Convergence of an Iterative Method for Non-Lipschitzian Nonself-Mappings

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Abstract : In this paper, we introduce an iterative method for approximating a fixed point of asymptotically nonexpansive in the intermediate sense nonself-mapping in a uniformly convex Banach space. We establish some strong and weak convergence theorems. The results generalize the corresponding some results of Khan and Hussian [S.H. Khan, N. Hussian, Convergence theorems for nonself asymptotically nonexpansive mappings, Comput. Math. Appl. 55 (2008) 2544–2553] and many authors.

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1   Introduction

Let C be a nonempty subset of normed space X and $T : C \rightarrow C$ a mapping. Recall the following concepts.

(i) T is nonexpansive if
$\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

(ii) T is asymptotically nonexpansive ([9]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
$\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in C$ and $n \geq 1$.

(iii) T is uniformly Lipschitzian if there exists a constant $L > 0$ such that
$\|T^n x - T^n y\| \leq L \|x - y\|$ for all $x, y \in C$ and $n \geq 1$.

(iv) T is asymptotically nonexpansive in the intermediate sense [2] provided T is uniformly continuous and

$$\limsup_{n \to \infty} \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \qquad (1.1)$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically nonexpansive

in the intermediate sense. It is know [15] that if C is a nonempty closed bounded subset of a uniformly convex Banach space and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense, then $F(T) \neq \emptyset$.

Iterative methods for approximation of asymptotically nonexpansive mappings have been further studied by various authors (see e.g. [3, 4, 5, 13, 16, 17, 19, 21, 22, 23, 24, 28, 31, 33] and references therein).

The class of symptotically nonexpansive in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2] and iterative methods for the approximation of fixed points such types of non-Lipschitzian mappings have been studied by Agarwal, O'Regan and Sahu [1], Bruck, Kuczumow and Reich [2], Chidume, Shahzad and Zegeye [7], Kim and Kim [14] and many authors.

Resent results on approximation of fixed points of nonexpansive and asymptotically nonexpansive self and nonself single mappings can be found in ([5, 6, 10, 11, 18, 25, 27, 29, 30, 31, 34]).

The concept of asymptotically nonexpansive in the intermediate sense nonself-mappings was introduced by Chidume et al.[7] as an important generalization of nonself asymptotically nonexpansive mappings.

Definition 1.1 ([7])**.** *Let C be a nonempty subset of a Banach space X. A mapping $T : C \to X$ is said to be asymptotically nonexpansive in the intermediate sense provided T is uniformly continuous and*

$$\limsup_{n\to\infty} \sup_{x,y\in C}(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|) \leq 0, \qquad (1.2)$$

where P is a nonexpansive retraction of X onto C.

Remark 1.2. *If T is a self-map, then $PT = T$. So that (1.2) coincide with (1.1).*

Very recently, Khan and Hussian [12] introduced the following iterative method and use it for the strong and weak convergence theorems for nonself asymptotically nonexpansive mappings in a uniformly convex Banach space.

$$\begin{cases} x_1 \in C, \\ z_n = P\left(a_n T(PT)^{n-1}x_n + (1 - a_n)x_n\right), \\ y_n = P\left(b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1 - b_n - c_n)x_n\right), \\ x_{n+1} = P\left(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n)x_n\right), \end{cases} \qquad (1.3)$$

for all $n \geq 1$. Where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ in $[0, 1]$ are such that $b_n + c_n$ and $\alpha_n + \beta_n$ remain in $[0, 1]$ and satisfy certain conditions.

The purpose of this paper is to establish several strong and weak convergence theorems using (1.3) for asymptotically nonexpansive in the intermediate sense (which not necessarily Lipschitzian) nonself mappings in a uniformly convex Banach space. As remarked earlier, Khan and Hussian [12] have established strong and weak convergence theorems for nonself asymptotically nonexpansive mappings while Chidume et al. [5] studied the Mann iterative process for the case of nonself mappings. Our results generalize corresponding results of Khan and Hussian [12] and others for nonself mappings.

2   Preliminaries

Definition 2.1. *Let C be a nonempty closed and convex subset of a Banach space X. A mapping $T : C \to X$ is called demiclosed at $y \in X$ if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \to x$ weakly and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.*

In the sequel, the following lemmas are needed to prove our main results.

Lemma 2.2 ([28])**.** *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n = 1, 2, ...$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then
(1) $\lim_{n\to\infty} a_n$ exists.
(2) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.3 ([32])**.** *Let $p > 0$ and $r > 0$ be two fixed real numbers. Then a Banach space X is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \le \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|),$$

for all $x, y \in B_r = \{x \in X : \|x\| \le r\}$, $r > 0$ and $0 \le \lambda \le 1$, where $w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 2.4 ([8])**.** *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \le r\}$, $r > 0$. Then there exists a continuous strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$.

Lemma 2.5 ([7])**.** *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X, and let $P : X \to C$ be a nonexpansive retraction. Let $T : C \to X$ be a mapping which is uniformly continuous and asymptotically nonexpansive in the intermediate sense. If $\{x_n\}$ is a sequence in C converging weakly to x and if $\lim_{j\to\infty}(\limsup_{k\to\infty} \|x_k - T(PT)^{j-1}x_k\|) = 0$, then $Tx = x$.*

Lemma 2.6 ([26])**.** *Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3   Main Results

In this section, we prove weak and strong convergence theorems of the iterative process (1.3) for asymptoticaly nonexpansive in the intermediate sense nonself-mappings in a uniformly convex Banach space. In order to prove this, the following lemma is needed.

Lemma 3.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive mapping in the intermediate sense with $F(T) \neq \emptyset$. Put*

$$G_n = \sup_{x,y \in C} \left(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\| \right) \vee 0, \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.3). Then we have the following:

1 If $q \in F(T)$ then $\lim_{n \to \infty} \|x_n - q\|$ exists.

2 If $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty}(\alpha_n + \beta_n) < 1$ and $0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty}(b_n + c_n) < 1$ then $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$.

Proof. (1) Let $q \in F(T)$. For each $n \geq 1$, then by (1.3) it follows that

$$
\begin{aligned}
\|z_n - q\| &\leq \|x_n - q\| + G_n, \\
\|y_n - q\| &\leq \|x_n - q\| + 3G_n, \\
\|x_{n+1} - q\| &\leq \|x_n - q\| + 6G_n.
\end{aligned}
\tag{3.1}
$$

Since $\sum_{n=1}^{\infty} G_n < \infty$, it follows from Lemma 2.2 that $\lim_{n \to \infty} \|x_n - q\|$ exists.
(2) Let $q \in F(T)$. Then By (1.3) and Lemma 2.3, we have

$$
\begin{aligned}
\|z_n - q\|^2 &= \|P(a_n T(PT)^{n-1}x_n + (1 - a_n)x_n) - Pq\|^2 \\
&\leq \|a_n T(PT)^{n-1}x_n + (1 - a_n)x_n - q\|^2 \\
&= \|a_n(T(PT)^{n-1}x_n - q) + (1 - a_n)(x_n - q)\|^2 \\
&\leq a_n\|T(PT)^{n-1}x_n - q\|^2 + (1 - a_n)\|x_n - q\|^2 \\
&\quad - \omega(a_n)g_1(\|T(PT)^{n-1}x_n - x_n\|) \\
&\leq a_n[\|x_n - q\| + G_n]^2 + (1 - a_n)\|x_n - q\|^2 \\
&\leq \|x_n - q\|^2 + 2G_n\|x_n - q\| + G_n^2,
\end{aligned}
\tag{3.2}
$$

Now, by (1.3), (3.1) and Lemma 2.4, we have

$$
\begin{aligned}
\|y_n - q\|^2 &= \|P(b_n T(PT)^{n-1} z_n + c_n T(PT)^{n-1} x_n + (1 - b_n - c_n)x_n) - Pq\|^2 \\
&\leq \|b_n T(PT)^{n-1} z_n + T(PT)^{n-1} x_n + (1 - b_n - c_n)x_n - q\|^2 \\
&= \|b_n (T(PT)^{n-1} z_n - q) + c_n (T(PT)^{n-1} x_n - q) \\
&\quad + (1 - b_n - c_n)(x_n - q)\|^2 \\
&\leq b_n \|T(PT)^{n-1} z_n - q\|^2 + c_n \|T(PT)^{n-1} x_n - q\|^2 \\
&\quad + (1 - b_n - c_n)\|x_n - q\|^2 - b_n(1 - b_n - c_n)g_2(\|T(PT)^{n-1} z_n - x_n\|) \\
&\leq b_n[\|z_n - q\| + G_n]^2 + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad + c_n[\|x_n - q\| + G_n]^2 \\
&\quad - b_n(1 - b_n - c_n)g_2(\|T(PT)^{n-1} z_n - x_n\|) \\
&= b_n[\|z_n - p\|^2 + 2G_n\|z_n - q\| + G_n^2] \\
&\quad + c_n[\|x_n - q\|^2 + 2G_n\|x_n - p\| + G_n^2] + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad - b_n(1 - b_n - c_n)g_2(\|T(PT^{n-1} z_n - x_n\|) \\
&\leq b_n[(\|x_n - q\|^2 + 2G_n\|x_n - q\| + G_n^2) \\
&\quad + 2G_n(\|x_n - q\| + G_n) + G_n^2] \\
&\quad + c_n[\|x_n - p\|^2 + 2G_n\|x_n - p\| + G_n^2] + (1 - b_n - c_n)\|x_n - q\|^2 \\
&\quad - b_n(1 - b_n - c_n)g_2(\|T^n z_n - x_n\|) \\
&\leq \|x_n - q\|^2 + 6G_n\|x_n - q\| + 5G_n^2 \\
&\quad - b_n(1 - b_n - c_n)g_2(\|T(PT)^{n-1} z_n - x_n\|). \quad\quad (3.3)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \|P(\alpha_n T(PT)^{n-1} y_n + \beta_n T(PT)^{n-1} z_n + (1 - \alpha_n - \beta_n)x_n) - Pq\|^2 \\
&\leq \|\alpha_n T(PT)^{n-1} y_n + \beta_n T(PT)^{n-1} z_n + (1 - \alpha_n - \beta_n)x_n - q\|^2 \\
&= \|\alpha_n (T(PT)^{n-1} y_n - q) + \beta_n (T(PT)^{n-1} z_n - q) \\
&\quad + (1 - \alpha_n - \beta_n)(x_n - q)\|^2 \\
&\leq \alpha_n \|T(PT)^{n-1} y_n - q\|^2 + \beta_n \|T(PT)^{n-1} z_n - q\|^2 \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T(PT)^{n-1} y_n - x_n\|) \\
&\leq \alpha_n(\|y_n - p\| + G_n)^2 + \beta_n(\|z_n - q\| + G_n)^2 \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T(PT)^{n-1} y_n - x_n\|) \\
&= \alpha_n[\|y_n - q\|^2 + 2G_n\|y_n - q\| + G_n^2] \\
&\quad + \beta_n[\|z_n - q\|^2 + 2G_n\|z_n - q\| + G_n^2]\|^2 \\
&\quad + (1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g_2(\|T(PT)^{n-1} y_n - x_n\|)
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & \alpha_n[(\|x_n - q\|^2 + 6G_n\|x_n - q\| + 5G_n^2) \\
& + 2G_n(\|x_n - q\| + 3G_n) + G_n^2] \\
& + \beta_n[(\|x_n - q\|^2 + 2G_n\|x_n - q\| + G_n^2) \\
& + 2G_n(\|x_n - q\| + G_n) + G_n^2] \\
& + (1 - \alpha_n - \beta_n)\|x_n - q\|^2 \\
& - \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T(PT)^{n-1}y_n - x_n\|) \\
\leq \quad & \|x_n - q\|^2 + 12G_n\|x_n - q\| + 16G_n^2 \\
& - \alpha_n(1 - \alpha_n - \beta_n)g_2(\|T(PT)^{n-1}y_n - x_n\|) \quad\quad (3.4)
\end{aligned}
$$

which imply that

$$
\begin{aligned}
\alpha_n(1 - \alpha_n - \beta_n - \lambda)g_2(\|T(PT)^{n-1}y_n - x_n\|) \quad \leq \quad & \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
& + 12LG_n + 16G_n^2 \quad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
b_n(1 - b_n - c_n - \mu_n)g_2(\|T(PT)^{n-1}z_n - x_n\|) \quad \leq \quad & \|x_n - q\|^2 - \|x_{n+1} - p\|^2 \\
& + 12LG_n + 16G_n^2, \quad (3.6)
\end{aligned}
$$

where $L = \sup\{\|x_n - q\| : n \geq 1\}$.

If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0, 1)$ such that

$$
0 < \eta < \alpha_n \quad \text{and} \quad \alpha_n + \beta_n < \eta' < 1 \quad \text{for all} \quad n \geq n_0.
$$

This implies by (3.5) that

$$
\begin{aligned}
\eta(1 - \eta')g_2(\|T(PT)^{n-1}y_n - x_n\|) \quad \leq \quad & \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 12LG_n + 16G_n^2 \\
\leq \quad & \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 12LG_n + 16MG_n \\
\leq \quad & \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 28KG_n, \quad (3.7)
\end{aligned}
$$

where $K = \max\{M = \sup_{n\geq 1} G_n, L\}$, for all $n \geq n_0$.

It follows from (3.7) that for $m \geq n_0$

$$
\begin{aligned}
\sum_{n=n_0}^{m} g_2(\|T(PT)^{n-1}y_n - x_n\|) \quad \leq \quad & \frac{1}{\eta(1 - \eta')}\left(\sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2)\right. \\
& \left. + \sum_{n=n_0}^{m} 28KG_n\right) \\
\leq \quad & \frac{1}{\eta(1 - \eta')}\left(\|x_{n_0} - q\|^2 + 28K\sum_{n=n_0}^{m} G_n\right) (3.8)
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} G_n < \infty$. Let $m \to \infty$ in inequality (3.8) we get that

$\sum_{n=n_0}^{\infty} g_2(\|T(PT)^{n-1}y_n - x_n\|) < \infty$, and therefore $\lim_{n\to\infty} g_2(\|T(PT)^{n-1}y_n - x_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0$.

If $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n) < 1$, then by the using a similar method together with inequality (3.6), it can be shown that

$$\lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0.$$

Now, we have

$$\lim_{n\to\infty} \|T(PT)^{n-1}y_n - x_n\| = 0 \ \text{ and } \ \lim_{n\to\infty} \|T(PT)^{n-1}z_n - x_n\| = 0. \qquad (3.9)$$

From $y_n = P(b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1 - b_n - c_n)x_n)$, we have

$$
\begin{aligned}
\|y_n - x_n\| &= \|P(b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1 - b_n - c_n)x_n) - P(x_n)\| \\
&\leq \|b_n T(PT)^{n-1}z_n + c_n T(PT)^{n-1}x_n + (1 - b_n - c_n)x_n - x_n\| \\
&= \|b_n(T(PT)^{n-1}z_n - x_n) + c_n(T(PT)^{n-1}x_n - x_n)\| \\
&\leq b_n\|T(PT)^{n-1}z_n - x_n\| + c_n\|T(PT)^{n-1}x_n - x_n\| \qquad (3.10)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|T(PT)^{n-1}x_n - x_n\| &= \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n + T(PT)^{n-1}y_n - x_n\| \\
&\leq \|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\| + \|T(PT)^{n-1}y_n - x_n\| \\
&\leq \|x_n - y_n\| + G_n + \|T(PT)^{n-1}y_n - x_n\| \\
&\leq b_n\|T(PT)^{n-1}z_n - x_n\| + c_n\|T(PT)^{n-1}x_n - x_n\| \\
&\quad + G_n + \|T(PT)^{n-1}y_n - x_n\|
\end{aligned}
$$

and so

$$
\begin{aligned}
(1 - c_n)\|T(PT)^{n-1}x_n - x_n\| &\leq b_n\|T(PT)^{n-1}z_n - x_n\| + G_n \\
&\quad + \|T(PT)^{n-1}y_n - x_n\|
\end{aligned}
$$

Since $\limsup_{n\to\infty} c_n < 1$, it follows from (3.9) and $\sum_{n=1}^{\infty} G_n < \infty$ that

$$\lim_{n\to\infty} \|T(PT)^{n-1}x_n - x_n\| = 0. \qquad (3.11)$$

It follows from (3.9), (3.10) and (3.11) that $\lim_{n\to\infty} \|y_n - x_n\| = 0$.
From $x_{n+1} = P(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n)x_n)$, we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|P(\alpha_n T(PT)^{n-1}y_n + \beta_n T(PT)^{n-1}z_n + (1 - \alpha_n - \beta_n)x_n) - P(x_n)\| \\
&\leq \alpha_n\|T(PT)^{n-1}y_n - x_n\| + \beta_n\|T(PT)^{n-1}z_n - x_n\| \ \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

And

$$
\begin{aligned}
\|x_{n+1} - T(PT)^{n-1}x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T(PT)^{n-1}x_{n+1} - T(PT)^{n-1}x_n\| \\
&\quad + \|T(PT)^{n-1}x_n - x_n\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Since

$$\|x_{n+1} - Tx_{n+1}\| \leq \|x_{n+1} - T(PT)^n x_{n+1}\| + \|Tx_{n+1} - T(PT)^{n-1} x_{n+1}\|$$

and by uniform continuity of T and (3.11), it follows that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. $\square$

The following theorem generalizes Theorem 1 of Khan and Hussian [12].

Theorem 3.2. *Let C, X, T and $\{x_n\}$ be as in Lemma 3.1. If, in addition, T is either completely or demicompact and $F(T) \neq \emptyset$, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Proof. Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, $\{x_{n_k}\}$ converges. Let $\lim_{k \to \infty} x_{n_k} = q$. By continuity of T and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, we have that $Tq = q$, so q is a fixed point of T. By Lemma 3.1 (i), $\lim_{n \to \infty} \|x_n - q\|$ exists. But $\lim_{k \to \infty} \|x_{n_k} - q\| = 0$. Thus $\lim_{n \to \infty} \|x_n - q\| = 0$. Since $\|y_n - x_n\| \to 0$ as $n \to \infty$, and

$$\begin{aligned} \|z_n - x_n\| &= \|P(a_n T(PT)^{n-1} x_n + (1 - a_n) x_n) - P(x_n)\| \\ &\leq \|T(PT)^{n-1} x_n - x_n\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

it follows that $\lim_{n \to \infty} y_n = q$ and $\lim_{n \to \infty} z_n = q$.

Next, assume that T is demicompact. Since $\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k \to \infty} x_{n_k} = q^*$ (say). By Lemma 2.5, it implies that $q^* = Tq^*$. Moreover, as $\lim_{n \to \infty} \|x_n - q'\|$ exists for all $q' \in F(T)$, therefore $\{x_n\}$ converges strongly to q^*. That is $\lim_{n \to \infty} \|x_n - q^*\| = 0$.
An argument similar to the above case proves that $\{y_n\}$ and $\{z_n\}$ also converge strongly to a fixed q' of T. This completes the proof. $\square$

Remark 3.3. *If T is nonself asymptotically nonexpansive mapping and $c_n = \beta_n = 0$, then Theorem 3.2 generalize theorem 2 of Khan and Hussian [12]. And also if T is an asymptotically nonexpansive self-mapping and $c_n = \beta_n = 0$, then Theorem 3.2 generalize Theorem 2.4 of Suantai [26] and Theorem 2.1 of Xu and Noor [33].*

The following theorem generalizes Theorem 3 of Khan and Hussian [12]

Theorem 3.4. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive mapping in the intermediate sense with $F(T) \neq \emptyset$. Put*

$$G_n = \sup_{x,y \in C} (\|T(PT)^{n-1} x - T(PT)^{n-1} y\| - \|x - y\|) \vee 0, \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. *Let* $\{b_n\}$ *and* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *be such that* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ *and* $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$. *Let the sequence* $\{x_n\}$ *be defined as follows:*

$$\begin{cases} x_1 \in C, \\ y_n = P\left(b_n T(PT)^{n-1} x_n + (1 - b_n)x_n\right), \\ x_{n+1} = P\left(\alpha_n T(PT)^{n-1} y_n + (1 - \alpha_n)x_n\right), n \ge 1, \end{cases}$$

If T *is completely continuous and* $F(T) \ne \emptyset$, *then* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a fixed point of* T.

Proof. The choice $a_n = c_n = \beta_n = 0$ in Theorem 3.2 leads to the conclusion. $\quad\square$

Remark 3.5. *Theorem 2.5 of Suantai [26] and Theorem 3 of Rhoades [22] has been generalized as Theorem 3.4.*

The following theorem generalizes Theorem 4 of Khan and Hussian [12].

Theorem 3.6. *Let* X *be a uniformly convex Banach space, and let* C *be a nonempty closed and convex subset of* X. *Let* $T : C \to X$ *be an asymptotically nonexpansive mapping in the intermediate sense with* $F(T) \ne \emptyset$. *Put*

$$G_n = \sup_{x,y \in C}\left(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|\right) \vee 0, \forall n \ge 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. *Let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *be such that* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. *Let the sequence* $\{x_n\}$ *be defined as follows:*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P\left(\alpha_n T(PT)^{n-1} x_n + (1 - \alpha_n)x_n\right), n \ge 1, \end{cases} \tag{3.12}$$

If T *is completely continuous and* $F(T) \ne \emptyset$, *then* $\{x_n\}$ *converge strongly to a fixed point of* T.

Proof. Put $a_n = b_n = c_n = \beta_n = 0$ in Theorem 3.2. $\quad\blacksquare$

Remark 3.7. *Theorem 2.2 of Schu [23], Theorem 2.6 of Suantai [26], Theorem 2 of Rhoades [22] and Theorem 1.5 of Schu [24] has been generalized as Theorem 3.6.*

In the same way, we can prove Lemma 3.1 under the conditions used by Chidume et al. [5] to get the following:

Theorem 3.8. *Let* X *be a uniformly convex Banach space, and let* C *be a nonempty closed and convex subset of* X. *Let* $T : C \to X$ *be an asymptotically nonexpansive mapping in the intermediate sense with* $F(T) \ne \emptyset$. *Put*

$$G_n = \sup_{x,y \in C}\left(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\|\right) \vee 0, \forall n \ge 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Define a sequence $\{x_n\}$ in C as in (1.3) where $\{a_n\}$, $\{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{b_n + c_n\}, \{\alpha_n + \beta_n\}$ are in $[\epsilon, 1 - \epsilon]$ for all $n \geq 1$ and for some ϵ in (0, 1). If T is completely continuous and $F(T) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

This theorem immediately gives the following:

Corollary 3.9 ([12] Theorem 5)**.** *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let $T : C \to X$ be a nonself asymptotically nonexpansive mapping with the nonempty fixed points set $F(T)$. Define a sequence $\{x_n\}$ in C as in (1.3) where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$, $\{b_n + c_n\}, \{\alpha_n + \beta_n\}$ are in $[\epsilon, 1 - \epsilon]$ for all $n \geq 1$ and for some ϵ in (0, 1). If T is completely continuous and $F(T) \neq \emptyset$, then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.*

Remark 3.10. *Theorem 3.7 of Chidume [7] has been generalized as Theorem 3.8.*

Now we turn our attention towards weak convergence. A Banach space X is said to satisfy *Opial's condition* [20] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Actually, if T is not taken to be completely continuous but X satisfy *Opial's condition*, then we have the following:

Theorem 3.11. *Let X be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed and convex subset of X. Let $T : C \to X$ be an asymptotically nonexpansive mapping in the intermediate sense. Put*

$$G_n = \sup_{x,y \in C} \left(\|T(PT)^{n-1}x - T(PT)^{n-1}y\| - \|x - y\| \right) \vee 0, \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} G_n < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ such that $b_n + c_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.3). If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. It follows from Lemma 3.1 (2) that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \to u$ weakly as $n \to \infty$, without loss of generality. By Lemma 2.5, it implies that $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.5, we have $u, v \in F(T)$. By Lemma 3.1 (1), $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. It follows from Lemma 2.6 that $u = v$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. $\square$

This theorem immediately gives the following:

Corollary 3.12 (Khan and Hussian [12]). *Let X be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty closed and convex subset of X. Let $T : C \to X$ be a nonself asymptotically nonexpansive mapping. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences in $[0,1]$ such that $\{b_n + c_n\}$ and $\{\alpha_n + \beta_n\}$ are in $[0,1]$ for all $n \geq 1$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined as in (1.3). If $F(T) \neq \emptyset$, then $\{x_n\}$ converges weakly to a fixed point of T.*

Remark 3.13. *Theorem 3.11 contain Theorem 6 of Khan and Hussian [12] and Theorem 2.8. Corollary 2.9-2.11 of Suantai [26] as spacial cases when T is self-mapping.*

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