7 (2009) Number 2 : 361–366

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Quasi Multiplication Modules

F. Farzalipour and P. Ghiasvand

Abstract : In this paper we study weakly prime submodules of a module over a commutative ring with identity. First, a number of results concerning weakly prime submodules are given. Second, for a commutative ring R, we define the notion quasi multiplication module over R. Also, we give a number of results concerning quasi multiplication modules.

Keywords : Multiplication; prime; secondary; weakly prime
2000 Mathematics Subject Classification : 47H09; 47H10 (2000 MSC)

1 Introduction

Throughout this work R will denote a commutative ring with non-zero identity and all modules are unitary. Several authors have extended the notion of prime ideal to modules, see, for example [3], [6]. A proper ideal P of R to be weakly prime ideal if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Weakly prime ideals in a commutative ring with non-zero identity have been introduced and studied by D. D. Anderson and E. Smith in [1]. A proper submodule N of a module M over a commutative ring R is said to be weakly prime submodule if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$ (see for example [4]). Here we study some properties of weakly prime submodules. For example, we show that weakly prime submodules of secondary modules are secondary.
Now we define the concepts that we will use. A commutative ring R is called a quasi local ring if it has a unique maximal ideal P, and denoted by (R, P). If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ be denoted by $(N :_R M)$. A proper submodule N of a module M over a commutative ring R is said to be prime submodule if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$ (see [3]). An R-module M is called a secondary module provided that for every element $r \in R$, the R-endomorphism of M produced by multiplication by r is either surjective or nilpotent. This implies

that $\sqrt{(0 :_R M)} = P$ is a prime ideal of R, and M is said to be P-secondary [7]. Recall that if R is an integral domain with the quotient field K, the rank of an R-module M ($rank\ M$) is defined to be the maximal number of elements of M linearly independent over R. We have $rank\ M$ = the dimention of the vector space KM over K, that is $rank\ M = rank_K\ KM$. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N :_R M)$. An R-module M is called weak multiplication if spec$M = \emptyset$ or for every prime submodule N of M, $N = IM$ where I is an ideal of R ([2]). If R is a ring and M an R-module, the subset $T(M)$ of M is defined by $T(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$. Obviously, if R is an integral domain, then $T(M)$ is a submodule of M.

2 Weakly prime submodules

It is clear that every prime submodule is a weakly prime submodule. However, since 0 is always weakly prime(by definition), a weakly prime submodule need not be prime, but we have the following results:

Proposition 2.1. *Let M be an R-module with $T(M) = 0$. Then every weakly prime submodule of M is prime.*

Proof. Let N be weakly prime submodule of M. Suppose that $rm \in N$ where $r \in R$, $m \in M$. If $0 \neq rm \in N$, N weakly prime gives $m \in N$ or $rM \subseteq N$. If $rm = 0$, then $r = 0$ or $m = 0$ since $T(M) = 0$. So N is prime. □

Proposition 2.2. *Let M be a module over a commutative ring R. Then*
(1) If every proper submodule of M is a weakly prime submodule and $T(M) \neq M$, then (R, P) is a quasi local ring with $P^2 = 0$ or $R = F_1 \times F_2$ where F_1 and F_2 are fields.
(2) If M is an R-module over a quasi local domain (R,P) with $P^2 = 0$, then every proper submodule of M is weakly prime.

Proof. (1) Let $a \in M \setminus T(M)$, $Ann(a) = 0$. It is easy to see that every proper submodule of $M^* = Ra$ is a weakly prime submodule of M^*, and $M^* \cong R/(0 : a) \cong R$ as R-modules. Therefore every proper ideal of R is a weakly prime ideal, hence by [1, Theorem 8], (R, P) is a quasi local ring with $P^2 = 0$ or $R = F_1 \times F_2$.
(2) Let N be a proper submodule of M. Assume that $0 \neq rm \in N$ where $r \in R$ and $m \in M$. If r is a unite, then $m \in N$. Let r is not a unite. Then $r \in P$, and $r^2 \in P^2 = 0$, hence $r = 0$ since R is a domain, which is a contradiction. Therefore N is a weakly prime submodule of M. □

Proposition 2.3. *Let M be a module over a quasi local ring (R,P) with $PM = 0$. Then every proper submodule of M is weakly prime.*

Proof. Let N be a proper submodule of M, and $0 \neq rm \in N$ where $r \in R$ and $m \in M$. If r is a unite, then $m \in N$. Let r is not a unite, so $rm \in PM = 0$, a contradiction. Hence N is weakly prime. □

Lemma 2.4. *Let M be an R-module. Assume that N and K are submodules of M such that $K \subseteq N$ with $N \neq M$. Then the following hold:*
(i) If N is a weakly prime submodule of M, then N/K is a weakly prime submodule of M/K.
(ii) If K and N/K are weakly prime submodules, then N is weakly prime.

Proof. (i) Let $0 \neq r(m+K) = rm + K \in N/K$ where $r \in R$ and $m \in M$. If $rm = 0$, then $r(m+K) = 0$, which is a contradiction. If $rm \neq 0$, N weakly prime gives either $m \in N$ or $r \in (N :_R M)$; hence either $m + K \in N/K$ or $r \in (N/K :_R M/K)$ (since we have $(N :_R M) = (N/K :_R M/K)$), as required.
(ii) Let $0 \neq rm \in N$ where $r \in R$ and $m \in M$, so $r(m+K) = rm + K \in N/K$. If $rm \in K$, then K weakly prime gives either $m \in K \subseteq N$ or $r \in (K :_R M) \subseteq (N :_R M)$. So we may assume that $rm \notin K$. Then $0 \neq r(m+K) \in N/K$. Since N/K is weakly prime, we get either $m \in N$ or $r \in (N/K :_R M/K) = (N : M)$, as required.

Theorem 2.5. *Let M be a secondary R-module and N a non-zero weakly prime R-submodule of M. Then N is secondary.*

Proof. Let $r \in R$. If $r^n M = 0$ for some $n \in N$, then $r^n N \subseteq r^n M = 0$, so r is nilpotent on N. Suppose that $rM = M$; we show that r divides N. Assume that $n \in N$. So $n = rm$ for some $m \in M$. We may assume that $0 \neq rm$. Hence $0 \neq rm \in N$, then N weakly prime gives $m \in N$. Thus $rN = N$, as needed. □

Corollary 2.6. *Let M be an R-module, N a secondary R-submodule of M and K a weakly prime submodule of M. Then $N \cap K$ is secondary.*

Proof. The proof is straightforward. □

Proposition 2.7. *Let M be a module over a commutative ring R, and S a multiplicatively closed subset of R. Let N be a weakly prime submodule of M such that $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a weakly prime submodule of $S^{-1}M$.*

Proof. Let $0/1 \neq r/s.m/t \in S^{-1}N$ where $r/s \in S^{-1}R$ and $m/t \in S^{-1}M$. So $0/1 \neq rm/st = n/t'$ for some $n \in N$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'rm = s'stn \in N$ (because if $s't'rm = 0$, $rm/st = s't'rm/s't'st = 0/1$, a contradiction) and $s't' \notin (N : M)$, so N weakly prime gives $0 \neq rm \in N$. Hence $m \in N$ or $r \in (N : M)$, thus $r/s \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $m/t \in S^{-1}N$, as needed. □

Lemma 2.8. *Let M be a module over a quasi local ring (R, P). Then $M_P = 0$ if and only if $M = 0$.*

Proof. Let $M_P = 0$. Assume that $M \neq 0$, and $0 \neq m \in M$. Hence $m/1 \neq 0/1$, because if $m/1 = 0/1$, then there exists $t \in S$ such that $tm = 0$. So $t \in (0 : m) \cap S \subseteq P \cap S = \emptyset$, a contradiction. Thus $M_P \neq 0$, a contradiction. So $M = 0$. The converse is clear. □

Lemma 2.9. *Let M be a module over a quasi local ring (R, P). Let N be a weakly prime submodule of R-module M, then $(N :_R M)_P = (N_P :_{R_P} M_P)$.*

Proof. Let $r/s \in (N_P :_{R_P} M_P)$ and $m \in M$. We show that $rm \in N$. We may assume that $rm \neq 0$. We have $r/s.m/1 \in S^{-1}N$; so $rm/s = n/t$ for some $t \in S$ and $n \in N$. There exists $t' \in S$ such that $t'trm = t'sn \in N$. If $t'trm = 0$, then $tt' \in (0 : rm) \cap P \subseteq P \cap S = \emptyset$, a contradiction. So $0 \neq tt'rm \in N$ and $tt' \notin (N : M)$; then $rm \in N$. Thus $(N_P :_{R_P} M_P) \subseteq (N :_R M)_P$. Clearly, $(N :_R M)_P \subseteq (N_P :_{R_P} M_P)$, so the proof is complete. □

Theorem 2.10. *Let M be a module over a quasi local ring (R, P). Then there exists a one to one correspondence between the weakly prime submodules of M and the weakly prime submodules of R_P-module M_P.*

Proof. Let K be a weakly prime submodule of M_P. So $K = N_P$ for some submodule N of M. We show that N is weakly prime submodule of M. Let $0 \neq rm \in N$, so $0/1 \neq rm/1 \in N_P$ (if $rm/1 = 0/1$, then $srm = 0$ for some $s \in S$, $s \in (0 : rm) \cap P \subseteq P \cap S = \emptyset$, a contradiction). Hence $r/1 \in (N_P :_{R_P} M_P) \subseteq (N :_P M)_P$ by Lemma 2.9 or $m/1 \in N_P$ since N_P is weakly prime. Thus $r \in (N : M)$ or $m \in N$, as required. Let N be a weakly prime submodule of M, then by Proposition 2.7, N_P is weakly prime submodule of M_P. □

3 Quasi multiplication modules

An R-module M is called quasi multiplication module if for every weakly prime submodule N of M,we have $N = IM$, where I is an ideal of R. One can easily show that if M is a quasi multiplication module, then $N = (N : M)M$ for every weakly prime submodule N of M.

Clearly, every multiplication module is quasi multiplication and every quasi multiplication is weak multiplication.

Lemma 3.1. *Let M be weak multiplication R-module with $T(M) = 0$. Then M is quasi multiplication R-module.*

Proof. It is clear by Proposition 2.1. □

As seen in [2], Q is a weak multiplication Z-module which is not multiplication. Since $T(Q) = 0$, by Lemma 3.1, Q is quasi multiplication Z-module, so quasi multiplication modules need not be multiplication module.

Proposition 3.2. *Let M be quasi multiplication R-module and K a weakly prime submodule of M, then M/K is quasi multiplication R-module.*

Proof. Let L be weakly prime submodule of M/K, so $L = N/K$ for some weakly prime submodule N of M by Lemma 2.4. Therefore, $N = IM$ for some ideal I of R since M is quasi multiplication. Thus $N/K = I(M/K)$, as required. □

Theorem 3.3. *Let M be a module over a quasi local ring (R, P). Then M is quasi multiplication R-module if and only if M_P is quasi multiplication R_P-module.*

Proof. Let M be a quasi multiplication and K be a weakly prime submodule of M_P. Hence $K = N_P$ for some weakly prime submodule of M by Theorem 2.10. So $N = IM$ for some ideal I of R since M is quasi multiplication. Therefore, $K = N_P = (IM)_P = I_P M_P$, as required. Conversely, let M_P is quasi multiplication module and N a weakly prime submodule of M. So N_P is weakly prime submodule of M_P by Theorem 2.10. Hence $N_P = JM_P$ for some ideal J of R_P. Thus $N_P = I_P M_P = (IM)_P$, then $(N/IM)_P = 0$, so $N/IM = 0$ by Lemma 2.8. Therefore, M is quasi multiplication. □

Proposition 3.4. *Let M be quasi multiplication module over a quasi local domain (R, P) with $P^2 = 0$. Then M is multiplication.*

Proof. This follows from Proposition 2.2. □

Corollary 3.5. *Every finitely generated quasi multiplication module is a multiplication module.*

Proof. The proof follows from [2, Theorem 2.7] and the fact that every quasi multiplication module is weak multiplication module. □

Corollary 3.6. *Let M be a quasi multiplication module over an integral domain. Then:*
(i) If M is a non-zero torsion-free, then $rankM = 1$.
(ii) If M is a torsion module, then $rankM = 0$.
(iii) M is either torsion or torsion free.

Proof. Since every quasi multiplication module is weak multiplication, so by [2, Proposition 2.4] the proof is hold. □

Acknowledgement(s) : I would like to thank the referee(s) for his comments and suggestions on the manuscript. This work was supported by the National Research Council of Thailand and Mathematical Association of Thailand.

References

[1] D. D. Anderson and E. Smith, Weakly prime ideals, Hoston J. of Math. 29(2003), 831-840.

[2] A. Azizi, Weak multiplication modules, Czechoslovak Mathematical Journal, 53(128)(2003), 520-534.

[3] J. Dauns, Prime modules, J. reine Angew. Math. 2(1978), 156-181.

[4] S. Ebrahimi Atani and F.Farzalipour, On weakly prime submodules, Tamkang J. of Math. 38(2007), 247-252.

[5] S. Ebrahimi Atani and F.Farzalipour, On weakly primary ideals, Georgian Math. Journal, Vol. 3, 2003, no. 7, 705-709.

[6] C. P. Lu, Spectra of modules, Comm. Algebra 23(1995), 3741-3752.

[7] I. G. Macdonald, Secondary represention of modules over commutative rings, Sympos. Math. XI, 23-42, (1973). Hungarica, 17(1-2)(1966), 115-123.

(Received 30 December 2009)

Farkhonde Farzalipour
Department of Mathematics,
Faculty of Science,
Payame Noor University (PNU),
Langrud, IRAN
e-mail : p_ghiasvand@pnu.ac.ir

Peyman Ghiasvand
Department of Mathematics,
Faculty of Science,
Payame Noor University (PNU),
Manjil 1161, IRAN
e-mail : p_ghiasvand@pnu.ac.ir and p.ghiasvand@gmail.com