Thai Journal of Mathematics
Volume 22 Number 2 (2024)
Pages 335–361

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Halpern's Proximal Point Algorithm in p-Uniformly Convex Metric Spaces

Chinedu Izuchukwu[1,2], Grace Nnennaya Ogwo[1] and Oluwatosin Temitope Mewomo[1,*]

[1] *School of Mathematics, Statistics and Computer Science, University of Kwazulu-Natal, Durban, South Africa*
e-mail : izuchukwuc@ukzn.ac.za, izuchukwu_c@yahoo.com (C. Izuchukwu);
219095374@stu.ukzn.ac.za, graceogwo@aims.ac.za (G.N. Ogwo);
mewomoo@ukzn.ac.za (O.T. Mewomo)

[2] *DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg, South Africa*

Abstract The main purpose of this paper is to introduce a Halpern-type proximal point algorithm, comprising a nonexpansive mapping and a finite composition of p-resolvent mappings associated with proper convex and lower semicontinuous functions. A strong convergence of the proposed algorithm to a common solution of a finite family of minimization problems and fixed point problems for a nonexpansive mapping is established in a complete p-uniformly convex metric space. Also, numerical examples of the proposed algorithm in nonlinear settings are given to illustrate the applicability of the obtained results.

MSC: 47H09; 47H10; 49J20; 49J40

Keywords: p-uniformly convex metric spaces; p-resolvent; convex functions; Halpern's iterations; proximal point algorithm

Submission date: 25.01.2022 / Acceptance date: 17.08.2023

1. Introduction

Let X be a geodesic space (to be defined in Section 2) and f be any real-valued function defined on X. The Minimization Problem (MP) is the problem of finding a point $\bar{v} \in X$ such that $f(\bar{v}) = \min_{v \in X} f(v)$. In this case, $\bar{v}$ is called a minimizer of f and is denoted by $\bar{v} := \operatorname{argmin}_{v \in X} f(v)$, where $\operatorname{argmin}_{v \in X} f(v)$ denotes the set of minimizers of f. MPs are very useful in optimization theory, convex and nonlinear analysis. One of the most popular and effective approach for solving MPs is the Proximal Point Algorithm (PPA), introduced by Martinet [37] in 1970 and further developed by Rockafellar [52] in Hilbert spaces. Rockafellar [52] proved that the PPA converges weakly to a minimizer of f when f is a proper convex and lower semicontinuous functional (to be defined in Section 2). In 2000 Kamimura and Takahashi [30] modified the PPA into a Halpern-type PPA and proved that it converges strongly to a minimizer of f when f is a proper convex

*Corresponding author.

and lower semicontinuous functional. Since then, different modifications of the PPA and several iterative methods for solving MP and its related optimization problems have been introduced, well-developed and extensively studied in Hilbert spaces and Banach spaces (see [1–4, 25, 28, 31, 32, 42, 43, 46–48, 50, 51, 60] and the references therein).
The study of the PPA for solving MPs and other related problems has recently been generalized from Hilbert spaces to nonlinear spaces, in particular, the Hadamard manifolds and Hilbert unit balls (see for example [7, 21, 35] and the references therein). This study was further generalized to the setting of Hadamard spaces (complete CAT(0) spaces) by Bačák [12] in 2013, as follows: For arbitrary point x_1 in a Hadamard space X, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = J^f_{\lambda_n}(x_n), \tag{1.1}$$

where $\lambda_n > 0$ for all $n \geq 1$, and $J^f_\lambda : X \to X$ is the Moreau-Yosida resolvent of a proper convex and lower semicontinuous functional defined by

$$J^f_\lambda(x) = \arg\min_{v\in X} \left(f(v) + \frac{1}{2\lambda} d^2(v, x) \right). \tag{1.2}$$

Bačák [12] proved that the PPA Δ-converges (to be defined in Section 2) to a minimizer of f provided that $\sum_{n=1}^{\infty} \lambda_n = \infty$ and f has a minimizer in X. Since then, there has been increased interest in the study of PPA by numerous researchers in Hadamard spaces (see for example, [9, 11, 27, 44, 45, 49, 59] and the references therein).
It is worth mentioning that, although the PPA was first introduced and studied in the linear settings, it is known to have important metric characteristics. Thus, these generalizations (that is, generalizing the study of PPA from Banach spaces to nonlinear spaces) are ideal and very important. More precisely, the PPA can be applied in Hadamard spaces for computing medians and means which are very important in computational phylogenetics, difussion tensor imaging, censensus algorithms and modeling of airway systems in human lungs and blood vessels (see [11, 23, 24] for details). Also, many non-convex problems in the setting of Banach spaces (particularly Hilbert spaces) can be seen as convex problems in the nonlinear settings (see for example, [20, Problem 4.1] and Section 4 of this paper). Furthermore, it is known that the energy functional is an example of a convex and lower semicontinuous functional on a CAT(0) space and minimizers of this functional (called the harmonic maps) are very useful in geometry and analysis (see [12]). However, we know that Hilbert spaces are the only Banach spaces which are Hadamard spaces. Thus, there is a need to further generalize the study of the PPA to more general nonlinear spaces which include other Banach spaces. The study of the PPA in such nonlinear spaces, in particular, p-uniformly convex metric spaces is our interest in this paper. The notion of p-uniformly convex metric spaces was first introduced by Noar and Silberman [38] in 2011, as follows (see also [33]): Let $p > 1$, a metric space (X, d) is called p-uniformly convex with parameter $c > 0$ if and only if (X, d) is a geodesic space and

$$d(v, (1-t)x \oplus ty)^p \leq (1-t)d(v, x)^p + td(v, y)^p - \frac{c}{2}t(1-t)d(x, y)^p, \ \forall x, y, v \in X, \tag{1.3}$$

$t \in [0, 1]$. The notion of p-uniformly convex metric spaces is an obvious generalization of the classical notion of p-uniformly convex Banach spaces (see [16, 33]). More precisely, L^p-spaces with $p \geq 2$ are typical examples of p-uniformly convex metric spaces. Furthermore, when $p = 2 = c$ in (1.3), we obtain the CAT(0) inequality (see [15, 38]). In fact, every CAT(0) space is 2-uniformly convex with parameter $c = 2$ and every CAT(k) space $(k > 0)$

with $\text{diam}(X) < \frac{\pi}{2\sqrt{k}}$ is 2-uniformly convex with parameter $c = (\pi - 2\sqrt{k}\epsilon)\tan(\sqrt{k}\epsilon)$ for any $0 < \epsilon \leq \frac{\pi}{2\sqrt{k}} - \text{diam}(X)$ (see [34, 38]). It is also interesting to note that, some recent results obtained in p-uniformly convex metric spaces have already found numerous applications in L^p-Wasserstein spaces, Finsler geometry and metric geometry; the non-linearization of the geometry of Banach space and other related fields (see for example [33, 40, 41]). For more details on p-uniformly convex metric spaces, see [33, 40, 41] and the references therein.
Let X be a p-uniformly convex metric space. Choi and Ji [16] introduced the notion of p-resolvent mapping of a proper, convex and lower semicontinuous functional f defined on X as follows: For $x \in X$ and $\lambda > 0$, $J_\lambda^f : X \to X$ is defined by

$$J_\lambda^f(x) = \arg\min_{v \in X}\left(f(v) + \frac{1}{2\lambda}d(v,x)^p\right). \tag{1.4}$$

Clearly, if $p = 2$, then (1.4) reduces to the Moreau-Yosida resolvent (1.2). Using (1.4), they proved that the PPA converges to a minimizer of f in a p-uniformly convex metric space. In fact, they proved the following theorem.

Theorem 1.1. [16, Theorem 3.6] *Let X be a p-uniformly convex metric space with parameter $c > 0$ and diameter $\alpha > 0$. Let $f : X \to (-\infty, \infty]$ be a proper uniformly convex, lower semicontinuous function, and $\{\lambda_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{n}{(\sum_{i=1}^n \lambda_i)} = 0$. Suppose that the sequence $\{x_n\}$ in X is generated by the following PPA:*

$$x_n = J_{\lambda_n}^f(x_{n-1}), \ n \geq 1, \tag{1.5}$$

where $J_{\lambda_n}^f$ is as defined in (1.4). Then, $\{x_n\}$ converges to a minimizer of f.

Kuwae [33] defined the p-resolvent mapping in a p-uniformly convex metric space slightly different from the one in (1.4) as follows:

$$J_\lambda^f(x) = \arg\min_{v \in X}\left(f(v) + \frac{1}{p\lambda^{p-1}}d(v,x)^p\right). \tag{1.6}$$

Clearly, (1.6) is more general than (1.4), and known to be applicable in obtaining solutions of initial boundary value problems for p-harmonic maps (see [33] for more details). Kuwae [33] also established the unique existence of the p-resolvent mapping (1.6) under Assumption 3.21 of [33] (see [33, Proposition 3.26]). Izuchukwu *et al.* [26] improved on this result by removing the Assumption 3.21 used in [33]. More precisely, Izuchukwu *et al.* [26] established the following result.

Theorem 1.2. *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for any $\lambda > 0$ and $x \in X$, there exists a unique point, say $J_\lambda^f(x) \in X$ such that*

$$f(J_\lambda^f(x)) + \frac{1}{p\lambda^{p-1}}d(J_\lambda^f(x),x)^p = \inf_{v \in X}\left(f(v) + \frac{1}{p\lambda^{p-1}}d(v,x)^p\right).$$

Furthermore, Izuchukwu *et al.* [26] introduced both the backward-backward algorithm and the alternating proximal algorithm, and proved that they converge to a minimizer of the sum of two proper convex and lower semicontinuous functions in the setting of complete p-uniformly convex metric spaces.

Motivated by the importance of p-uniformly convex metric spaces and the results obtained therein, we further investigate the study of PPAs in p-uniformly convex metric spaces. Furthermore, we introduce a Halpern-type PPA which comprises a nonexpansive mapping and a finite composition of resolvent mappings associated with proper convex and lower semicontinuous functions, and prove that it converges strongly to a common minimizer of a finite family of proper convex and lower semicontinuous functions, which is also a fixed point of a nonexpansive mapping in a complete p-uniformly convex metric space. To show the applicability of our results, we give numerical examples of our proposed algorithm in nonlinear settings.

This paper is organized as follows: In Section 2, we recall the geometry of geodesic spaces and the definitions of convex functions. We also give some remarks (and improve) on existing results on PPA in p-uniformly convex metric spaces. In Section 3, we prove several results needed in carrying out our strong convergence analysis. We then propose a Halpern-type PPA and prove that it converges strongly to a common solution of a finite family of MPs and a fixed point problem for a nonexpansive mapping in a complete p-uniformly convex metric space. In Section 4, we give numerical examples of our algorithm in nonlinear settings, to show the applicability of our main result. Finally, we summarize the significance of the paper in Section 5.

2. Preliminaries

2.1. Geometry of Geodesic Metric Spaces

Definition 2.1. A metric space X is called a geodesic space, if every two points $x, y \in X$ are joined by a geodesic path $c : [0, d(x, y)] \to X$ such that $c(0) = x$ and $c(d(x, y)) = y$. In this case, c is an isometry and the image of c is called a geodesic segment joining x to y. The space X is said to be uniquely geodesic, if every two points of X are joined by exactly one geodesic segment. Let $x, y \in X$ and $t \in [0, 1]$, we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = (1 - t)d(x, y) \text{ and } d(y, z) = td(x, y). \tag{2.1}$$

Remark 2.2. p-uniformly convex metric spaces are uniquely geodesic spaces (see [26]).

Definition 2.3. Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \{\bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v\in X} \limsup_{n\to\infty} d(v, x_n)\}.$$

The sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$, if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = \bar{v}$ and we say that $\bar{v}$ is the Δ-limit of $\{x_n\}$. The notion of Δ-convergence in a metric space was introduced by Lim [36], and it is known as an analogue of the notion of weak convergence in a Banach space. Thus, it is sometimes referred to as the notion of weak convergence in metric spaces.

Definition 2.4. [8] Let X be a complete convex metric space and $T : X \to X$ be any nonlinear mapping. T is said to be Δ-demiclosed at 0, if for any bounded sequence $\{x_n\}$ in X such that $\Delta\text{-}\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $z = Tz$.

Definition 2.5. Let X be a geodesic space. A nonlinear mapping $T : X \to X$ is called nonexpansive, if

$$d(Tx, Ty) \leq d(x, y) \ \forall x, y \in X.$$

A point $x \in X$ is called a fixed point of a nonlinear mapping T, if $x = Tx$. Throughout this paper, we shall denote by $F(T)$, the set of fixed points of T.

Lemma 2.6. [58, Remark 2.4] *Let X be a complete p-uniformly convex metric space and $T : X \to X$ be a nonexpansive mapping, then T is Δ-demiclosed at 0.*

Definition 2.7. [53, 55] A continuous linear functional μ defined on l_∞ (where l_∞ is the Banach space of bounded real sequences) is called a Banach limit, if

$$||\mu|| = \mu(1, 1, \dots) = 1 \text{ and } \mu_n(a_n) = \mu_n(a_{n+1}) \ \forall a_n \in l_\infty.$$

Lemma 2.8. [53, 55] *Let $(a_1, a_2, \dots) \in l_\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limits μ, and $\limsup\limits_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then, $\limsup\limits_{n\to\infty} a_n \leq 0$.*

Definition 2.9. For $p > 1$, let X be a p-uniformly convex metric space with parameter $c > 0$. A mapping $T : X \to X$ is said to be a firmly nonexpansive-type mapping, if

$$d(Tx, Ty)^p \leq \frac{1}{c}\left[d(Tx, y)^p + d(Ty, x)^p - d(Tx, x)^p - d(Ty, y)^p\right] \ \forall x, y \in X. \quad (2.2)$$

Remark 2.10. The p-resolvent of a proper convex and lower semicontinuous function is a firmly nonexpansive-type mapping (see [26, Lemma 2.8]).

Remark 2.11. It follows from Lemma 2.19 that if T is a firmly nonexpansive-type mapping and $c \geq 2$, then T is nonexpansive. Indeed, for all $x, y \in X$, we obtain from (2.2) and Lemma 2.19 that

$$\begin{aligned} d(Tx, Ty)^p &\leq \frac{1}{c}\left(\frac{2}{c}\left(d(Tx, Ty)^p + d(Tx, x)^p + d(Ty, y)^p + d(x, y)^p\right) - d(Tx, x)^p - d(Ty, y)^p\right) \\ &\leq \frac{1}{2}\left(d(Tx, Ty)^p + d(x, y)^p\right), \end{aligned}$$

which implies that T is nonexpansive.

2.2. Convex Functions

Let X be a geodesic space. A mapping $f : D \subseteq X \to (-\infty, \infty]$ is called convex, if for any geodesic $[x, y] := \{tx \oplus (1-t)y : 0 \leq t \leq 1\}$ joining $x, y \in X$, we have that

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y).$$

We also recall that $f : D \subseteq X \to (-\infty, \infty]$ is called proper, if its domain $D := \{v \in X : f(v) < +\infty\} \neq \emptyset$, and f is said to be lower semi-continuous at a point $\bar{v} \in D$, if $f(\bar{v}) \leq \liminf\limits_{n\to\infty} f(x_n)$ for each sequence $\{x_n\}$ in D such that $\lim\limits_{n\to\infty} x_n = \bar{v}$.
Inequality (1.3) ensures that the function $x \mapsto d(., x)^p : X \to [0, \infty)$ is a convex and lower semicontinuous function.

2.3. Some Remarks on Proximal Point Algorithms in p-Uniformly Convex Metric Spaces

Lemma 2.12. [58, Lemma 3.1] *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Let J_λ^f be the p-resolvent mapping (1.4) such that $F(J_\lambda^f) \neq \emptyset$, then for $\lambda > 0$, we have the following:*

- *(a) $x^* \in F(J_\lambda^f)$ if and only if x^* is a minimizer of f;*
- *(b) $d(x^*, J_\lambda^f x)^p + d(J_\lambda^f x, x)^p \leq d(x^*, x)^p$ for all $x \in X$ and $x^* \in F(J_\lambda^f)$;*
- *(c) J_λ^f is a generalized quasi-nonexpansive mapping, i.e.,*

$$d(J_\lambda^f x, x^*)^p \leq d(x, x^*)^p, \ \forall x \in X, \ x^* \in F(J_\lambda^f).$$

Using Lemma 2.12, the authors in [58] proved the following theorem for approximating a common solution of a finite family of MPs, which is also a fixed point of a nonexpansive mapping in the framework of a complete p-uniformly convex metric space.

Theorem 2.13. [58, Theorem 3.3] *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let the p-resolvent $J_{\lambda^{(i)}}^{f_i}$ of f be Δ-demiclosed at 0 for each $i = 1, 2, \ldots, N$, and $T : X \to X$ be a nonexpansive mapping. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} f_i(y)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}}^{f_N} \circ J_{\lambda_n^{(N-1)}}^{f_{N-1}} \circ \cdots \circ J_{\lambda_n^{(2)}}^{f_2} \circ J_{\lambda_n^{(1)}}^{f_1}(x_n), \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{2.3}$$

where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \ldots, N$, $n \geq 1$ and $\{\alpha_n\}$ is a sequence in $[a, \ b]$, for some $a, b \in (0, 1)$. Then, $\{x_n\}$ Δ-converges to some $x^* \in \Gamma$.

Remark 2.14. We point out here that Theorem 2.13 can be established without the assumption that the p-resolvent $J_{\lambda^{(i)}}^{f_i}$ of f is Δ-demiclosed at 0 for each $i = 1, 2, \ldots, N$. One way to achieve this is to establish that the p-resolvent is nonexpansive rather than establishing that it is a generalized quasi-nonexpansive mapping as obtained in Lemma 2.12 (c) (see also the conclusion in [58, Page 8]). However, as observed by the authors in [58] (see Remark 3.5 of their paper for more details), unless f or the space X has some nicer properties which is mainly determined by the restrictions on p and/or the smoothness constant c, one may not get that the p-resolvent of f is nonexpansive. In fact, to extend existing results on PPA to p-uniformly convex metric spaces, a natural obstacle (among others) one has to overcome is the smoothness constant (parameter) $c \in (0, \infty)$.
When $c \geq 2$ as in the case of Theorem 2.13, the authors in [26] proved that the p-resolvent of f is nonexpansive. More precisely, they proved the following result.

Lemma 2.15. [26] *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, the p-resolvent mapping (1.6) (which is more general than (1.4)) is nonexpansive.*

With Lemma 2.15, Theorem 2.13 can be established without the demiclosedness assumption on the p-resolvent of f. Precisely, we have the following theorem.

Theorem 2.16. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} f_i(y)\right) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{2.4}$$

where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \ldots, N$, $n \geq 1$ and $\{\alpha_n\}$ is a sequence in $[a, \ b]$, for some $a, b \in (0, 1)$. Then, $\{x_n\}$ Δ-converges to some $x^* \in \Gamma$.

Proof. By Lemma 2.15, we obtain that $J^{f_i}_{\lambda^{(i)}}$ is nonexpansive for each $i = 1, 2, \ldots, N$. Thus, by Lemma 2.6, $J^{f_i}_{\lambda^{(i)}}$ is Δ-demiclosed at 0 for each $i = 1, 2, \ldots, N$. Therefore, the desired conclusion follows from Theorem 2.13. ∎

Remark 2.17. We also like to point out here that, in infinite dimensional spaces, strong convergence results are often much more desirable than Δ-convergence results. Moreover, strong convergence implies Δ-convergence but the converse is not always true. For example, see [52] for the question of interest raised by Rockafellar as to whether the PPA can be improved from weak convergence (an analogue of Δ-convergence) to strong convergence in Hilbert space settings. Several counterexamples have been provided to establish that the PPA do not always converge strongly without additional assumptions on either the underlying space or the proper convex and lower semicontinuous function (see [13, 14, 29]).
Therefore, it is our intention to further improve Theorem 2.13 from a Δ-convergence result to the following strong convergence result by modifying Algorithm (2.3) into a Halpern-type algorithm.

Theorem 2.18. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} f_i(y)\right) \neq \emptyset$. For arbitrary $u, x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) \left[\beta_n x_n \oplus (1 - \beta_n) T y_n\right], \ n \geq 1, \end{cases} \tag{2.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence in $(0, \infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that

(i) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a \leq \beta_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to $P_\Gamma u$, where P_Γ is the nearest point map (projection) of X onto Γ.

Before we can give the proof of Theorem 2.18, we shall first establish in the next section, some important results needed in proving it.

We now end this section by recalling the following important lemmas which will be very useful in the proof of our main results.

Lemma 2.19. [26, Lemma 2.4] *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c > 0$. Then, for all $w, x, y, z \in X$, we have*

$$d(w,x)^p + d(y,z)^p \leq \frac{2}{c}\left(d(w,y)^p + d(w,z)^p + d(x,y)^p + d(x,z)^p\right).$$

Lemma 2.20. [26, Lemma 2.11] *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for $0 < \lambda_1 < \lambda_2$, we have*

$$d(J^f_{\lambda_1}x, x) \leq d(J^f_{\lambda_2}x, x) \ \forall x \in X.$$

Lemma 2.21. [8, 22]. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c > 0$. Then,*

(i) every bounded sequence in X has a unique asymptotic center,
(ii) every bounded sequence in X has a Δ-convergent subsequence.

Lemma 2.22. [54] *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with condition $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \ n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \geq 0$, then, $\lim_{n\to\infty} a_n = 0$.

3. Main Results

For the rest of this paper, we shall use the p-resolvent mapping (1.6) in all our analysis.

Lemma 3.1. *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for $0 < \lambda_1 \leq \lambda_2$, we have*

$$d(J^f_{\lambda_1}x, J^f_{\lambda_2}x) \leq \left[\frac{2}{c}\left(1 - \frac{\lambda_1^{p-1}}{\lambda_2^{p-1}}\right)\right]^{\frac{1}{p}} d(x, J^f_{\lambda_2}x) \ \forall x \in X.$$

Proof. From (1.6), we obtain that

$$f(J^f_{\lambda_2}x) + \frac{1}{p{\lambda_2}^{p-1}}d(J^f_{\lambda_2}x, x)^p \leq f(v) + \frac{1}{p{\lambda_2}^{p-1}}d(v, x)^p \ \forall v \in X.$$

Let $v = (1-t)J^f_{\lambda_1}x \oplus tJ^f_{\lambda_2}x, \ t \in [0, 1)$. Then, we obtain from the convexity of f and the inequality (1.3) that

$$\begin{aligned} f(J^f_{\lambda_2}x) + \frac{1}{p{\lambda_2}^{p-1}}d(J^f_{\lambda_2}x, x)^p \ &\leq \ (1-t)f(J^f_{\lambda_1}x) + tf(J^f_{\lambda_2}x) + \frac{(1-t)}{p{\lambda_2}^{p-1}}d(J^f_{\lambda_1}x, x)^p \\ &\quad + \frac{t}{p{\lambda_2}^{p-1}}d(J^f_{\lambda_2}x, x)^p - \frac{ct(1-t)}{2p{\lambda_2}^{p-1}}d(J^f_{\lambda_1}x, J^f_{\lambda_2}x)^p, \end{aligned}$$

which implies that

$$\begin{aligned} f(J^f_{\lambda_2}x) + \frac{1}{p{\lambda_2}^{p-1}} d(J^f_{\lambda_2}x, x)^p &\leq f(J^f_{\lambda_1}x) + \frac{1}{p{\lambda_2}^{p-1}} d(J^f_{\lambda_1}x, x)^p \\ &- \frac{ct}{2p{\lambda_2}^{p-1}} d(J^f_{\lambda_1}x, J^f_{\lambda_2}x)^p. \end{aligned} \tag{3.1}$$

Letting $t \to 1$ in (3.1), we obtain that

$$\frac{c}{2p{\lambda_2}^{p-1}} d(J^f_{\lambda_1}x, J^f_{\lambda_2}x)^p \leq f(J^f_{\lambda_1}x) - f(J^f_{\lambda_2}x) + \frac{1}{p{\lambda_2}^{p-1}} \left[d(J^f_{\lambda_1}x, x)^p - d(J^f_{\lambda_2}x, x)^p \right]. \tag{3.2}$$

Similarly, we obtain that

$$\frac{c}{2p{\lambda_1}^{p-1}} d(J^f_{\lambda_2}x, J^f_{\lambda_1}x)^p \leq f(J^f_{\lambda_2}x) - f(J^f_{\lambda_1}x) + \frac{1}{p{\lambda_1}^{p-1}} \left[d(J^f_{\lambda_2}x, x)^p - d(J^f_{\lambda_1}x, x)^p \right]. \tag{3.3}$$

Adding (3.2) and (3.3), and noting that $\lambda_1 \leq \lambda_2$, we obtain that

$$\begin{aligned} \frac{c}{2p}\left(\frac{1}{{\lambda_1}^{p-1}} + \frac{1}{{\lambda_2}^{p-1}}\right) d(J^f_{\lambda_1}x, J^f_{\lambda_2}x)^p &\leq \frac{1}{p}\left(\frac{1}{{\lambda_2}^{p-1}} - \frac{1}{{\lambda_1}^{p-1}}\right) d(J^f_{\lambda_1}x, x)^p \\ &+ \frac{1}{p}\left(\frac{1}{{\lambda_1}^{p-1}} - \frac{1}{{\lambda_2}^{p-1}}\right) d(J^f_{\lambda_2}x, x)^p \\ &\leq \frac{1}{p}\left(\frac{1}{{\lambda_1}^{p-1}} - \frac{1}{{\lambda_2}^{p-1}}\right) d(J^f_{\lambda_2}x, x)^p, \end{aligned}$$

which after further simplification implies that

$$d(J^f_{\lambda_1}x, J^f_{\lambda_2}x) \leq \left[\frac{2}{c}\left(1 - \frac{\lambda_1^{p-1}}{\lambda_2^{p-1}}\right)\right]^{\frac{1}{p}} d(x, J^f_{\lambda_2}x).$$

■

Lemma 3.2. *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $S : X \to X$ be a nonexpansive mapping. Let $u \in X$ be fixed, then for each $t, t' \in (0,1)$, the mapping $f_{tt'} : X \to X$ defined by*

$$f_{tt'}x = tu \oplus (1-t)\Big[t'x \oplus (1-t')Sx\Big] \ \forall x \in X, \tag{3.4}$$

has a unique fixed point $x_{tt'} \in X$, that is

$$x_{tt'} = f_{tt'}x_{tt'} = tu \oplus (1-t)\Big[t'x_{tt'} \oplus (1-t')Sx_{tt'}\Big]. \tag{3.5}$$

Proof. From (3.4), (1.3) and noting that $c \geq 2$, we obtain

$$\begin{aligned}
&d(f_{tt'}x, f_{tt'}y)^p \\
&\leq td(f_{tt'}x, u)^p + (1-t)d(f_{tt'}x, t'y \oplus (1-t')Sy)^p - \frac{c}{2}t(1-t)d(u, t'y \oplus (1-t')Sy)^p \\
&\leq t^2 d(u,u)^p + t(1-t)d(t'x \oplus (1-t')Sx, u)^p - \frac{c}{2}t^2(1-t)d(u, t'x \oplus (1-t')Sx)^p \\
&+ t(1-t)d(u, t'y \oplus (1-t')Sy)^p + (1-t)^2 d(t'x \oplus (1-t')Sx, t'y \oplus (1-t')Sy)^p \\
&- \frac{c}{2}t(1-t)^2 d(u, t'x \oplus (1-t')Sx)^p - \frac{c}{2}t(1-t)d(u, t'y \oplus (1-t')Sy)^p \\
&= \Big[t(1-t) - \frac{c}{2}t^2(1-t) - \frac{c}{2}t(1-t)^2\Big] d(u, t'x \oplus (1-t')Sx)^p \\
&+ \Big[t(1-t) - \frac{c}{2}t(1-t)\Big] d(u, t'y \oplus (1-t')Sy)^p \\
&+ (1-t)^2 d(t'x \oplus (1-t')Sx, t'y \oplus (1-t')Sy)^p \\
&\leq (1-t)^2 d(t'x \oplus (1-t')Sx, t'y \oplus (1-t')Sy)^p. \qquad (3.6)
\end{aligned}$$

Again, using (1.3) and noting that $c \geq 2$, we obtain

$$\begin{aligned}
&d(t'x \oplus (1-t')Sx, t'y \oplus (1-t')Sy)^p \\
&\leq t'd(x, t'y \oplus (1-t')Sy)^p + (1-t')d(Sx, t'y \oplus (1-t')Sy)^p - \frac{c}{2}t'(1-t')d(x, Sx)^p \\
&\leq t'^2 d(x,y)^p + t'(1-t')d(x, Sy)^p - \frac{c}{2}t'^2(1-t')d(y, Sy)^p + t'(1-t')d(Sx, y)^p \\
&+ (1-t')^2 d(Sx, Sy)^p - \frac{c}{2}t'(1-t')^2 d(y, Sy)^p - \frac{c}{2}t'(1-t')d(x, Sx)^p \\
&\leq t'^2 d(x,y)^p + t'(1-t')d(x, Sy)^p + t(1-t')d(Sx, y)^p + (1-t')^2 d(Sx, Sy)^p \\
&- t'(1-t')d(y, Sy)^p - t'(1-t')d(x, Sx)^p \\
&= t'^2 d(x,y)^p + t'(1-t')\Big[d(x, Sy)^p + d(Sx, y)^p - d(y, Sy)^p - d(x, Sx)^p\Big] \qquad (3.7) \\
&+ (1-t')^2 d(Sx, Sy)^p.
\end{aligned}$$

By the nonexpansivity of S and (1.3), we obtain

$$\begin{aligned}
d(Sx, Sy)^p &\leq d(x,y)^p \\
&\leq d(x,y)^p + 4t'd\Big(\frac{1}{2}x \oplus \frac{1}{2}Sy, \frac{1}{2}Sx \oplus \frac{1}{2}y\Big)^p \\
&\leq d(x,y)^p + t'\Big[d(x,y) + d(Sx,Sy) + d(x,Sx) + d(y,Sy) - d(x,Sy) - d(y,Sx)\Big],
\end{aligned}$$

which implies that

$$d(Sx, Sy)^p \leq \frac{1+t'}{1-t'}d(x,y)^p + \frac{t'}{1-t'}\Big[d(x,Sx)^p + d(y,Sy)^p - d(x,Sy)^p - d(y,Sx)^p\Big]. \qquad (3.8)$$

Combining (3.7) and (3.8), we obtain

$$\begin{aligned}
&d(t'x \oplus (1-t')Sx, t'y \oplus (1-t')Sy)^p \\
&\le t'^2 d(x,y)^p + t'(1-t')\Big[d(x,Sy)^p + d(Sx,y)^p - d(y,Sy)^p - d(x,Sx)^p\Big] \\
&+ (1-t')^2\Big[\frac{1+t'}{1-t'}d(x,y)^p + \frac{t'}{1-t'}\Big(d(x,Sx)^p + d(y,Sy)^p - d(x,Sy)^p - d(y,Sx)^p\Big)\Big] \\
&= d(x,y)^p + t'(1-t')\Big[d(x,Sy)^p + d(Sx,y)^p - d(y,Sy)^p - d(x,Sx)^p\Big] \\
&- t'(1-t')\Big[d(x,Sy)^p + d(y,Sx)^p - d(y,Sy)^p - d(x,Sx)^p\Big] \\
&= d(x,y)^p. \qquad (3.9)
\end{aligned}$$

Using this in (3.6), we obtain

$$d(f_{tt'}x, f_{tt'}y)^p \le (1-t)^2 d(x,y)^p.$$

Thus, $f_{tt'}$ is a contraction with constant $(1-t)^{\frac{2}{p}}$, and by Banach contraction mapping principle, we obtain the desired conclusion. ∎

Lemma 3.3. *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c \ge 2$ and $S : X \to X$ be a nonexpansive mapping. Then $F(S) \neq \emptyset$ if and only if $\{x_{tt'}\}$ defined by (3.5) is bounded as $t \to 0$ (where $t' \in [a,b] \subset (0,1)$). Furthermore, we have the following*

(i) $\{x_{tt'}\}$ converges to $v = P_{F(S)}u$, where $P_{F(S)}$ is the the nearest point map (projection) of X onto $F(S)$.

(ii) $d(u,v)^p \le \frac{2}{c}\mu_n\left(d(u,x_n)^p\right)$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with $\{x_n - Sx_n\}$ converging strongly to 0.

Proof. Let $F(S) \neq \emptyset$, then it is easy to show that $\{x_{tt'}\}$ is bounded. Now suppose that $\{x_{tt'}\}$ is bounded, we prove that $F(S) \neq \emptyset$. Let $\{t_n\}$ and $\{t'_n\}$ be sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and $0 < a \le t_n \le b < 1$. Then, $\{x_{t_nt'_n}\}$ is bounded. Thus, by Lemma 2.21 (i), there exists $v \in X$ such that $A(\{x_{t_nt'_n}\}) = \{v\}$, that is

$$\limsup_{n\to\infty} d(v, x_{t_nt'_n}) = \inf_{y\in X}\limsup_{n\to\infty} d(y, x_{t_nt'_n}). \qquad (3.10)$$

From (2.1) and (3.9), we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} d(x_{t_nt'_n},\ t'_nv \oplus (1-t'_n)Sv)^p \\
\le\ &\limsup_{n\to\infty}\Big(d(x_{t_nt'_n},\ t'_nx_{t_nt'_n} \oplus (1-t'_n)Sx_{t_nt'_n}) \\
&+d(t'_nx_{t_nt'_n} \oplus (1-t'_n)Sx_{t_nt'_n},\ t'_nv \oplus (1-t'_n)Sv)\Big)^p \\
=\ &\limsup_{n\to\infty}\Big(t_nd(u, t'_nx_{t_nt'_n} \oplus (1-t'_n)Sx_{t_nt'_n}) \\
&+d(t'_nx_{t_nt'_n} \oplus (1-t'_n)Sx_{t_nt'_n},\ t'_nv \oplus (1-t'_n)Sv)\Big)^p \\
=\ &\limsup_{n\to\infty} d\Big(t'_nx_{t_nt'_n} \oplus (1-t'_n)Sx_{t_nt'_n},\ t'_nv \oplus (1-t'_n)Sv\Big)^p \\
\le\ &\limsup_{n\to\infty} d(x_{t_nt'_n},\ v)^p. \qquad (3.11)
\end{aligned}$$

Setting $t = \frac{1}{2}$ in (1.3), and $h_n v = t'_n v \oplus (1-t'_n)Sv$, we obtain

$$d\Big(x_{t_n t'_n}, \frac{1}{2}h_n v \oplus \frac{1}{2}v\Big)^p \leq \frac{1}{2}d(x_{t_n t'_n}, h_n v)^p + \frac{1}{2}d(x_{t_n t'_n}, v)^p - \frac{c}{8}d(h_n v, v)^p. \tag{3.12}$$

Since $A\Big(\{x_{t_n t'_n}\}\Big) = \{v\}$, then by setting $y = \frac{1}{2}h_n v \oplus \frac{1}{2}v$ in (3.10), we obtain from (3.12) that

$$\begin{aligned}\limsup_{n\to\infty} d(x_{t_n t'_n}, v)^p &\leq \limsup_{n\to\infty} d(x_{t_n t'_n}, \ \frac{1}{2}h_n v \oplus \frac{1}{2}v)^p \\ &\leq \frac{1}{2}\limsup_{n\to\infty} d(x_{t_n t'_n}, h_n v)^p + \frac{1}{2}\limsup_{n\to\infty} d(x_{t_n t'_n}, v)^p \\ &- \frac{c}{8}\limsup_{n\to\infty} d(h_n v, v)^p,\end{aligned}$$

which implies from (3.11) that

$$\begin{aligned}\limsup_{n\to\infty} d(h_n v, v)^p &\leq 2\limsup_{n\to\infty} d(x_{t_n t'_n}, h_n v)^p - 2\limsup_{n\to\infty} d(x_{t_n t'_n}, v)^p \\ &\leq 2\limsup_{n\to\infty} d(x_{t_n t'_n}, v)^p - 2\limsup_{n\to\infty} d(x_{t_n t'_n}, v)^p = 0. \end{aligned}\tag{3.13}$$

Combining (2.1) and (3.13), we obtain

$$\limsup_{n\to\infty}(1-t'_n)^p d(Sv, v)^p = \limsup_{n\to\infty} d(h_n v, v)^p \leq 0.$$

Since $t_n \leq b$, we obtain that

$$(1-b)^p d(Sv, v)^p \leq \limsup_{n\to\infty}(1-t'_n)^p d(Sv, v)^p \leq 0.$$

Hence, $v \in F(S)$. Therefore, $F(S) \neq \emptyset$.

We now give the proofs of (i) and (ii).
(i) Let $v = P_{F(S)}u$, then $\{x_{tt'}\}$ is bounded. Since $c \geq 2$, we obtain from (3.5) and (1.3) that

$$\begin{aligned}d(v, x_{tt'})^p &\leq d\Big(v, tu \oplus (1-t)\Big[t'x_{tt'} \oplus (1-t')Sx_{tt'}\Big]\Big)^p \\ &\leq td(v,u)^p + (1-t)d(v, t'x_{tt'} \oplus (1-t')Sx_{tt'})^p \\ &- \frac{c}{2}t(1-t)d(u, t'x_{tt'} \oplus (1-t')Sx_{tt'})^p \\ &\leq td(v,u)^p + (1-t)d(v, x_{tt'})^p - \frac{c}{2}t(1-t)d(u, t'x_{tt'} \oplus (1-t')Sx_{tt'})^p,\end{aligned}$$

which implies that

$$d(v, x_{tt'})^p \leq d(v,u)^p + (t-1)d(u, t'x_{tt'} \oplus (1-t')Sx_{tt'})^p. \tag{3.14}$$

Now let $\{t_m\}$ be any sequence in $(0,1)$ such that $t_m \to 0$, as $m \to \infty$. Since $\{x_{t_m t'_m}\}$ is bounded, it follows from Lemma 2.21 (ii) that there exists a subsequence $\{x_{t_{m_k} t'_{m_k}}\}$ of $\{x_{t_m t'_m}\}$ that Δ-converges to $q \in X$. Thus, by Lemma 2.21 (i), we obtain that $A\left(\{x_{t_m t'_m}\}\right) = \{q\}$. By the same argument as in (3.10)-(3.13), we obtain that $q \in F(S)$. Now, observe that $d(x_{t_m t'_m}, hx_{t_m t'_m}) \to 0$, as $m \to \infty$, where $hx_{t_m t'_m} = t'x_{t_m t'_m} \oplus$

$(1-t')Sx_{t_m t'_m}$. Thus, we may assume that the subsequence $\{hx_{t_{m_k} t'_{m_k}}\}$ of $\{hx_{t_m t'_m}\}$ Δ-converges to $q \in F(S)$ and

$$\lim_{k\to\infty} d(u, hx_{t_{m_k} t'_{m_k}})^p = \liminf_{m\to\infty} d(u, hx_{t_m t'_m})^p. \tag{3.15}$$

It then follows from the Δ-lower semicontinuity of $d(u,\cdot)^p$ and (3.15) that

$$d(u,q)^p \le \liminf_{k\to\infty} d(u, hx_{t_{m_k} t'_{m_k}})^p = \lim_{k\to\infty} d(u, hx_{t_{m_k} t'_{m_k}})^p = \liminf_{m\to\infty} d(u, hx_{t_m t'_m})^p. \tag{3.16}$$

Since $v = P_{F(S)}u$ and $q \in F(S)$, we have that $d(v,u) \le d(q,u)$. Thus, letting $d_{t_m t'_m} = d(v,u)^p + (t_m - 1)d(u, hx_{t_m t'_m})^p$, we obtain from (3.16) that

$$\begin{aligned}\limsup_{m\to\infty} d_{t_m t'_m} &= d(v,u)^p + \limsup_{m\to\infty}(-d(u, hx_{t_m t'_m})^p)\\ &\le d(q,u)^p - \liminf_{m\to\infty} d(u, hx_{t_m t'_m})^p \le 0.\end{aligned} \tag{3.17}$$

From (3.14) and (3.17), we obtain that $\limsup_{m\to\infty} d(v, x_{t_m t'_m})^p \le 0$. Therefore, $\lim_{m\to\infty} d(v, x_{t_m t'_m}) = 0$, and this implies that $\{x_{t_m t'_m}\}$ converges strongly to v. Hence, it follows that $\{x_{tt'}\}$ converges strongly to $v = P_{F(S)}u$.

(ii) The proof of (ii) is very similar to the proof in [53, Lemma 2.2 (2)]. However, we shall give the proof here for readers convenience.
Let $\{x_{t_m t'_m}\}$ be a sequence defined as in (3.5). Then by (i), $\lim_{m\to\infty} x_{t_m t'_m} = v$, where $v = P_{F(S)}u$.
From (3.5) and (1.3), we obtain that

$$\begin{aligned}&d(x_n, x_{t_m t'_m})^p\\ &\le t_m d(x_n,u)^p + (1-t_m)d(x_n, hx_{t_m t'_m})^p - \frac{c}{2}t_m(1-t_m)d(u, hx_{t_m t'_m})^p\\ &\le t_m d(x_n,u)^p + (1-t_m)\Big[d(x_n, hx_n) + d(hx_n, hx_{t_m t'_m})\Big]^p\\ &- \frac{c}{2}t_m(1-t_m)d(u, hx_{t_m t'_m})^p\\ &\le t_m d(x_n,u)^p + (1-t_m)\Big[(1-t'_n)d(x_n, Sx_n) + d(x_n, x_{t_m t'_m})\Big]^p\\ &- \frac{c}{2}t_m(1-t_m)d(u, hx_{t_m t'_m})^p.\end{aligned} \tag{3.18}$$

Since μ is a Banach limit, then (3.18) becomes

$$\begin{aligned}\mu_n\Big(d(x_n, x_{t_m t'_m})^p\Big) &\le t_m\mu_n\Big(d(x_n,u)^p\Big) + (1-t_m)\mu_n\Big(d(x_n, x_{t_m t'_m})^p\Big)\\ &- \frac{c}{2}t_m(1-t_m)d(u, hx_{t_m t'_m})^p,\end{aligned}$$

which implies that

$$\mu_n\Big(d(x_n, x_{t_m t'_m})^p\Big) \le \mu_n\Big(d(x_n,u)^p\Big) - \frac{c}{2}(1-t_m)d(u, hx_{t_m t'_m})^p. \tag{3.19}$$

By letting $m\to\infty$ in (3.19), we obtain

$$\mu_n\Big(d(x_n,v)^p\Big) \le \mu_n\Big(d(x_n,u)^p\Big) - \frac{c}{2}d(u,v)^p,$$

which gives the desired conclusion. ∎

Lemma 3.4. *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $S : X \to X$ be a nonexpansive mapping. Let $T_i : X \to X,\ i = 1, 2, \dots N$ be a finite family of firmly nonexpansive-type mappings such that $F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1) \neq \emptyset$. Then*

$$F(S \circ T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1) = F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1).$$

Proof. Clearly, $F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1) \subseteq F(S \circ T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1)$.
So, we only show that $F(S \circ T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1) \subseteq F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1)$.
Let $F_N = T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1$ and $F_0 = I$ (where I is the identity mapping on X), then for any $x \in F(S \circ F_N)$ and $y \in F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1)$, we have that

$$\begin{aligned} d(x, y)^p &= d(SF_N x, SF_N y)^p \\ &\leq d(F_N x, y)^p. \end{aligned} \tag{3.20}$$

Since $c \geq 2$ and T_N is a firmly nonexpansive-type mapping, we obtain from (2.2), (3.20) and Remark 2.11 that

$$\begin{aligned} d(T_N(F_{N-1}x), F_{N-1}x)^p &\leq d(F_{N-1}x, y)^p - d(T_N(F_{N-1}x), y)^p \\ &\leq d(x, y)^p - d(F_N x, y)^p \\ &\leq d(F_N x, y)^p - d(F_N x, y)^p, \end{aligned}$$

which implies

$$F_N x = F_{N-1}x. \tag{3.21}$$

By similar argument, we also obtain that

$$\begin{aligned} d(T_{N-1}(F_{N-2}x), F_{N-2}x)^p &\leq d(F_{N-2}x, y)^p - d(T_{N-1}(F_{N-2}x), y)^p \\ &\leq d(x, y)^p - d(F_{N-1}x, y)^p \\ &\leq d(F_N x, y)^p - d(F_N x, y)^p, \end{aligned}$$

which implies

$$F_{N-1}x = F_{N-2}x. \tag{3.22}$$

By repeating the same process as in (3.21)-(3.22), we obtain

$$F_N x = F_{N-1}x = F_{N-2}x = F_{N-3}x = \cdots = F_2 x = F_1 x = F_0 x = x. \tag{3.23}$$

From (3.23), we obtain

$$x = T_1 x. \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$x = F_2 x = T_2(T_1 x) = T_2 x. \tag{3.25}$$

By repeating the same process as in (3.24)-(3.25), we can show that

$$x = T_1 x = T_2 x = \cdots = T_{N-1}x = T_N x. \tag{3.26}$$

Also, from (3.23), we get

$$x = S(F_N x) = Sx. \tag{3.27}$$

Thus, we have from (3.26) and (3.27) that $F(S \circ T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1) \subseteq F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1)$. ■

We are now ready to give the proof of Theorem 2.18. For the sake of readers convenience, we restate the theorem here before giving the proof.

Theorem 3.5. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} \arg\min_{y \in X} f_i(y)\right) \neq \emptyset$. For arbitrary $u, x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n{}^{(N)}} \circ J^{f_{N-1}}_{\lambda_n{}^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n{}^{(2)}} \circ J^{f_1}_{\lambda_n{}^{(1)}}(x_n) \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left[\beta_n x_n \oplus (1-\beta_n) T y_n\right], \ n \geq 1, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence in $(0, \infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a \leq \beta_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$, where P_Γ is the nearest point map (projection) of X onto Γ.

Proof. First, we show that $\{x_n\}$ is bounded.
Set $w_n^{(i+1)} = J^{f_i}_{\lambda_n^{(i)}} w_n^{(i)}$, $i = 1, 2, \ldots, N$, where $w_n^{(1)} = x_n$, for all $n \geq 1$. Then, $w_n^{(2)} = J^{f_i}_{\lambda_n^{(1)}}(x_n)$, $w_n^{(3)} = J^{f_i}_{\lambda_n^{(2)}} \circ J^{f_i}_{\lambda_n^{(1)}}(x_n)$, $\ldots$, $w_n^{(N)} = J^{f_i}_{\lambda_n^{(N-1)}} \circ J^{f_i}_{\lambda_n^{(2)}} \circ J^{f_i}_{\lambda_n^{(1)}}(x_n)$ and $w_n^{(N+1)} = y_n$.
Now, let $v \in \Gamma$, then we obtain from (1.3), (3.28), Lemma 2.12 (a) and Lemma 2.15 that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(u,v)^p + (1-\alpha_n) d(\beta_n x_n \oplus (1-\beta_n) T y_n,\ v)^p \\ &\leq \alpha_n d(u,v)^p + (1-\alpha_n)\Big[\beta_n d(x_n, v)^p + (1-\beta_n) d(T y_n, v)^p \\ &\quad -\frac{c}{2}\beta_n(1-\beta_n) d(x_n, T y_n)^p\Big] && (3.29) \\ &\leq \alpha_n d(u,v)^p + (1-\alpha_n)\left[\beta_n d(x_n, v)^p + (1-\beta_n) d(w_n^{(N+1)}, v)^p\right] && (3.30) \\ &= \alpha_n d(u,v)^p + (1-\alpha_n)\left[\beta_n d(x_n, v)^p + (1-\beta_n) d(J^{f_N}_{\lambda_n^{(N)}} w_n^{(N)}, v)^p\right] \\ &\ \ \vdots \\ &\leq \alpha_n d(u,v)^p + (1-\alpha_n)\left[\beta_n d(x_n, v)^p + (1-\beta_n) d(x_n, v)^p\right] \\ &\leq \max\{d(u,v)^p, d(x_n, v)^p\}, \end{aligned}$$

which implies by induction that

$$d(x_n, v)^p \leq \max\{d(u,v)^p, d(x_1, v)^p\} \ \forall n \geq 1. \tag{3.31}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ and $\{T y_n\}$ are also bounded.

From Lemma 2.12(b), we obtain for each $i = 1, 2, \dots, N$ that

$$d(w_n^{(i+1)}, v)^p \leq d(w_n^{(i)}, v)^p - d(w_n^{(i)}, w_n^{(i+1)})^p. \tag{3.32}$$

Let $z_n = \beta_n x_n \oplus (1-\beta_n)Ty_n$. Then from (1.3), we obtain

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(u,v)^p + (1-\alpha_n)d(z_n, v)^p - \frac{c}{2}\alpha_n(1-\alpha_n)d(u, z_n)^p \\ &\leq \alpha_n d(u,v)^p + (1-\alpha_n)d(x_n, v)^p - \frac{c}{2}\alpha_n(1-\alpha_n)d(u, z_n)^p \\ &= (1-\alpha_n)d(x_n, v)^p + \alpha_n\Big(d(u,v)^p - \frac{c}{2}(1-\alpha_n)d(u, z_n)^p\Big) \\ &= (1-\alpha_n)d(x_n, v)^p + \alpha_n d_n, \end{aligned} \tag{3.33}$$

where $d_n = d(u,v)^p - \frac{c}{2}(1-\alpha_n)d(u,z_n)^p$.
According to Lemma 2.22, to conclude the proof, it suffices to show that $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{d(x_{n_k}, v)^p\}$ of $\{d(x_n, v)^p\}$ satisfying the condition:

$$\liminf_{k\to\infty} \Big(d(x_{n_{k+1}}, v)^p - d(x_{n_k}, v)^p\Big) \geq 0. \tag{3.34}$$

Now, to show that $\limsup_{k\to\infty} d_{n_k} \leq 0$, let $\{d(x_{n_k}, v)^p\}$ be a subsequence of $\{d(x_n, v)\}$ such that (3.34) holds. Then by setting $i = N$ in (3.32), we obtain from (3.30) that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(u,v)^p + (1-\alpha_n)\beta_n d(x_n, v)^p + (1-\alpha_n)(1-\beta_n)d(w_n^{(N+1)}, v)^p \\ &\leq \alpha_n d(u,v)^p + (1-\alpha_n)\beta_n d(x_n, v)^p + (1-\alpha_n)(1-\beta_n)d(w_n^{(N)}, v)^p \\ &\quad -(1-\alpha_n)(1-\beta_n)d(w_n^{(N)}, w_n^{(N+1)})^p \\ &\leq \alpha_n d(u,v)^p + (1-\alpha_n)d(x_n, v)^p - (1-\alpha_n)(1-\beta_n)d(w_n^{(N)}, w_n^{(N+1)})^p, \end{aligned}$$

which together with (3.34) imply that

$$\begin{aligned} &\limsup_{k\to\infty}\Big[(1-\alpha_{n_k})(1-\beta_{n_k})d(w_{n_k}^{(N)}, w_{n_k}^{(N+1)})^p\Big] \\ \leq\ &\limsup_{k\to\infty}\Big[\Big(d(x_{n_k}, v)^p - d(x_{n_{k+1}}, v)^p\Big) + \alpha_{n_k}\left(d(u,v)^p - d(x_{n_k}, v)^p\right)\Big] \\ =\ &-\liminf_{k\to\infty}\Big[d(x_{n_{k+1}}, v)^p - d(x_{n_k}, v)^p\Big] \\ \leq\ &0. \end{aligned}$$

Hence, by conditions (i) and (ii), we obtain

$$\lim_{k\to\infty} d(w_{n_k}^{(N)}, w_{n_k}^{(N+1)}) = 0. \tag{3.35}$$

Also, by setting $i = N-1$ in (3.32), and following the steps as above, we obtain

$$\lim_{k\to\infty} d(w_{n_k}^{(N-1)}, w_{n_k}^{(N)}) = 0. \tag{3.36}$$

Repeating the same process, we can show that

$$\lim_{k\to\infty} d(w_{n_k}^{(i)}, w_{n_k}^{(i+1)}) = 0,\ \ i = 1, 2, \dots, N-2. \tag{3.37}$$

This, together with (3.35) and (3.36), gives

$$\lim_{k\to\infty} d(w_{n_k}^{(i)}, w_{n_k}^{(i+1)}) = 0,\ \ i = 1, 2, \dots, N. \tag{3.38}$$

From (3.38), and applying triangle inequality, we obtain for each $i = 1, 2, \ldots, N+1$, that

$$\lim_{k\to\infty} d(x_{n_k}, w_{n_k}^{(i)}) = \lim_{k\to\infty} d(w_{n_k}^{(1)}, w_{n_k}^{(i)}) = 0. \tag{3.39}$$

Since $0 < \lambda^{(i)} \leq \lambda_n^{(i)}$ for all $n \geq 1$, we obtain from Lemma 2.20 and (3.38) that

$$d\left(w_{n_k}^{(i)}, J_{\lambda^{(i)}}^{f_i} w_{n_k}^{(i)}\right) \quad \leq \quad d\left(w_{n_k}^{(i)}, J_{\lambda_{n_k}^{(i)}}^{f_i} w_{n_k}^{(i)}\right) \to 0, \text{ as } k \to \infty, \; i = 1, 2, \ldots, N. \tag{3.40}$$

Furthermore, we obtain from (3.29) that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(u, v)^p + (1-\alpha_n) d(x_n, v)^p - \frac{c}{2}(1-\alpha_n)\beta_n(1-\beta_n) d(x_n, Ty_n)^p \\ &\leq \alpha_n d(u, v)^p + d(x_n, v)^p - \frac{c}{2}\beta_n(1-\alpha_n)(1-\beta_n) d(x_n, Ty_n)^p. \end{aligned}$$

This together with (3.34) implies

$$\begin{aligned} & \limsup_{k\to\infty} \left[\frac{c}{2}\beta_{n_k}(1-\alpha_{n_k})(1-\beta_{n_k}) d(x_{n_k}, Ty_{n_k})^p\right] \\ \leq \quad & \limsup_{k\to\infty} \left[\alpha_{n_k} d(u, v)^p + d(x_{n_k}, v)^p - d(x_{n_{k+1}}, v)^p\right] \\ = \quad & -\liminf_{k\to\infty} \left(d(x_{n_{k+1}}, v)^p - d(x_{n_k}, v)^p\right) \leq 0. \end{aligned}$$

This gives

$$\lim_{k\to\infty} d(x_{n_k}, Ty_{n_k}) = 0. \tag{3.41}$$

From (2.1), we get

$$d(x_{n_{k+1}}, z_{n_k}) = d(\alpha_{n_k} u \oplus (1-\alpha_{n_k}) z_{n_k}, z_{n_k}) = \alpha_{n_k} d(u, z_{n_k}) \to 0, \text{ as } k \to \infty. \tag{3.42}$$

Similarly, we get that

$$d(z_{n_k}, x_{n_k}) = (1-\beta_{n_k}) d(Ty_{n_k}, x_{n_k}) \to 0, \text{ as } k \to \infty. \tag{3.43}$$

Setting $i = N+1$ in (3.39), we obtain that

$$\lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = 0. \tag{3.44}$$

From (3.42) and (3.43), we get

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 0. \tag{3.45}$$

Again, we obtain from (3.39), (3.40) and (3.41) that

$$\begin{aligned} d(x_{n_k}, TJ_{\lambda^{(i)}}^{f_i} x_{n_k}) \quad &\leq \quad d(x_{n_k}, Ty_{n_k}) + d(Ty_{n_k}, TJ_{\lambda^{(i)}}^{f_i} x_{n_k}) \\ &\leq \quad d(x_{n_k}, Ty_{n_k}) + d(w_{n_k}^{(N+1)}, J_{\lambda^{(i)}}^{f_i} w_{n_k}^{(1)}) \\ &\leq \quad d(x_{n_k}, Ty_{n_k}) + d(w_{n_k}^{(N+1)}, w_{n_k}^{(1)}) + d(w_{n_k}^{(1)}, w_{n_k}^{(i)}) \\ &\qquad + d(w_{n_k}^{(i)}, J_{\lambda^{(i)}}^{f_i} w_{n_k}^{(i)}) + d(J_{\lambda^{(i)}}^{f_i} w_{n_k}^{(i)}, J_{\lambda^{(i)}}^{f_i} w_{n_k}^{(1)}) \\ &\leq \quad d(x_{n_k}, Ty_{n_k}) + d(w_{n_k}^{(N+1)}, w_{n_k}^{(1)}) + 2d(w_{n_k}^{(1)}, w_{n_k}^{(i)}) \\ &\qquad + d(w_{n_k}^{(i)}, J_{\lambda^{(i)}}^{f_i} w_{n_k}^{(i)}) \to 0, \text{ as } k \to \infty. \end{aligned} \tag{3.46}$$

We now show that $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$.
Let $\bar{v} = \lim\limits_{t\to 0} x_{tt'}$, where $\{x_{tt'}\}$ is as defined in (3.5) with $S = T \circ J_{\lambda^{(N)}}^{f_N} \circ J_{\lambda^{(N-1)}}^{f_{N-1}} \circ \cdots \circ J_{\lambda^{(1)}}^{f_1}$. Then, by Lemma 3.3, Lemma 3.4 and Remark 2.10, we obtain that $\bar{v} = P_{F(S)} u = P_\Gamma u$.

Replacing v by $\bar{v}$ and using Lemma 3.3 (ii), we obtain that $\mu_{n_k}(d(u,\bar{v})^p - \frac{2}{c}d(u,x_{n_k})^p) \leq 0$ for all Banach limits μ. Also, we obtain from (3.45) that

$$\limsup_{k\to\infty}\left(\left[d(u,\bar{v})^p - \frac{2}{c}d(u,x_{n_{k+1}})^p\right] - \left[d(u,\bar{v})^p - \frac{2}{c}d(u,x_{n_k})^p\right]\right) \leq 0.$$

Hence, it follows from Lemma 2.8 that

$$\limsup_{k\to\infty}\left(d(u,\bar{v})^p - \frac{2}{c}d(u,x_{n_k})^p\right) \leq 0. \tag{3.47}$$

From (3.43) and (3.47), we obtain that

$$\limsup_{k\to\infty} d_{n_k} = \limsup_{k\to\infty}\left(d(u,\bar{v})^p - \frac{c}{2}(1-\alpha_{n_k})d(u,z_{n_k})^p\right) \leq 0.$$

Thus, replacing v by $\bar{v}$ in (3.33) and applying Lemma 2.22 to (3.33), we get $\lim\limits_{n\to\infty} d(x_n,\bar{v}) = 0$. Therefore, $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$. ■

The following corollary of Theorem 3.5 generalizes the results of [57, Theorem 3.1] and [53, Theorem 2.3] in CAT(0) spaces.

Corollary 3.6. *Let X be a complete 2-uniformly convex metric space with parameter $c = 2$ (in particular, a complete CAT(0) space) and $f_i : X \to (-\infty,\infty]$, $i = 1,2,\ldots,N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y\in X} f_i(y)\right) \neq \emptyset$. For arbitrary $u, x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left[\beta_n x_n \oplus (1-\beta_n)Ty_n\right], \ n \geq 1, \end{cases} \tag{3.48}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n^{(i)}\}$, $i = 1,2,\ldots,N$ is a sequence in $(0,\infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < a \leq \beta_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$, where P_Γ is the nearest point map (projection) of X onto Γ.

By setting $T \equiv I$ in Theorem 3.5 (where I is the identity mapping on X), we obtain the following result which is similar to the result of [26, Theorem 3.15] and generalizes the result of [26, Theorem 3.13].

Corollary 3.7. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty,\infty]$, $i = 1,2,\ldots,N$ be a finite family of proper convex and lower semicontinuous functions. Let $\Gamma := \cap_{i=1}^N \arg\min_{y\in X} f_i(y) \neq \emptyset$. For arbitrary $u, x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n)\left[\beta_n x_n \oplus (1-\beta_n)y_n\right], \ n \geq 1, \end{cases} \tag{3.49}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n^{(i)}\}$, $i = 1,2,\ldots,N$ is a sequence in $(0,\infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that *(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < a \leq \beta_n \leq b < 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{v} = P_\Gamma u$*, where* P_Γ *is the nearest point map (projection) of* X *onto* Γ*.*

4. Numerical Examples

In this section, we present two numerical examples of Algorithm (3.5) (in comparison with the methods studied by Okeke and Izuchukwu [49] and Suparatulatorn *et al.* [57]) in an Hadamard space and in a p-uniformly convex metric space, to show its applicability and advantage. Throughout this section, we shall take $\alpha_n = \frac{1}{3n+1}$, $\beta_n = \frac{n}{100n+1}$ and $\lambda_n^{(i)} = \frac{in+1}{in}$, $i = 1, 2, 3, 4$, for all $n \geq 1$. Hence, Algorithm (3.28) becomes

$$\begin{cases} \bar{z}_n = \arg\min_{v\in X} \left(f_1(v) + \left(\frac{1}{p\lambda_n^{p-1}}\right) d_X(v, x_n)^p \right), \\ w_n = \arg\min_{v\in X} \left(f_2(v) + \left(\frac{1}{p\lambda_n^{p-1}}\right) d_X(v, z_n)^p \right), \\ v_n = \arg\min_{v\in X} \left(f_3(v) + \left(\frac{1}{p\lambda_n^{p-1}}\right) d_X(v, w_n)^p \right), \\ y_n = \arg\min_{v\in X} \left(f_4(v) + \left(\frac{1}{p\lambda_n^{p-1}}\right) d_X(v, v_n)^p \right), \\ x_{n+1} = \frac{u}{3n+1} \oplus (\frac{3n}{3n+1}) \left[(\frac{n}{100n+1}) x_n \oplus (\frac{99n+1}{100n+1}) T y_n \right], \ n \geq 1, \end{cases} \tag{4.1}$$

the algorithm studied by Okeke and Izuchukwu [49] becomes

$$\begin{cases} w_n = \arg\min_{v\in X} \left(f(v) + \left(\frac{1}{p\lambda^{p-1}}\right) d_X(v, x_n)^p \right), \\ y_n = \arg\min_{v\in X} \left(f(v) + \left(\frac{1}{p\lambda^{p-1}}\right) d_X(v, w_n)^p \right), \\ x_{n+1} = \frac{u}{3n+1} \oplus (\frac{3n}{3n+1}) T y_n, \ n \geq 1, \end{cases} \tag{4.2}$$

and the algorithm studied by Suparatulatorn *et al.* [57] becomes

$$\begin{cases} y_n = \arg\min_{v\in X} \left(f(v) + \left(\frac{1}{p\lambda_n^{p-1}}\right) d_X(v, x_n)^p \right), \\ x_{n+1} = \frac{u}{3n+1} \oplus (\frac{3n}{3n+1}) T y_n, \ n \geq 1. \end{cases} \tag{4.3}$$

Example 4.1. Here, we present a numerical example in an Hadamard space to illustrate the performance of our algorithm. Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by

$$d_X(x, y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \ \forall x, y \in \mathbb{R}^2.$$

Then, $(\mathbb{R}^2, d_X)$ is a complete CAT(0) space (see [61, Example 5.2]) with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2) \right).$$

Therefore, $(\mathbb{R}^2, d_X)$ is a complete p-uniformly convex metric space with $p = 2$ and parameter $c = 2$.

Now define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_1, 2x_1^2 - x_2)$. Clearly, T is not a nonexpansive mapping in the classical sense. However, it is easy to check that T is nonexpansive in $(\mathbb{R}^2, d_X)$. Indeed, for all $x, y \in \mathbb{R}^2$,

$$\begin{aligned} d_X(Tx, Ty) &= \sqrt{(x_1 - y_1)^2 + (x_1^2 - (2x_1^2 - x_2) - y_1^2 + (2y_1^2 - y_2))^2} \\ &= \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \\ &= d_X(x, y). \end{aligned}$$

Again, define $f_1 : \mathbb{R}^2 \to \mathbb{R}$ by $f_1(x_1, x_2) = 100((x_2 + 1) - (x_1 + 1)^2)^2 + x_1^2$. Then f_1 is a proper convex and lower semicontinuous function in $(\mathbb{R}^2, d_X)$ but not convex in the classical sense (see [61]). Also, define $f_i : \mathbb{R}^2 \to \mathbb{R}$ by $f_i(x_1, x_2) = 50ix_1^2,\ i = 2, 3, 4$. Then f_i is a proper convex and lower semicontinuous function for each $i = 2, 3, 4$.

We consider the following 4 cases for our numerical experiments.
Case 1: $x_1 = (0.5, -0.25)^T$ and $u = (0.5, 3)^T$,
Case 2: $x_1 = (-1.5, -3)^T$ and $u = (0.5, 3)^T$,
Case 3: $x_1 = (0.5, 3)^T$ and $u = (-1.5, -3)^T$,
Case 4: $x_1 = (0.5, 3)^T$ and $u = (0.5, -0.25)^T$.

Example 4.2. We now give an example in p-uniformly convex metric space to further show the efficiency and advantage of our Algorithm. Let $\mathbf{P}(n)$ be the space of $n \times n$ Hermitian positive definite matrices. For $1 \le p < \infty$, the geodesic distance between A and B in $\mathbf{P}(n)$ (also called the p-Schatten distance) $d_p : \mathbf{P}(n) \times \mathbf{P}(n) \to [0, \infty)$ is defined by (see [18], [39, Chapter 2] and [17, Example 5.2])

$$\begin{aligned} d_p(A, B) &= \inf\{L(c) \mid c : [0, 1] \to \mathbf{P}(n) \text{ is a smooth curve with } c(0) = A \text{ and } c(1) = B\} \\ &= ||\log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}||_p \\ &= \left(\sum_{i=1}^{m} \log^p \mu_i(A^{-1}B) \right)^{\frac{1}{p}}, \end{aligned}$$

where $\mu_i(A^{-1}B)$ is the eigenvalue of $A^{-1}B$, $L(c) := \int_0^1 ||c(t)^{-\frac{1}{2}} c'(t) c(t)^{-\frac{1}{2}}||_p dt$, $||A||_p := (\text{tr}|A|^p)^{\frac{1}{p}}$, tr is the the usual trace functional and $|A| = (A^H A)^{\frac{1}{2}}$ (where A^H is the conjugate transpose of A). The pair $(\mathbf{P}(n), d_p)$ is a p-uniformly convex metric space with geodesic joining A to B in $\mathbf{P}(n)$ given by (see [18, 19, 39])

$$(1 - t)x \oplus ty = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}},\ 0 \le t \le 1.$$

Now, define $T : \mathbf{P}(n) \to \mathbf{P}(n)$ by $TA = D^H A D$, where $D \in GL(n)$ (the set of $n \times n$ invertible matrices). Then T is a nonexpansive mapping (see [39, Chapter 2]). Also, define $f_1 : \mathbf{P}(n) \to \mathbb{R}$ by $f_1 A = \left(\sum_{i=1}^{m} \log^p \mu_i(A^{-1}e^A) \right)^{\frac{1}{p}}$, where $\mu_i(A^{-1}e^A)$ is the eigenvalue of $A^{-1}e^A$. Then f_1 is convex and lower semicontinuous (see [5]). Again, define $f_2, f_3, f_4 : \mathbf{P}(n) \to \mathbb{R}$ by $f_2 A = -\log det A$, $f_3 A = \text{tr}(A)$ and $f_4 A = \text{tr}(e^A)$ respectively, then f_i is convex and lower semicontiunous for each $i = 2, 3, 4$ (see [5, 56]).
We now consider the following 4 cases for our numerical experiments.

Case I: $x_1 = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$ and $u = \begin{bmatrix} 5 & 1+i \\ 1-i & 4 \end{bmatrix}$,

Case II: $x_1 = \begin{bmatrix} 2 & 2-i \\ 2+i & 4 \end{bmatrix}$ and $u = \begin{bmatrix} 5 & 1+i \\ 1-i & 4 \end{bmatrix}$,

Case III: $x_1 = \begin{bmatrix} 2 & 2-i \\ 2+i & 4 \end{bmatrix}$ and $u = \begin{bmatrix} 1 & 4+i \\ 4-i & 3 \end{bmatrix}$,

Case IV: $x_1 = \begin{bmatrix} 3 & -3-i \\ -3+i & 4 \end{bmatrix}$ and $u = \begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix}$.

Remark 4.3. Using different choices of the initial vectors x_1 and u for Example 4.1 (that is, **Case 1-Case 4**), and matrices x_1 and u for Example 4.2 (that is, **Case I-Case IV**), we obtain the numerical results shown in Figure 1, Figure 2, Table 1 and Table 2. We see in the figures that the error values converge to 0, suggesting that by choosing arbitrary starting vectors, the sequence $\{x_n\}$ converges to the common minimizer of $f_i,\ i = 1, 2, 3, 4$ which is also a fixed point of T. In all our comparisons (see the tables and graphs), we see that our algorithm performs better than the algorithms studied in [49] and [57].

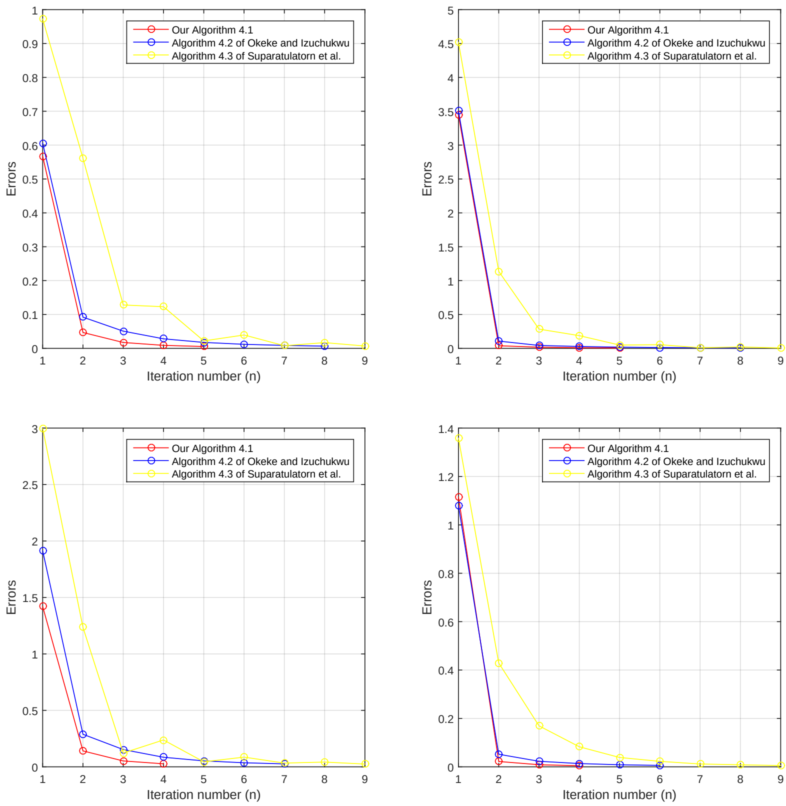

FIGURE 1. Errors vs Iteration numbers for **Example 4.1**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

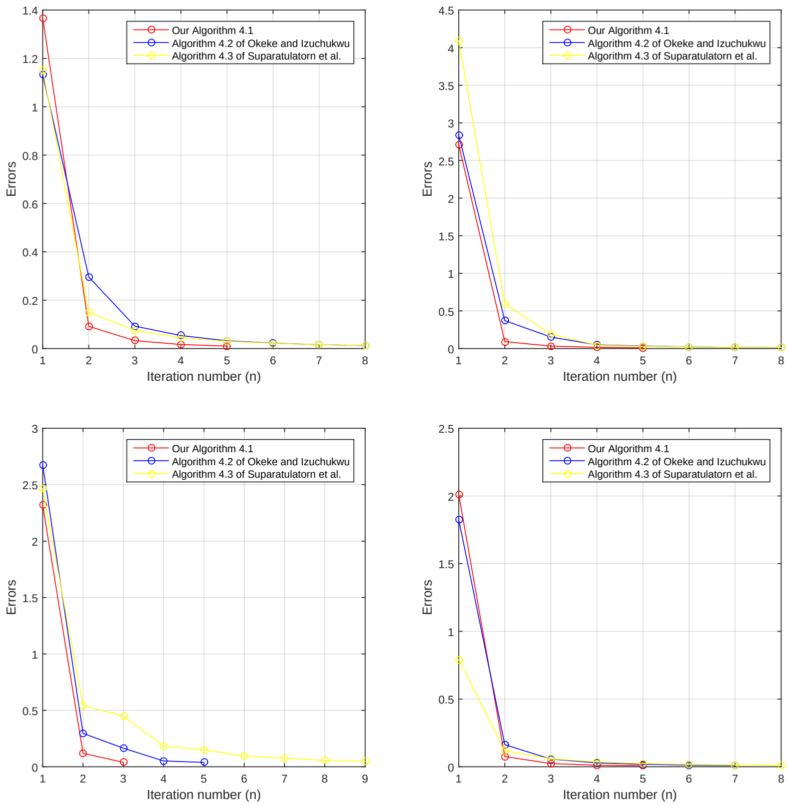

FIGURE 2. Errors vs Iteration numbers for **Example 4.2**: **Case I** (top left); **Case II** (top right); **Case III** (bottom left); **Case IV** (bottom right).

Table 1. Showing CPU time and iteration number for **Example 4.1**

Initial Vectors	Our Algorithm 4.1		Algorithm 4.2		Algorithm 4.3	
with Tol=10^{-3}	Iter	CPU	Iter	CPU	Iter	CPU
Case 1	6	1.0612	11	1.3291	15	1.3830
Case 2	6	1.0501	11	1.3130	15	1.2911
Case 3	7	1.0333	18	1.3909	25	1.8001
Case 4	5	1.0202	8	1.0891	11	1.3201

Table 2. Showing CPU time and iteration number for **Example 4.2**

Initial Vectors with Tol=10^{-2}	Our Algorithm 4.1		Algorithm 4.2		Algorithm 4.3	
	Iter	CPU	Iter	CPU	Iter	CPU
Case I	7	1.2080	10	1.4310	11	1.3801
Case II	7	1.1981	10	1.3811	11	1.2997
Case III	7	1.1859	11	1.3993	21	2.0190
Case IV	6	1.0001	8	1.2886	9	1.1984

In the tables above, Iter denotes iteration number, CPU denotes the CPU time in seconds and Tol denotes tolerance (stopping criterion).

5. Conclusion

Strong convergence of Halpern-type proximal point algorithm to a common minimizer of a finite family of proper convex and lower semicontinuous functions which is also a fixed point of a nonexpansive mapping is established under some mild assumptions in the framework of a complete p-uniformly convex metric space. The established results of this paper improve the results in [58] from Δ-convergence results to strong convergence results since strong convergence results are much more desirable than Δ-convergence results in infinite dimensional spaces (see Remark 2.17). Moreover, the obtained results show that the Δ-demicloseness assumption imposed on the p-resolvent operators in [58] can be dispensed with (see Remark 2.14). The obtained results in this paper also generalize several contemporary results from Hadamard spaces to p-uniformly convex metric spaces which was inspired by the importance and the applicability of the p-uniformly convex metric spaces (for example, see Example 4.2). To further show the applicability and the advantage of the obtained results, two numerical examples of our method in comparison (in terms of CPU time and number of iterations as seen in Tables 1 and 2, and in terms of Errors vs number of iterations as seen in Figures 1 and 2) with other methods in the literature, were considered in an Hadamard space and in a p-uniformly convex metric space. In all our comparisons, the numerical results show that our method performs better and have competative advantage than the methods in [49] and [57].

Declaration

The authors declare that they have no competing interests.

Acknowledgement

The authors sincerely thank the editor and anonymous referee for the careful reading, constructive comments and fruitful suggestions that substantially improved the manuscript. The first author acknowledges with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. The second author acknowledges the doctoral bursary and financial support from the University of KwaZulu-Natal, Durban, South Africa. The third author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

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