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F-Contraction-Type Fixed Point Theorems in b-Metric Spaces

Muhammad Sirajo Abdullahi[1,2,*], Abor Isa Garba[1], Jamilu Abubakar[1,2] and Poom Kumam[2,3]

[1] *Department of Mathematics, Faculty of Physical and Computing Sciences, Usmanu Danfodiyo University, Sokoto, Sokoto State, Nigeria*
e-mail : abdullahi.sirajo@udusok.edu.ng (M.S. Abdullahi); garba.isa@udusok.edu.ng (A.I. Garba); abubakar.jamilu@udusok.edu.ng (J. Abubakar)

[2] *KMUTTFixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, SCL 802 Fixed Point Laboratory, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*
e-mail : poom.kumam@mail.kmutt.ac.th (P. Kumam)

[3] *Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Science Laboratory Building, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*

Abstract In this paper, we present a generalization and improvement of some recent results concerning F-contraction. We establish some fixed point theorems in the setting of b-metric spaces. The obtained results are proper generalization of many results in the literature. Finally, we constructed some examples to support our findings.

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1. Introduction

The most fundamental result in metric fixed point theory was established by Stefan Banach [1] in 1922.

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

where $\lambda \in [0, 1)$. Then T has a unique fixed point and for each $x \in X$, the sequence $\{T^n x\}$ converges to the fixed point.

In 1968, Kannan [2] gave the following contractive-type result:

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*Corresponding author.

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Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a Kannan contraction, i.e.,*

$$d(Tx, Ty) \leq \lambda(d(x, Tx) + d(y, Ty)) \text{ for all } x, y \in X, \tag{1.2}$$

where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point and for each $x \in X$, the sequence $\{T^n x\}$ converges to the fixed point.

Afterwards in 1972, Chatterjea [3] presented the following contractive-type result:

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a Chatterjea contraction, i.e.,*

$$d(Tx, Ty) \leq \lambda(d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X, \tag{1.3}$$

where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point and for each $x \in X$, the sequence $\{T^n x\}$ converges to the fixed point.

In [4] Hardy and Rogers considered combining the right hand sides of (1.1), (1.2) and that of (1.3) to obtain the following generalized type of contractive maps $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) + \lambda_4 d(x, Tx) + \lambda_5 d(y, Ty), \tag{1.4}$$

for all $x, y \in X$, where $\sum_{i=1}^{\infty} \lambda_i < 1$.

They established a nice result for the existence of a unique fixed point for maps satisfying (1.4) among others. In particular, we present it as follows.

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ satisfies (1.4). Then T has a unique fixed point and for each $x \in X$, the sequence $\{T^n x\}$ converges to the fixed point.*

In [5], Wardowski defined a new type of mappings as follows:

Definition 1.5. Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying:

- F_1) F is strictly increasing. I.e. for all $u, v \in (0, +\infty)$, $u < v \implies F(u) < F(v)$;
- F_2) for each sequence $\{u_n\}_{n=1}^{\infty}$ of positive numbers, $lim_{n\to\infty} u_n = 0$ if and only if $\lim_{n\to\infty} F(u_n) = -\infty$;
- F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} u^k F(u) = 0$.

Definition 1.6. [5] Let (X, d) be a metric space. A mapping $T : X \to X$ is called an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \tag{1.5}$$

Example 1.7. [5] Define a map $F : (0, +\infty) \to (-\infty, +\infty)$ by $F(\gamma) = \ln \gamma$. It is easy to see that the map F satisfies the conditions of Definition 1.5, for any $k \in (0, 1)$. Hence any mapping $T : X \to X$ satisfying (1.5) is an F-contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x, y),$$

for all $x, y \in X$, with $Tx \neq Ty$. More so, we can observe that, for $x, y \in X$ with $Tx = Ty$ the following holds

$$d(Tx, Ty) \leq e^{-\tau} d(x, y).$$

More precisely, we say that T is a Banach contraction.

Wardowski [5] gave a new generalization of Banach contraction principle as follows:

Theorem 1.8. *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to $x^* \in X$.*

Several articles study the generalizations and improvements of results in [5], we refer the reader to [6–10] and references therein. In Particular, Cosentino and Vetro in the paper [6] presented the following.

Theorem 1.9. *Let (X, d) be a complete metric space and $T : X \to X$ be a map. Suppose there exists $\tau > 0$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tx,Ty) > 0 \implies & \tau + F(d(Tx,Ty)) \\ & \leq F(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty) + \alpha_4 d(x,Ty) + \alpha_5 d(y,Tx)), \end{aligned} \tag{1.6}$$

where $F \in \mathcal{F}$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non-negative numbers with $\alpha_3 \neq 1, \alpha_1+\alpha_2+\alpha_3+2\alpha_4 = 1$ and $\alpha_1+\alpha_4+\alpha_5 \leq 1$. Then T has a unique fixed point and for each $x \in X$, the sequence $\{T^n x\}$ converges to the fixed point.

Recently, Popescu and Stan [10] gave a generalization of Theorem 1.1, 1.8 and 1.9, which is given below:

Theorem 1.10. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-map. Suppose there exists $\tau > 0$ such that for all $x, y \in X$,*

$$\begin{aligned} d(Tx,Ty) > 0 \implies & \tau + F(d(Tx,Ty)) \\ & \leq F(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty) + \alpha_4 d(x,Ty) + \alpha_5 d(y,Tx)), \end{aligned} \tag{1.7}$$

where $F : (0, +\infty) \to (-\infty, +\infty)$ is an increasing mapping, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are non-negative numbers with $\alpha_4 < \frac{1}{2}, \alpha_3 < 1, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 = 1, 0 < \alpha_1 + \alpha_4 + \alpha_5 \leq 1$. Then T has a unique fixed point and for each $x \in X$, the sequence $\{T^n x\}$ converges to the fixed point.

For some other results related to b-metric and its variants, we refer the reader to [11–18] and references therein.

In this paper, motivated by the work of Popescu and Stan [10], we formulate our fixed point theorems in b-metric space. We study the problem of finding sufficient conditions on the contractive constants in b-metric spaces which guarantees the existence and uniqueness of fixed point for the map T and the convergence of the Picard iterative sequence $\{T^n x\}$ to the fixed point of T for any point $x \in X$.

2. Preliminaries

Let us recall the definition and some basic concepts of the b-metric space. Throughout, we denote $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{N}$ to represent the sets of real numbers, non-negative real numbers and natural numbers, respectively.

Definition 2.1. Let X be a nonempty set. A function $d_m : X \times X \to \mathbb{R}^+$ is called a metric on X if for all $x, y, z \in X$, d_m satisfies the following:

m_1. $d_m(x, y) = 0$ if and only if $x = y$;
m_2. $d_m(x, y) = d_m(y, x)$;
m_3. $d_m(x, y) \leq d_m(x, z) + d_m(z, y)$.

If d_m is a metric on X, then the pair (X, d_m) is called a metric space.

Modifying the inequality in Definition 2.1, we obtain the following concept introduced by Stefan Czerwik [19] in 1993.

Definition 2.2. [19]. Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}^+$, is called a b-metric on X, if there exists a real number $s \geq 1$ such that for all $x, y, z \in X$, d satisfies the following:

b_1. $d(x, y) = 0$ if and only if $x = y$;
b_2. $d(x, y) = d(y, x)$;
b_3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

If d is a b-metric on X, then the pair (X, d) is called a b-metric space.

Note that, every metric space is a b-metric space when $s = 1$. However, in general the converse is not true.

Here, we give an example of b-metric space that is not a metric space.

Example 2.3. Let $X = [0, 2]$ and d be defined on X by $d(x, y) = (x - y)^2$, for every $x, y \in X$. Then (X, d) is a b-metric space with coefficient $s > 1$. However, (X, d) is not a metric space, because the condition m_3. fails. For instance, we have

$$4 = d(0, 2) \nleq d(0, 1) + d(1, 2) = 2.$$

Definition 2.4. [19] Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. Then,

i. A sequence $\{x_n\}$ b-converges to x if and only if there exist x such that

$$\lim_{n\to\infty} d(x_n, x) = 0;$$

ii. A sequence $\{x_n\}$ is called b-Cauchy if and only if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0;$$

iii. A b-metric space (X, d) is called b-complete b-metric space if every b-Cauchy sequence in (X, d) b-converges in (X, d).

Theorem 2.5. [19] *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Then the following holds:*

i. Any b-convergent sequence has a unique limit;
ii. Every b-convergent sequence is b-Cauchy;
iii. In general, a b-metric is not continuous.

Definition 2.6. Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$. Then, a point $x \in X$ is called a fixed point of T if $x = Tx$.

3. Fixed Point Theorems

Now, we present a new variant of F-contraction of Banach-type in the setting of b-metric spaces:

Theorem 3.1. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a map. Suppose there exists $\tau > 0$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(\alpha\, d(x, y)), \tag{3.1}$$

where $F:(0,+\infty)\to\mathbb{R}$ *is an increasing mapping,* $0\le\alpha\in\mathbb{R}$ *such that* $\alpha s^2<1$*. Then* T *has a unique fixed point* $x^*\in X$ *and for each* $x\in X$*, the sequence* $\{T^n\}$ *converges to the fixed point.*

Proof. **(Existence:)** Let $x\in X$ be arbitrary fixed. Define the sequence $\{x_n\}$ by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. If there exists some $N_0\in\mathbb{N}$ such that $x_{N_0}=x_{N_0+1}$, then $x_{N_0}=T(x_{N_0})$, implying that x_{N_0} is a fixed point of T.

So, we suppose that $x_n\neq x_{n+1}$ for all $n\ge 0$. By (3.1), we have

$$\tau+F(d(Tx_{n-1},Tx_n))\le F(\alpha\, d(x_n,x_{n+1})).$$

This implies that

$$\begin{aligned}F(d(Tx_{n-1},Tx_n))&\le F(\alpha\, d(x_n,x_{n+1})-\tau\\ &< F(\alpha\, d(x_n,x_{n+1})),\end{aligned}\tag{3.2}$$

which further implies that

$$d(x_n,x_{n+1})<\alpha\, d(x_n,x_{n+1}),$$

and

$$(1-\alpha)\, d(x_n,x_{n+1})<0.$$

This suggests that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0.\tag{3.3}$$

Now, we assume that the sequence $\{x_n\}$ is not b-Cauchy, then there exists $\epsilon>0$ and integers $n_i,m_i\in\mathbb{N}$ such that $m_i>n_i\ge i$ and

$$d(x_{n_i},x_{m_i})\ge\epsilon,\quad \text{for } i\in\mathbb{N}.$$

By choosing m_i as small as possible, we may assume that

$$d(x_{n_i},x_{m_{i-1}})<\epsilon.$$

Therefore, for each $i\in N$, we have

$$\begin{aligned}\epsilon\le d(x_{n_i},x_{m_i})&\le s\Big(d(x_{n_i},x_{m_{i-1}})+d(x_{m_{i-1}},x_{m_i})\Big)\\ &= s\, d(x_{n_i},x_{m_{i-1}})+s\, d(x_{m_{i-1}},x_{m_i})\\ &<\epsilon+s\, d(x_{m_{i-1}},x_{m_i}).\end{aligned}$$

Now, from (3.3) and the above inequality, we have

$$\lim_{i\to\infty} d(x_{n_i},x_{m_i})=\epsilon.$$

By the triangle inequality, we have

$$\begin{aligned}d(x_{n_i-1},x_{m_i-1})&\le s\Big(d(x_{n_i-1},x_{n_i})+d(x_{n_i},x_{m_{i-1}})\Big)\\ &\le s\Big(d(x_{n_i-1},x_{n_i})+s\Big[d(x_{n_i},x_{m_i})+d(x_{m_i},x_{m_i-1})\Big]\Big)\\ &\le s^2\Big(d(x_{n_i-1},x_{n_i})+d(x_{n_i},x_{m_i})+d(x_{m_i},x_{m_i-1})\Big).\end{aligned}$$

Thus by (3.1), we obtain

$$\begin{aligned}\tau + F(d(x_{n_i}, x_{m_i})) &= \tau + F(d(Tx_{n_i-1}, Tx_{m_i-1})) \\ &\leq F(\alpha\, d(x_{n_i-1}, x_{m_i-1})) \\ &\leq F\Big(\alpha s^2 \Big[d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_i-1})\Big]\Big) \\ &= F\Big(\alpha s^2\, d(x_{n_i}, x_{m_i}) + \alpha s^2\, d(x_{n_i-1}, x_{n_i}) + \alpha s^2\, d(x_{m_i-1}, x_{m_i})\Big).\end{aligned}$$

Letting $n \to \infty$ in the above inequality and taking the limit, we have

$$\tau + F(\epsilon + 0) \leq F(\epsilon + 0) < F(\epsilon + 0),$$

a contradiction. Hence, the sequence $\{x_n\}$ is b-Cauchy and since X is b-complete, we conclude that the sequence $\{x_n\}$ b-converges to a point say $x^* \in X$ as $n \to \infty$.

Now, it is left to show that $x^* = Tx^*$. If there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that $x_{n_i+1} = Tx_{n_i} = Tx^*$, then $\lim_{i\to\infty} x_{n_i+1} = x^*$, hence $Tx^* = x^*$. Otherwise, there exists $N \in \mathbb{N}$ such that $x_{n+1} = Tx_n \neq Tx^*$, for all $n \geq \mathbb{N}$. Now, suppose that $Tx^* \neq x^*$. Then, we have

$$\tau + F(d(Tx_n, Tx^*)) \leq F(\alpha\, d(x_n, x^*)).$$

Since F is increasing and by taking the limit as $n \to \infty$, we have

$$d(Tx_n, Tx^*) < \alpha\, d(x_n, x^*),$$

and

$$d(Tx^*, Tx^*) < \alpha\, d(x^*, x^*),$$

a contradiction. Hence $x^* = Tx^*$.

(Uniqueness:) Let x' be a fixed of T different from x^*. BIt follows from (3.1) that

$$\tau + F(d(x^*, x')) = \tau + F(d(Tx^*, Tx')) \leq F(\alpha\, d(x^*, x')) < F(d(x^*, x')),$$

which is a contradiction. Hence the fixed point x^* is unique.

For each $x \in X$, the convergence of $T^n x$ to x^* follows immediately. ■

Corollary 3.2. [1, 5] *Let (X, d) be a complete metric space and $T : X \to X$ be a map. Suppose there exists $\tau > 0$ such that for all $x, y \in X$,*

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(\alpha\, d(x, y)), \tag{3.4}$$

where $F : (0, +\infty) \to \mathbb{R}$ is an increasing mapping, $0 < \alpha < 1$. Then T has a unique fixed point $x^* \in X$ and for each $x \in X$, the sequence $\{T^n\}$ converges to the fixed point.

Now, we present a more generalized result in the form of a new variant results of F-contraction of Hardy-Rogers type in the setting of b-metric spaces. This generalizes some results in [6] and [10]:

Theorem 3.3. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a map. Suppose there exists $\tau > 0$ such that for all $x, y \in X$,*

$$\begin{aligned}d(Tx,Ty) > 0 \implies &\tau + F(d(Tx, Ty)) \\ &\leq F\Big(\alpha_1 d(x, y) + \alpha_2 d(x, Tx) + \alpha_3 d(y, Ty) + \alpha_4 d(x, Ty) + \alpha_5 d(y, Tx)\Big),\end{aligned} \tag{3.5}$$

where $F:(0,+\infty)\to\mathbb{R}$ is an increasing mapping, $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ are non-negative real numbers such that $\alpha_3+\alpha_4<1, \alpha_1+\alpha_2+\alpha_4 s<1, \alpha_1+\alpha_2+\alpha_3+2\alpha_4 s=1$ and $\alpha_1 s^2+\alpha_4 s+\alpha_5 s\le 1$. Then T has a unique fixed point $x^*\in X$ and for each $x\in X$, the sequence $\{T^n\}$ converges to the fixed point.

Proof. **(Existence:)** Let $x\in X$ be arbitrary fixed. Define the sequence $\{x_n\}$ by $x_{n+1}=Tx_n$ for all $n\in\mathbb{N}$. If there exists some $N_0\in\mathbb{N}$ such that $x_{N_0}=x_{N_0+1}$, then $x_{N_0}=T(x_{N_0})$, implying that x_{N_0} is a fixed point of T.

So, we suppose that $x_n\neq x_{n+1}$ for all $n\ge 0$. By (3.5), we have

$$\begin{aligned}\tau+&F(d(Tx_{n-1},Tx_n))\\ &\le F\Big(\alpha_1 d(x_n,x_{n+1})+\alpha_2 d(x_n,Tx_n)+\alpha_3 d(x_{n+1},Tx_{n+1})\\ &\qquad +\alpha_4 d(x_n,Tx_{n+1})+\alpha_5 d(x_{n+1},Tx_n)\Big)\\ &= F\Big(\alpha_1 d(x_n,x_{n+1})+\alpha_2 d(x_n,x_{n+1})+\alpha_3 d(x_{n+1},x_{n+2})\\ &\qquad +\alpha_4 d(x_n,x_{n+2})+\alpha_5 d(x_{n+1},x_{n+1})\Big)\\ &= F\Big((\alpha_1+\alpha_2)\,d(x_n,x_{n+1})+\alpha_3 d(x_{n+1},x_{n+2})+\alpha_4 d(x_n,x_{n+2})\Big)\\ &\le F\Big((\alpha_1+\alpha_2+\alpha_4\,s)\,d(x_n,x_{n+1})+(\alpha_3+\alpha_4\,s)\,d(x_{n+1},x_{n+2})\Big).\end{aligned}$$

This implies

$$\begin{aligned}&F(d(Tx_{n-1},Tx_n))\\ &\quad\le F\Big((\alpha_1+\alpha_2+\alpha_4\,s)\,d(x_n,x_{n+1})+(\alpha_3+\alpha_4\,s)\,d(x_{n+1},x_{n+2})\Big)-\tau \qquad (3.6)\\ &\quad< F\Big((\alpha_1+\alpha_2+\alpha_4\,s)\,d(x_n,x_{n+1})+(\alpha_3+\alpha_4\,s)\,d(x_{n+1},x_{n+2})\Big),\end{aligned}$$

which further implies that

$$d(x_n,x_{n+1})<(\alpha_1+\alpha_2+\alpha_4\,s)\,d(x_n,x_{n+1})+(\alpha_3+\alpha_4\,s)\,d(x_{n+1},x_{n+2}),$$

and

$$\begin{aligned}d(x_n,x_{n+1})&<\frac{(\alpha_3+\alpha_4\,s)}{1-(\alpha_1+\alpha_2+\alpha_4\,s)}\,d(x_{n+1},x_{n+2})\\ &<d(x_{n+1},x_{n+2}).\end{aligned}$$

This suggests that there exists $p=\lim_{n\to\infty}d(x_n,x_{n+1})$. So, suppose that $p>0$. Then there exists $\lim_{x\to p+}F(x)=F(p+0)$ as F is increasing. Now, letting $n\to\infty$ in (3.6), we have

$$F(p+0)\le F(p+0)-\tau<F(p+0),$$

a contradiction. Therefore,

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0. \qquad (3.7)$$

Now, we assume that the sequence $\{x_n\}$ is not b-Cauchy, then there exists $\epsilon>0$ and integers $n_i,m_i\in\mathbb{N}$ such that $m_i>n_i\ge i$ and

$$d(x_{n_i},x_{m_i})\ge\epsilon,\quad \text{for } i\in\mathbb{N}.$$

By choosing m_i as small as possible, we may assume that

$$d(x_{n_i}, x_{m_{i-1}}) < \epsilon.$$

Therefore, for each $i \in N$, we have

$$\begin{aligned}\epsilon \leq d(x_{n_i}, x_{m_i}) &\leq s\Big(d(x_{n_i}, x_{m_{i-1}}) + d(x_{m_{i-1}}, x_{m_i})\Big)\\ &= s\, d(x_{n_i}, x_{m_{i-1}}) + s\, d(x_{m_{i-1}}, x_{m_i})\\ &< \epsilon + s\, d(x_{m_{i-1}}, x_{m_i}).\end{aligned}$$

Now, from (3.7) and the above inequality, we have

$$\lim_{i\to\infty} d(x_{n_i}, x_{m_i}) = \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned}d(x_{n_i-1}, x_{m_i-1}) &\leq s\Big(d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, x_{m_{i-1}})\Big)\\ &\leq s\Big(d(x_{n_i-1}, x_{n_i}) + s\Big[d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_i-1})\Big]\Big)\\ &\leq s^2\Big(d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_i-1})\Big).\end{aligned}$$

Thus by (3.5), we obtain

$$\begin{aligned}\tau + F(d(x_{n_i}, x_{m_i})) &= \tau + F(d(Tx_{n_i-1}, Tx_{m_i-1}))\\ &\leq F\Big(\alpha_1 d(x_{n_i-1}, x_{m_i-1}) + \alpha_2 d(x_{n_i-1}, Tx_{n_i-1})\\ &\qquad + \alpha_3 d(x_{m_i-1}, Tx_{m_i-1}) + \alpha_4 d(x_{n_i-1}, Tx_{m_i-1})\\ &\qquad + \alpha_5 d(x_{m_i-1}, Tx_{n_i-1})\Big)\\ &= F\Big(\alpha_1 d(x_{n_i-1}, x_{m_i-1}) + \alpha_2 d(x_{n_i-1}, x_{n_i})\\ &\qquad + \alpha_3 d(x_{m_i-1}, x_{m_i}) + \alpha_4 d(x_{n_i-1}, x_{m_i})\\ &\qquad + \alpha_5 d(x_{m_i-1}, x_{n_i})\Big)\\ &\leq F\Big(\alpha_1 s^2\Big[d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, x_{m_i}) + d(x_{m_i}, x_{m_i-1})\Big]\\ &\qquad + \alpha_2 d(x_{n_i-1}, x_{n_i}) + \alpha_3 d(x_{m_i-1}, x_{m_i})\\ &\qquad + \alpha_4 s\left[d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, x_{m_i})\right]\\ &\qquad + \alpha_5 s\left[d(x_{m_i-1}, x_{m_i}) + d(x_{m_i}, x_{n_i})\right]\Big)\\ &= F\Big((\alpha_1 s^2 + \alpha_4 s + \alpha_5 s)\, d(x_{n_i}, x_{m_i})\\ &\qquad + (\alpha_1 s^2 + \alpha_2 + \alpha_4 s)\, d(x_{n_i-1}, x_{n_i})\\ &\qquad + (\alpha_1 s^2 + \alpha_3 + \alpha_5 s)\, d(x_{m_i-1}, x_{m_i})\Big).\end{aligned}$$

Letting $n \to \infty$ in the above inequality and taking the limit, we have

$$\tau + F(\epsilon + 0) \leq F(\epsilon + 0) < F(\epsilon + 0),$$

a contradiction. Hence, the sequence $\{x_n\}$ is b-Cauchy and since X is b-complete, we conclude that the sequence $\{x_n\}$ b-converges to a point say $x^* \in X$ as $n \to \infty$.

Now, it is left to show that $x^* = Tx^*$. If there exists a sequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that $x_{n_i+1} = Tx_{n_i} = Tx^*$, then $\lim_{i\to\infty} x_{n_i+1} = x^*$, hence $Tx^* = x^*$. Otherwise, there exists $N \in \mathbb{N}$ such that $x_{n+1} = Tx_n \neq Tx^*$, for all $n \geq \mathbb{N}$. Now, suppose that $Tx^* \neq x^*$. Then, we have

$$\begin{aligned}
\tau+&F(d(Tx_n,Tx^*))\\
&\leq F\Big(\alpha_1 d(x_n,x^*) + \alpha_2 d(x_n,Tx_n) + \alpha_3 d(x^*,Tx^*)\\
&\qquad + \alpha_4 d(x_n,Tx^*) + \alpha_5 d(x^*,Tx_n)\Big)\\
&\leq F\Big(\alpha_1 d(x_n,x^*) + \alpha_2 d(x_n,x_{n+1}) + \alpha_3 d(x^*,Tx^*)\\
&\qquad + \alpha_4 d(x_n,Tx^*) + \alpha_5 d(x^*,x_{n+1})\Big).
\end{aligned}$$

Since F is increasing and by taking the limit as $n \to \infty$, we have

$$\begin{aligned}
d(Tx_n,Tx^*) < \alpha_1 d(x_n,x^*) &+ \alpha_2 d(x_n,x_{n+1}) + \alpha_3 d(x^*,Tx^*)\\
&+ \alpha_4 d(x_n,Tx^*) + \alpha_5 d(x^*,x_{n+1}),
\end{aligned}$$

and

$$\begin{aligned}
d(x^*,Tx^*) &< \alpha_1 d(x^*,x^*) + \alpha_2 d(x^*,x^*) + \alpha_3 d(x^*,Tx^*)\\
&\qquad + \alpha_4 d(x^*,Tx^*) + \alpha_5 d(x^*,x^*)\\
&= (\alpha_3+\alpha_4)\, d(x^*,Tx^*)\\
&< d(x^*,Tx^*),
\end{aligned}$$

a contradiction. Hence $x^* = Tx^*$.

(Uniqueness:) Let x' be a fixed of T different from x^*. It follows from (3.5) that

$$\begin{aligned}
\tau + F(d(x^*,x')) &= \tau + F(d(Tx^*,Tx'))\\
&\leq F\Big(\alpha_1 d(x^*,x') + \alpha_2 d(x^*,Tx^*) + \alpha_3 d(x',Tx')\\
&\qquad + \alpha_4 d(x^*,Tx') + \alpha_5 d(x',Tx^*)\Big)\\
&= F\Big(\alpha_1 d(x^*,x') + \alpha_2 d(x^*,x^*) + \alpha_3 d(x',x') + \alpha_4 d(x^*,x')\\
&\qquad + \alpha_5 d(x',x^*)\Big)\\
&= F((\alpha_1+\alpha_4+\alpha_5)\, d(x^*,x'))\\
&= F(d(x^*,x')),
\end{aligned}$$

which is a contradiction. Hence the fixed point x^* is unique.

For each $x \in X$, the convergence of $T^n x$ to x^* follows immediately. ■

Corollary 3.4. [10] *Let (X,d) be a complete metric space and $T : X \to X$ be a map. Suppose there exists $\tau > 0$ such that for all $x,y \in X$,*

$$\begin{aligned}
d(Tx,Ty) > 0 \implies &\tau + F(d(Tx,Ty))\\
&\leq F\Big(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty) + \alpha_4 d(x,Ty) + \alpha_5 d(y,Tx)\Big),
\end{aligned} \tag{3.8}$$

where $F:(0,+\infty)\to\mathbb{R}$ *is an increasing mapping,* $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ *are non-negative real numbers such that* $\alpha_3+\alpha_4<1,\alpha_1+\alpha_2+\alpha_4<1,\alpha_1+\alpha_2+\alpha_3+2\alpha_4=1$ *and* $\alpha_1+\alpha_4+\alpha_5\leq 1$. *Then* T *has a unique fixed point* $x^*\in X$ *and for each* $x\in X$, *the sequence* $\{T^n\}$ *converges to the fixed point.*

Corollary 3.5. [2] *Let* (X,d) *be a b-complete b-metric space with coefficient* $s\geq 1$ *and* $T:X\to X$ *be a map. Suppose there exists* $\tau>0$ *such that for all* $x,y\in X$,

$$d(Tx,Ty)>0 \implies \tau+F(d(Tx,Ty))\leq F\Big(\frac{1}{2}\,(d(x,Tx)+d(y,Ty))\Big), \tag{3.9}$$

where $F:(0,+\infty)\to\mathbb{R}$ *is an increasing mapping. Then* T *has a unique fixed point* $x^*\in X$ *and for each* $x\in X$, *the sequence* $\{T^n\}$ *converges to the fixed point.*

Corollary 3.6. [3] *Let* (X,d) *be a b-complete b-metric space with coefficient* $s\geq 1$ *and* $T:X\to X$ *be a map. Suppose there exists* $\tau>0$ *such that for all* $x,y\in X$,

$$d(Tx,Ty)>0 \implies \tau+F(d(Tx,Ty))\leq F\Big(\lambda\,(d(x,Ty)+d(y,Tx))\Big), \tag{3.10}$$

where $F:(0,+\infty)\to\mathbb{R}$ *is an increasing mapping,* $0\leq\lambda\leq\frac{1}{2s}$. *Then* T *has a unique fixed point* $x^*\in X$ *and for each* $x\in X$, *the sequence* $\{T^n\}$ *converges to the fixed point.*

Corollary 3.7. [20] *Let* (X,d) *be a b-complete b-metric space with coefficient* $s\geq 1$ *and* $T:X\to X$ *be a map. Suppose there exists* $\tau>0$ *such that for all* $x,y\in X$,

$$d(Tx,Ty)>0 \implies \tau+F(d(Tx,Ty))\leq F\Big(\lambda\,d(x,y)+\beta\,d(x,Tx)+\gamma\,d(y,Ty)\Big), \tag{3.11}$$

where $F:(0,+\infty)\to\mathbb{R}$ *is an increasing mapping,* λ,β,γ *are non-negative real numbers such that* $\gamma<1,\lambda+\beta<1,\lambda+\beta+\gamma=1$ *and* $\lambda s^2\leq 1$. *Then* T *has a unique fixed point* $x^*\in X$ *and for each* $x\in X$, *the sequence* $\{T^n\}$ *converges to the fixed point.*

Corollary 3.8. *Let* (X,d) *be a b-complete b-metric space with coefficient* $s\geq 1$ *and* $T:X\to X$ *be a map. Suppose there exists* $\tau>0$ *such that for all* $x,y\in X$,

$$\begin{aligned} d(Tx,Ty)>0 \implies \tau&+F(d(Tx,Ty)) \\ &\leq F\Big(\lambda\,d(x,y)+\beta\,d(x,Ty)+\gamma\,d(y,Tx)\Big), \end{aligned} \tag{3.12}$$

where $F:(0,+\infty)\to\mathbb{R}$ *is an increasing mapping,* λ,β,γ *are non-negative real numbers such that* $\beta<1,\lambda+\beta s<1,\lambda+2\beta s=1$ *and* $\lambda s^2+\beta s+\gamma s\leq 1$. *Then* T *has a unique fixed point* $x^*\in X$ *and for each* $x\in X$, *the sequence* $\{T^n\}$ *converges to the fixed point.*

4. Conclusion

In this paper, we established some b-metric fixed point theorems for some generalized F-contractive type mappings. Our results are proper generalization and improvement of results due to [10]. We have presented some sufficient conditions on the contractive constants for the existence and uniqueness of fixed point of some F-contractive type mappings. Finally, we have furnished some examples to support our findings.

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