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Exchange General Rings with Bounded Indices are Clean

W. Chen

Abstract : It is proven that an exchange general ring with bounded index is a clean general ring. In particular, an ideal I with bounded index in an exchange ring R is a clean ideal, which gives a positive answer to a question of H. Chen and M. Chen (Internat. J. Math. Math. Sci. 2003). Moreover, it is proven that each bounded matrix over an exchange ring R is a sum of an invertible matrix and an idempotent matrix, extending one of the main results of S. Wang and H. Chen (Bull. Korean Math. Soc. 43(1)(2006)).

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1 Introduction

In this note, the term a ring means an associative ring with unity and a general ring means an associative ring with or without unity (cf. [12]). A ring R is called an exchange ring if R_R has the exchange property introduced by Crawley and Jonsson in their fundamental work [7]. This property is left-right symmetric by Warfield [14]. And it is proven independently by Goodearl and Warfield [8], and Nicholson [11] that R is an exchange ring if and only if for each $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. Exchange rings arise in functional analysis. It is known that, for unital C^*-algebras, being an exchange ring is the same as having real rank zero (cf. [12]). The notion of clean rings was introduced by Nicholson [11]. A ring R is called clean if each element of R can be written as a sum of a unit and an idempotent. Clean rings are exchange rings and the converse is not true in general [4]. There are numerous papers to investigate exchange rings, clean rings and their generalizations. For instance, Ara [2] defined a general ring I to be an exchange ring if for each $a \in I$ there exist an idempotent $e \in I$ and $x, y \in I$ such that $e = ax = a + y - ay$, and proved that if I is a general ring and K is an ideal of I then I is an exchange general ring if and only if I/K and K are exchange general rings and idempotents in I/K can be lifted to I. In [3], Ara proved that a right (left) ideal of an exchange general ring is also an exchange general ring. Nicholson and Zhou [12] defined a general ring I to be clean if each element a of I can be written as $a = e + q$ where $e^2 = e \in I$ and $q \in Q(I) = \{q|q + p + qp = p + q + pq = 0 \ for\ some\ p\ in\ I\}$, and proved that clean general rings are exchange general rings in the scene of Ara [2]. In another situation, H. Chen and M. Chen [6] defined an ideal I of a ring R to be clean in case each element in I is a sum of a unit and an idempotent of R. However, it is an open question whether an exchange ideal with bounded index in a ring is a clean ideal (see [6, p. 3951]). The main purpose of this note is to give a positive answer to this question.

As usual, we use the symbol $J(I)$ to denote the Jacobson radical of a general ring I. $M_n(I)$ denotes the ring of $n \times n$ matrices over I, and $Id(I)$ denotes the set of idempotents of I. Let I be a general ring and K be an ideal of I, and $e \in Id(I)$. We write $(1 - e)K = \{a - ea \mid a \in K\}$ and $eK(1 - e) = \{ea - eae \mid a \in K\}$, and write $(1 - e)K(1 - e) = \{a - ea - ae + eae \mid a \in K\}$ (cf. [10, §1.5]). The meaning of other symbols like these is similar to that of the above. For example, if $a_1, a_2, \cdots, a_n, b \in K$, then we write $b - \Sigma_{i=1}^n a_i b + \Sigma_{i<j} a_i a_j b - \cdots + (-1)^n a_1 a_2 \cdots a_n b = (1 - a_1) \cdots (1 - a_n)b$ for simplicity.

2 The Results

We start this section with the following lemmas.

Lemma 2.1. *([12, Proposition 7 (1), (2)]) Let R be a ring and I be an ideal of R. Then I is a clean ideal in R if and only if I is a clean general ring.*

The next lemma was first observed by Nicholson and Zhou in [12].

Lemma 2.2. *Let I be a general ring. If all right primitive factor rings of I are right artinian, then I is a clean general ring.*

Proof. It is very similar to the proof of [5, Theorem 1], so we omit the details (cf. the proof of [12, Theorem 10 (2)]) . □

Proposition 2.3. *Let I be an exchange general ring and $e \in Id(I)$. Then $(1 - e)I(1 - e)$ is an exchange general subring of I. Moreover, if K is an ideal of I then $(1-e)K(1-e)$ is an exchange ideal of $(1-e)I(1-e)$ and $(1-e)K(1-e) = K \bigcap [(1 - e)I(1 - e)]$.*

Proof. It is known and easy to prove that $(1-e)I(1-e)$ is a subring of I (cf. [10, §1.5]). For any $b \in (1 - e)I(1 - e)$, then $b = a - ae - ea + eae$ with $a \in I$. Since I is an exchange general ring and $b \in I$, there exist $f \in Id(I)$ and $x, y \in I$ such that $f = xb = y + b - yb$. Clearly, $be = eb = 0$ and so $fe = xbe = 0$, which implies $(f - ef)^2 = f - ef$. Moreover, $f - ef = f - fe - ef + efe \in (1-e)I(1-e)$. Using $eb = 0$, then $f - ef = xb - exb = (x - ex - xe + exe)b \in (1-e)I(1-e)$ since $x - ex - xe + exe, b \in (1-e)I(1-e)$. On the other hand, since $f = y + b - yb$ and $eb = be = fe = 0$, we have $ye = 0$, and $f - ef = y + b - yb - e(y + b - yb) = (y - ey) + b + (-yb + eyb) = (y - ey - ye + eye) + b - (y - ey - ye + eye)b$ where $y - ey - ye + eye \in (1-e)I(1-e)$. Hence $(1-e)I(1-e)$ is an exchange general subring of I. Obviously, $(1-e)K(1-e)$ is an additive subgroup of $(1-e)I(1-e)$. For any $b \in (1-e)I(1-e)$, and $x \in (1-e)K(1-e)$, then $b = a - ea - ae + eae$ and $x = x_1 - ex_1 - x_1e + ex_1e$ where $a \in I$ and $x_1 \in K$. Since $e[(1-e)I(1-e)] = [(1-e)I(1-e)]e = 0$, $bx = bx_1 - bx_1e = bx_1 - bx_1e - ebx_1 + ebx_1e$. Note that $x_1 \in K$, we have $bx \in (1-e)K(1-e)$. Similarly, $xb \in (1-e)K(1-e)$ and so $(1-e)K(1-e)$ is an ideal of $(1-e)I(1-e)$. By [2, Theorem 2.2], $(1-e)K(1-e)$ is an exchange ideal of $(1-e)I(1-e)$. Clearly, $(1-e)K(1-e) \subseteq K \bigcap [(1-e)I(1-e)]$ holds. Conversely, for any $a \in K \bigcap [(1-e)I(1-e)]$, then $a = x - ex - xe + exe$ where $a \in K$ and $x \in I$. Since $ae = ea = 0$, we get $a = a - ae - ea + eae \in (1-e)K(1-e)$. The proof is completed. □

Lemma 2.4. *Let I be an exchange general ring and K be an ideal of I. Then every finite or countably infinite sequence of orthogonal idempotents in I/K can be orthogonally lifted to I.*

Proof. Let $f_1, f_2, \cdots, f_n, \cdots$ be any sequence of orthogonal idempotents in I/K. Since I is an exchange general ring, there exists $e_1 \in Id(I)$ such that $\bar{e_1} = f_1$ in I/K. Now take t_2 in I such that $\bar{t_2} = f_2$ in I/K. Let $s_2 = t_2 - t_2e_1 - e_1t_2 + e_1t_2e_1$. Then we have $s_2 \in (1-e_1)I(1-e_1)$ and $\bar{s_2} = f_2$ in I/K and hence $s_2^2 - s_2 \in K$. This implies that $s_2^2 - s_2 \in K \bigcap [(1-e_1)I(1-e_1)] = (1-e_1)K(1-e_1)$. By Proposition 2.3, $(1-e_1)K(1-e_1)$ is an exchange ideal of $(1-e_1)I(1-e_1)$. Since $\bar{s_2}$ is an idempotent in $(1-e_1)I(1-e_1)/(1-e_1)K(1-e_1)$, there exists an idempotent e_2 in $(1-e_1)I(1-e_1)$ such that $\bar{e_2} = \bar{s_2}$. Note that $e_1[(1-e_1)I(1-e_1)] = [(1-e_1)I(1-e_1)]e_1 = 0$. We have $e_1e_2 = e_2e_1 = 0$ and $\bar{e_2} = f_2$ in I/K.

For any $n \geq 2$, assume that we have already obtained idempotents $e_1, e_2, \cdots, e_n$ such that $e_ie_j = 0$ whenever $i \neq j$, and $\bar{e_i} = f_i \in I/K$. Let $e = e_1 + e_2 + \cdots + e_n$. Then $e \in Id(I)$, similar to the proof of the above paragraph, there exists $e_{n+1} \in Id(I)$ such that $\overline{e_{n+1}} = f_{n+1} \in I/K$ and $ee_{n+1} = e_{n+1}e = 0$. This implies that $e_ie_{n+1} = e_{n+1}e_i = 0$ for all $i < n+1$. By induction, we obtain a sequence of orthogonal idempotents $e_1, e_2, \cdots, e_n, \cdots$ in I such that $\bar{e_i} = f_i \in I/K$. □

A general ring I is called semipotent if each right ideal not contained in $J(I)$ contains a nonzero idempotent [12]. In particular, in case $J(I) = 0$ then every nonzero right ideal contains a nonzero idempotent. It is proven by [12, Proposition 5] that every exchange general ring is semipotent. And it is well known that an exchange ring is a semiperfect ring if and only if it has no infinite sequence of nonzero orthogonal idempotents (cf. [4, Corollary 2]).

The next lemma is crucial to obtaining our main result of this note.

Lemma 2.5. *Let I be an exchange general ring such that $J(I) = 0$. If I has no infinite sequence of nonzero orthogonal idempotents, then I is a right artinian general ring.*

Proof. Assume that I is not a right artinian ring. Then I is not a minimal right ideal and it contains a nonzero proper right ideal I_1. By [3, Proposition 1.3], I_1 is an exchange general ring. And [1, Proposition 9.14] implies that $J(I_1) = 0$. So there exists a nonzero idempotent e_1 in I_1. Hence $I = e_1I \oplus (1-e_1)I$ is a direct sum of two right ideals of I. Clearly, e_1I and $(1-e_1)I$ are both nonzero since $e_1I \subset I_1$, and e_1I or $(1-e_1)I$ is not right artinian. If e_1I is not right artinian, then e_1I must contain an infinite of sequence of nonzero orthogonal idempotents of I. In fact, there exists a general ring surjective homomorphism $f : e_1I \to e_1Ie_1$ given by $f(e_1a) = e_1ae_1$ where $a \in I$. Obviously, $kerf = e_1I(1 - e_1)$. So $e_1I/e_1I(1 - e_1) \cong e_1Ie_1$. Since $[e_1I(1 - e_1)]^2 = 0$, we have $e_1I(1 - e_1) \subseteq J(e_1I) = 0$. Hence $e_1I \cong e_1Ie_1$ and e_1I is a ring with unity e_1. If e_1I is not right artinian, then it is not semiperfect since $J(e_1I) = 0$, and so it contains an infinite sequence of nonzero orthogonal idempotents of I ([4, Corollary 2]). Now assume that $(1-e_1)I$ is not right artinian. Since $J(I) = J(e_1I) \oplus J((1 - e_1)I)$ and $J(I) = 0$, we have $J((1 - e_1)I) = 0$ by [1, Proposition 9.19]. Similar to the above argument for I, there exists a nonzero idempotent e_2 in $(1 - e_1)I$ such that $e_2(1 - e_1)I$ and $(1 - e_2)(1 - e_1)I$ are both nonzero and $(1-e_1)I = e_2(1-e_1)I \oplus (1-e_2)(1-e_1)I$. Hence $I = e_1I \oplus e_2(1-e_1)I \oplus (1 - e_2)(1 - e_1)I$. It is easy to see that $e_1e_2 = 0$ since $e_2 \in (1 - e_1)I$. In this case, $e_2(1-e_1) = e_2-e_2e_1 \in I$ and $e_2(1-e_1)e_2(1-e_1) = e_2(e_2-e_1e_2)(1-e_1) = e_2(1-e_1)$. Since I and e_1I are not right artinian, $e_2(1 - e_1)I$ or $(1 - e_2)(1 - e_2)I$ is not right artinian. Hence $e_2(1 - e_1)I$ contains an infinite sequence of nonzero orthogonal idempotents of I provided that $e_2(1 - e_1)I$ is not right artinian. Now we can assume that $(1-e_1)(1-e_2)I$ is not right artinian. More generally, assume that we have already obtained nonzero idempotents $e_1, e_2, \cdots, e_n$ such that $I = e_1I \oplus e_2(1-e_1)I \oplus \cdots \oplus e_n(1-e_{n-1})\cdots(1-e_1)I \oplus (1-e_n)(1-e_{n-1})\cdots(1-e_1)I$ in which every direct summand is nonzero and $e_i \in (1-e_{i-1})\cdots(1-e_1)I$ for $i \geq 2$. Not lose the generality, we may assume that $(1-e_n)(1-e_{n-1})\cdots(1-e_1)I$ is not right artinian, so there exists a nonzero idempotent $e_{n+1} \in (1-e_n)(1-e_{n-1})\cdots(1-e_1)I$ such that $(1-e_n)(1-e_{n-1})\cdots(1-e_1)I = e_{n+1}(1-e_n)(1-e_{n-1})\cdots(1-e_1)I \oplus (1-e_{n+1})(1-e_n)\cdots(1-e_1)I$ with $e_ne_{n+1} = 0$. And so $I = e_1I \oplus e_2(1-e_1)I \oplus \cdots \oplus e_{n+1}(1-e_n)\cdots(1-e_1)I \oplus (1-e_{n+1})(1-e_n)\cdots(1-e_1)I$. By induction, we obtain an infinite sequence of nonzero idempotents $e_1, e_2, \cdots, e_n, \cdots$ such that $e_{i+1} \in (1-e_i)\cdots(1-e_1)I$ and $e_ie_{i+1} = 0$. We claim that $e_1, e_2(1-e_1), \cdots, e_n(1-e_{n-1})\cdots(1-e_1)\cdots$ is an infinite sequence of nonzero orthogonal idempotents of I. To prove this, first we prove that $e_ie_j = 0$ for all $i < j$. It is known that $e_ie_{i+1} = 0$ for any $i \geq 1$. Assume that $e_ie_k = 0$ for all $k < n$. Then $e_ie_n \in e_i(1-e_{n-1})\cdots(1-e_i)\cdots(1-e_1)I = e_i(1-e_i)\cdots(1-e_1)I = 0$. Second we prove that $e_1, e_2(1-e_1), \cdots, e_n(1-e_{n-1})\cdots(1-e_1), \cdots$ is an infinite sequence of nonzero orthogonal idempotents of I. For any $n \geq 1$, $e_n(1-e_{n-1})\cdots(1-e_1)e_n(1-e_{n-1})\cdots(1-e_1) = e_ne_n(1-e_{n-1})\cdots(1-e_1) = e_n(1-e_{n-1})\cdots(1-e_1)$. Moreover, for any $i < j$, $e_i(1-e_{i-1})\cdots(1-e_1)e_j(1-e_{j-1})\cdots(1-e_1) = e_ie_j(1-e_{j-1})\cdots(1-e_1) = 0$. And $e_j(1-e_{j-1})\cdots(1-e_i)\cdots(1-e_1)e_i(1-e_{i-1})\cdots(1-e_1) = e_j(1-e_{j-1})\cdots(1-

$e_i)e_i(1-e_{i-1})\cdots(1-e_1)=0$. At last we prove that $e_n(1-e_{n-1})\cdots(1-e_1)\neq 0$. Otherwise, we have $e_n=e_n(1-e_{n-1})\cdots(1-e_1)e_n=0$, a contradiction. From the above argument, if I is not right artinian then it must contain an infinite sequence of nonzero orthogonal idempotents, which contradicts the assumption. The proof is completed. □

Recall that the index of a nilpotent element x in a general ring I is the least positive integer n such that $x^n=0$. The index of a two-sided ideal K in I is the supremum of the indices of all nilpotent elements of K. If this supremum is finite, then K is said to have bounded index (cf. [9, p.71]).

Theorem 2.6. *Let I be an exchange general ring. If I has bounded index, then each right primitive factor ring of I is right artinian and so I is a clean general ring.*

Proof. Let k be the bounded index of I. Then for any right primitive ideal P of I, each set of nonzero orthogonal idempotents of I/P contains at most k elements. Otherwise, let $f_1, f_2, \cdots, f_k, f_{k+1}$ be a set of nonzero orthogonal idempotents of I/K. By lemma 2.4, we can orthogonally lift them to orthogonal idempotents $e_1, e_2, \cdots, e_k, e_{k+1}\in I$. Now P is a right primitive ideal of I implies that it is a prime ideal. Since $e_i\notin P$, by induction, we can obtain $x_1, x_2, \cdots, x_{k+1}\in I$ such that $e_1x_1e_2x_2\cdots e_{k+1}x_{k+1}\notin P$. Now let $y=e_1x_1e_2+e_2x_2e_3+\cdots+e_kx_ke_{k+1}$. Then $y^k=e_1x_1e_2x_2\cdots e_kx_ke_{k+1}$ and $y^{k+1}=0$. Hence we have $y^k=0$, that is, $e_1x_1e_2x_2\cdots e_kx_ke_{k+1}x_{k+1}=0$, a contradiction. Since $J(I/P)=0$, I/P is right artinian by Lemma 2.5. Hence I is a clean general ring by Lemma 2.2. □

Corollary 2.7. *Let R is an exchange ring and I be an ideal of R. If I has bounded index, then I is a clean ideal of R.*

Proof. By [2, Theorem 2.2], I is a exchange general ring. Now we obtain the desired result from Theorem 2.6 and Lemma 2.1. □

A ring R is called von Neumann regular if for each $a\in R$ there exists $b\in R$ such that $a=aba$. Is is well known that a von Neumann ring is an exchange ring. Let R be a ring. In [13], a $n\times n$ matrix $A\in M_n(R)$ is called a bounded matrix in case $M_n(R)AM_n(R)$ is a bounded ideal (an ideal with bounded index) of $M_n(R)$. And [13, Lemma 1] states that for a bounded matrix $A\in M_n(R)$ over a von Neumann regular ring R, there exists a bounded ideal I of R such that $A\in M_n(I)$. In fact, this conclusion is true for any ring R.

Lemma 2.8. *Let $A\in M_n(R)$ be a bounded matrix over any ring R. Then there exists a bounded ideal I of R such that $A\in M_n(I)$.*

Proof. Since A is a bounded matrix, $M_n(R)AM_n(R)$ is a bounded ideal of $M_n(R)$. Let e_{ij} be the usual matrix units for $1\leq i,j\leq n$ and $A=(a_{st})_{n\times n}\in M_n(R)$. Since $e_{ji}Ae_{jj}=e_{ji}\sum_{s=1}^{n}\sum_{t=1}^{n}a_{st}e_{st}e_{jj}=a_{ij}e_{jj}$, there exists a ring isomorphism:

$e_{ji}M_n(R)e_{jj} \cong R$. Thus $e_{ji}M_n(R)e_{jj}e_{ji}Ae_{jj}e_{ji}M_n(R)e_{jj} \cong Ra_{ij}R$, and so $Ra_{ij}R$ is a bounded ideal of R. Let $I = \sum_{i=1}^{n}\sum_{j=1}^{n} Ra_{ij}R$. We prove that I is a bounded ideal of R. It is sufficient to prove that if K and L are two ideals of R and A, B have the bounded indices m and n, respectively, then the bounded index of $K+L$ is no more then mn. Assume that $a \in K, b \in L$. Then $(a+b)^m = a^m + b_1 = b_1$ for some $b_1 \in L$ and so $(a+b)^{mn} = 0$. Hence the sum of finitely many ideals of bounded indices is an ideal of bounded index by induction. Now I is a bounded ideal of R and clearly $A \in M_n(I)$. □

Theorem 2.9. *Each bounded matrix over an exchange ring R is a sum of an invertible matrix and an idempotent matrix.*

Proof. Let $A \in M_n(R)$ be a bounded matrix over an exchange ring R. By Lemma 2.8, there exists a bounded ideal I of R such that $A \in M_n(I)$. Corollary 2.7 implies that I is a clean ideal of R and so $M_n(I)$ is a clean ideal of $M_n(R)$ by [6, Theorem 1.9]. Therefore A is a sum of an invertible matrix and an idempotent matrix. □

Corollary 2.10. *([13, Theorem 5]) Each bounded matrix over a von Neumann ring R is a sum of an invertible matrix and an idempotent matrix.*

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Weixing Chen
Mathematics and Information Science School,
Shandong Institute of Business and Technology,
Yantai 264005, China
e-mail : wxchen5888@163.com