Thai Journal of Mathematics
Volume 7 (2009) Number 2 : 329–337

www.math.science.cmu.ac.th/thaijournal
Online ISSN 1686-0209

Strong convergence theorem by the shrinking projection method for hemi-relatively nonexpansive mappings

K. Wattanawitoon, P. Kumam and U. W. Humphries

Abstract: In this paper, we prove a strong convergence theorem by the shrinking projection method for hemi-relatively nonexpansive mappings in Banach spaces. Using this result, we also discuss the problem of strong convergence quasi-nonexpansive mappings in a Hilbert space.

Keywords: strong convergence; nonexpansive mapping; shrinking projection method; hemi-nonexpansive mapping; quasi-nonexpansive mapping
2000 Mathematics Subject Classification: 446C05, 47D03, 47H09, 47H10, 47H20.

1 Introduction

Let E be a real Banach space, C be a nonempty closed convex subset of E, and $T : C \to C$ be a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \qquad \text{for all } \; x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : x = Tx\}$. A mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\| \qquad \text{for all } \; x \in C \quad \text{and} \quad y \in F(T).$$

It is easy to see that if T is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive. Recently, several articles have appeared providing methods for

Copyright © 2009 by the Mathematical Association of Thailand. All rights reserved.

approximating fixed points of relatively (quasi-)nonexpansive mappings [4, 5, 6, 8]. Matsushita and Takahashi [4] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \tag{1.1}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0,1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of T. Moreover, Matsushita and Takahashi [5] proposed the following modification of iteration (1.1):

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarity,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.2}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}(x_0)$.

In 2008, Takahashi et al. [8] proved the following theorem by a hybrid method. We call such a method the shrinking projection method.

Theorem 1.1. (Takahashi et al. [8]). *Let H be a hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\|) \le \|u_n - z\|\}, \\ u_{n+1} = \Pi_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $0 \le \alpha_n \le a < 1$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Very recently, Yongfu Su et al. [7] extended Theorem 1.1 from a closed relatively nonexpansive mapping to a closed hemi-relatively nonexpansive mapping. They proved a strong convergence theorem by the (CQ) hybrid method.

In this paper, motivated by Takahashi et al.'s result [8] and Yongfu Su et al.'s result [7], we prove a strong convergence theorem for fixed points of closed hemi-relatively nonexpansive mappings in a Banach space by using the shrinking projection method. Our results modify and improve the result of Matsushita and Takahashi [5] and Yongfu Su et al. [7].

2 Preliminaries

Let E be a real Banach space with dual E^*. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \tag{2.1}$$

for $x \in E$.

If C is a nonempty closed convex subset of real Hilbert space H and $P_C : H \to C$ is the metric projection, then P_C is nonexpansive. Alber [1] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue representation of the metric projection in Hilbert spaces.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \qquad \text{for } x, y \in E. \tag{2.2}$$

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = x^*$, where x^* is the solution to the minimization problem

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x),$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\| y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \qquad \text{for } x, y \in E. \tag{2.3}$$

Remark 2.1. ([7]). *If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if, and only if, $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From* (2.3)*, we have $\|x\| = \|y\|$. This implies $\langle y, Jx \rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$.*

Let C be a closed convex subset of E, and let T be a mapping from C into itself. The set of fixed points of T is denoted by $F(T)$. A mapping T is said to be hemi-relatively nonexpansive if

$$\phi(p, Tx) \leq \phi(p, x) \qquad \text{for all } \; x \in C \quad \text{and } \; p \in F(T).$$

A point p in C is said to be an asymptotic fixed point of T [2] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n \to \infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A hemi-relatively nonexpansive mapping T from C into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$.

Lemma 2.2. ([3]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3. ([1]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \qquad \forall y \in C. \tag{2.4}$$

Lemma 2.4. ([1]). *Let E be a reflexive, strictly convex, and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \qquad \forall y \in C. \tag{2.5}$$

Lemma 2.5. ([5]). *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3 Main Results

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed hemi-relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in [0,1] such that $\limsup_{n\longrightarrow\infty}\alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x), \qquad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

***Proof*.** We first show that C_{n+1} is closed and convex for each $n \geq 0$. From the definition of C_{n+1} it is obvious that C_{n+1} is closed for each $n \geq 0$. We show that C_{n+1} is convex for any $n \geq 0$. Since

$$\phi(z, y_n) \leq \phi(z, x_n) \Longleftrightarrow 2\langle z, Jx_n - Jy_n \rangle + \|y_n\|^2 - \|x_n\|^2 \leq 0,$$

and hence C_{n+1} is convex.

Next, we show that $F(T) \subset C_n$ for all $n \geq 0$. Indeed, let $p \in F(T)$ and T is hemi-relatively nonexpansive, we have

$$\begin{aligned} \phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n)) \\ &\leq \|p\|^2 - 2\langle p, \alpha_n Jx_n + (1-\alpha_n)JTx_n \rangle + \alpha_n \|x_n\|^2 \\ &\quad +(1-\alpha_n)\|Tx_n\|^2 \\ &\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2(1-\alpha_n)\langle p, JTx_n \rangle + \alpha_n \phi(p, x_n) \\ &\quad +(1-\alpha_n)\phi(p, Tx_n) \\ &= \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, Tx_n) \\ &\leq \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\ &\leq \phi(p, x_n). \end{aligned}$$

This means that, $p \in C_{n+1}$ for all $n \geq 0$. Thus, $\{x_n\}$ is well defined.
By definition of x_n, we obtain

$$\phi(x_n, x) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, \Pi_{C_n} x_0) \leq \phi(p, x),$$

for all $p \in F(T) \subset C_n$. Thus, $\phi(x_n, x_0)$ is bounded. So, $\{x_n\}$ and $\{Tx_n\}$ are bounded.
Since $x_n = \Pi_{C_{n+1}} x_0$ and $x_{n+1} \in C_{n+1} \subset C_n$, we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0),$$

for all $n \geq 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. Thus $\lim_{n \longrightarrow \infty} \phi(x_n, x)$ exists. By Lemma 2.4, we have

$$\begin{aligned} \phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{C_n} x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n}, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0), \end{aligned}$$

for all $n \geq 0$. Thus, $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Assuming not, hence there exists $\varepsilon_0 > 0$ and subsequence $\{n_k\}, \{m_k\} \subset \{n\}$ such that

$$\|x_{n_k + m_k} - x_{n_k}\| \geq \varepsilon_0,$$

for all $k \geq 1$. Applying Lemma 2.4 that

$$\phi(x_{n_k+m_k}, x_{n_k}) \leq \phi(x_{n_k+m_k}, x) - \phi(x_{n_k}, x) \to 0, \text{as} \;\; k \to \infty. \tag{3.2}$$

Since $\phi(x_n, x)$ is bounded and the limit of $\phi(x_n, x)$ exists, we obtain

$$\lim_{n \to \infty} \phi(x_{n_k+m_k} - x_{n_k}) = 0.$$

Hence, by Lemma 2.2, we have

$$\lim_{k \to \infty} \|x_{n_k+m_k} - x_{n_k}\| = 0.$$

This is a contradiction, so that $\{x_n\}$ is a Cauchy sequence, such that $\{x_n\}$ converges strongly to p.
From $x_{n+1} = \Pi_{C_{n+1}} \in C_{n+1}$, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n),$$

for all $n \geq 0$. It follows from (3.2) that

$$\phi(x_{n+1}, y_n) \to 0, \text{as} \;\; n \to \infty.$$

By using Lemma 2.2, we also have

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.3}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.4}$$

We observe that

$$\begin{aligned} \|Jx_{n+1} - Jy_n\| &= \|Jx_{n+1} - (\alpha_n Jx_n + (1-\alpha_n)JTx_n)\| \\ &= \|\alpha_n(Jx_{n+1} - Jx_n) + (1-\alpha_n)(Jx_{n+1}) - JTx_n\| \\ &= \|(1-\alpha_n)(Jx_{n+1} - JTx_n) - \alpha_n(Jx_n - Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1} - JTx_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|. \end{aligned}$$

It follows that

$$\|Jx_{n+1} - JTx_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

By (3.4) and $\limsup_{n\longrightarrow\infty}\alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JTx_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - Tx_n\| = 0. \tag{3.5}$$

By triangle inequality, we get

$$\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|.$$

From (3.3) and (3.5)

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Finally, we prove that $p = \Pi_{F(T)}x_0$. By Lemma 2.4, we have

$$\phi(p, \Pi_{F(T)}x_0) + \phi(\Pi_{F(T)}x_0, x_0) \leq \Pi(p, x_0).$$

Since $x_{n+1} = \Pi_{C_n x}$ and $F(T) \subset C_n$, for all n, we get from Lemma 2.4 that

$$\phi(\Pi_{F(T)}x_0, x_{n+1}) + \phi(x_{n+1}, x_0) \leq \phi(\Pi_{F(T)}x_0, x_0).$$

By the definition of $\phi(x, y)$, it follows that both $\phi(p, x_0) \leq \phi(\Pi_{F(T)}x_0, x_0)$ and $\phi(p, x_0) \geq \phi(\Pi_{F(T)}x_0, x_0)$, hence $\phi(p, x_0) = \phi(\Pi_{F(T)}x_0, x_0)$. Thus, it follows from the uniqueness of $\Pi_{F(T)}x_0$ that $p = \Pi_{F(T)}x_0$. □

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in* [0,1] *such that* $\limsup_{n\longrightarrow\infty}\alpha_n < 1$. *Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x), \qquad n \in \mathbb{N}, \end{cases} \tag{3.6}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

***Proof*.** Since every relatively nonexpansive mapping is a hemi-relatively nonexpansive. □

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed hemi-relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = Tx_n, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x), \qquad n \in \mathbb{N}. \end{cases} \tag{3.7}$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

***Proof*.** In Theorem 3.1 if $\alpha_n = 0$, then (3.6) reduced to (3.7). □

4 Deduced Theorems

In Hilbert spaces, hemi-relatively nonexpansive and quasi-nonexpansive mappings are the same.
We obtain the following Theorem:

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a sequence of quasi-nonexpansive mappings from C into C such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\longrightarrow\infty}\alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 = x \in C, C_0 = C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|z - y_n\| \leq \|z - x_n\|\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \qquad n \in \mathbb{N}. \end{cases} \tag{4.1}$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

***Proof*.** Since J is an identity operator, we have

$$\phi(x, y) = \|x - y\|^2$$

for every $x, y \in H$. Hence

$$\|Tx - z\| \leq \|x - z\| \Leftrightarrow \phi(z, Tx) \leq \phi(z, x)$$

for every $x \in C$ and $z \in F(T)$. Therefore, T is quasi-nonexpansive if and only if T is hemi-relatively nonexpansive. Thus, by Theorem 3.1, we obtain the theorem.□

Acknowledgement(s): I would like to express the authors thanks to the Faculty of Science KMUTT Research Fund for their financial support. The second author was supported by the Thailand Research Fund and commission on Higher Education.

References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications A*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, A.G. Kartsatos (Ed), Marcel Dekker, New York. **178** (1996) 15–50.

[2] D. Butnariu, S. Reich, and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. of Appl. Anal. **7** no.2 (2001) 151–174.

[3] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** no.3 (2002) 938–945.

[4] S. Matsushita, W. Takahashi, *Weakly and strong convergence theorems for relatively nonexpansive mappings in a Banach space*, Fixed Point Theory and Appl. **2004** (2004) 37–47.

[5] S. Matsushita, W. Takahashi, *A Strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. of Approximation Theory. **134** no.2 (2005) 257–266.

[6] S. Plutieng, K. Ungchittrakool, *Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space*, J. of Approximation Theory. **149** (2007) 103–115.

[7] Y. Su, D. Wang and M. Shang, *Strong convergence theorems of monotone hybrid algorithm for hemi-relatively nonexpansive mappings,* Fixed Point Theory and Appl. Vol 2008 (2008), Article ID 284613, 8 pages doi:10.1155/2008/284613.

[8] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341**(2008) 276–286.

(Received 3 July 2008)

Kriengsak Wattanawitoon
Department of Mathematics,

Faculty of Science,
King Mongkut's University of Technology Thonburi, KMUTT
Bangmod, Thrungkru, Bangkok 10140, THAILAND.
e-mail: s9510105@st.kmutt.ac.th

Poom Kumam
Department of Mathematics,
Faculty of Science,
King Mongkut's University of Technology Thonburi, KMUTT
Bangmod, Thrungkru, Bangkok 10140, THAILAND.
e-mail: poom.kum@kmutt.ac.th

Usa Wannasingha Humphries
Department of Mathematics,
Faculty of Science,
King Mongkut's University of Technology Thonburi, KMUTT
Bangmod, Thrungkru, Bangkok 10140, THAILAND.
e-mail: usa.wan@kmutt.ac.th