Thai Journal of Mathematics
Volume 22 Number 2 (2024)
Pages 311–322

http://thaijmath.in.cmu.ac.th

ISSN 1686-0209

On Primary and Regular Γ-Semihypergroups

Safoora J. Ansari[1,*] and Kishor F. Pawar[2]

[1] *Department of Mathematics, S.N.J.B.'s K.K.H.A. Arts, S.M.G.L. Commerce & S.P.H.J. Science College, Chandwad, District Nashik, 423 101, M.S., India*
e-mail : ansari.sjacs@snjb.org

[2] *Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, 425 001, M.S., India*
e-mail : kfpawar@nmu.ac.in

Abstract In this paper the notions of primary and semiprimary Γ-hyperideals in Γ-semihypergroup are introduced. It is shown that if $rad.(I)$ is a maximal Γ-hyperideal of S then I is a semiprimary Γ-hyperideal of S. It is proved that a Γ-semihypergroup S is semiprimary if and only if prime Γ-hyperideals of S forms a chain under set inclusion. Regular set in Γ-semihypergroup and regular Γ-hypergroup are defined with several examples. Characterizations of regular Γ-semihypergroup is established in terms of left and right Γ-hyperideals.

MSC: 20N20
Keywords: Γ-semihypergroups; primary Γ-hyperideal; semiprimary; Γ-hyperideal; regular Γ-semihypergroup

Submission date: 03.05.2020 / Acceptance date: 19.09.2023

1. Introduction

Study of hyperstructures was initiated by a French mathematician Marty [1] in the year 1934 when he defined hypergroups based on the notion of hyperoperation at the 8^{th} Congress of Scandinavian Mathematicians. Since then a number of different algebraic hyperstructures are being studied. In a non-empty set equipped with binary operation, the composition of two elements is an element while in an algebraic hyperstructure the composition of two elements yields a non-empty set. There are many researchers in several countries who study hyperstructures and extend their contributions through research articles and books. Corsini wrote many articles and books on different algebraic hyperstructures [2–5]. He has given applications of hyperstructures in various subjects like cryptography, coding theory, automata, probability, lattice theory, graph theory and rough sets [4]. A book [6] on hyperrings is written by Davvaz and Leoreanu-Fotea in 2007 which gives detailed insight on fundamentals of hyperring theory. The notion of Γ-semigroup was first defined by Sen and Saha [7] as a suitable generalization of semigroup

*Corresponding author.

Published by The Mathematical Association of Thailand.

and ternary semigroup and thereafter many mathematicians began to study Γ-semigroups, extended and generalized many concepts and notions of semigroups to Γ-semigroups [8].

The study of Γ-semihypergroup was initiated by Davvaz et al. [9–11] as a generalization of three algebraic structures semigroup, semihypergroup and Γ-semigroup. They have given many examples and studied Γ-semihypergroups considering several notions. In this paper the notions of primary Γ-hyperideal and semiprimary Γ-hyperideal are introduced and studied along with regular Γ-semihypergroups.

2. Preliminaries

We begin with recalling some basic definitions and results from [10, 11] required for our purpose. For further study reader is requested to refer [9, 12].

Definition 2.1. [9] Let H be a non-empty set and $\circ : H \times H \to \wp^*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The pair $(H, \circ)$ is called a *hypergroupoid.*

For any two non-empty subsets A and B of H and $x \in H$,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2. [13] A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$ we have, $(a \circ b) \circ c = a \circ (b \circ c)$. In addition, if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a *hypergroup.*

For more details of hypergroups, semihypergroups see [14].

Definition 2.3. [11] Let S and Γ be two non-empty sets. Then S is called a Γ-*semihypergroup* if every $\gamma \in \Gamma$ is a hyperoperation on S, that is $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property

$$x\alpha(y\beta z) = (x\alpha y)\beta z.$$

Let A and B be two non-empty subsets of S and $\gamma \in \Gamma$, we denote the following:

$$A\gamma B = \bigcup_{a \in A, b \in B} a\gamma b.$$

Also,

$$A\Gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

A Γ-semihypergroup S is said to be *commutative* if for every $x, y \in S$ and $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$.
If (S, γ) is a hypergroup for every $\gamma \in \Gamma$ then S is called a Γ-*hypergroup.*

Example 2.4. [9] Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$ we define $\gamma : S \times S \longrightarrow \wp^*(S)$ by $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$. Then γ is a hyperoperation on S and $x\alpha(y\beta z) = \left[0, \frac{xyz}{\alpha\beta}\right] = (x\alpha y)\beta z$. This means that S is a Γ-semihypergroup.

Definition 2.5. [11] A non-empty subset A of Γ-semihypergroup S is said to be a Γ-*subsemihypergroup* if $A\Gamma A \subseteq A$ i.e. $a\gamma b \subseteq A$ for every $a, b \in A$ and $\gamma \in \Gamma$.

Definition 2.6. [11] A non-empty subset A of a Γ-semihypergroup S is said to be a *left(right)* Γ-*hyperideal* if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$).

A is said to be a *two sided* Γ-*hyperideal* or simply a Γ-*hyperideal* if it is both left and right Γ-hyperideal.

S is called a *left(right) simple* Γ-*semihypergroup* if it has no proper left (right) Γ-hyperideal. S is said to be a *simple* Γ-*semihypergroup* if it has no proper Γ-hyperideal.

Example 2.7. In Example 2.4, let $T = [0, t]$ where $t \in [0, 1]$. Then T is left (right) Γ-hyperideal of S.

Definition 2.8. [10] Let A be a non-empty subset of a Γ-semihypergroup S. Then intersection of all Γ-hyperideals of S containing A is a Γ-hyperideal of S generated by A, and denoted by $< A >$.

Definition 2.9. [12] A Γ-hyperideal A of a Γ-semihypergroup S is said to be a *principal* Γ-*hyperideal* if A is a Γ-hyperideal generated by single element a and is denoted by (a).

To study more examples on Γ-semihypergroup and the notions of fundamental relations on Γ-semihypergroups, quotient Γ-semihypergroups, right Noetherian Γ-semihypergroups etc. see [11].

3. Primary and Semiprimary Γ-Hyperideal in a Γ-Semihypergroup

Anjaneyulu studied primary ideals in semigroups [15, 16]. In this section the notions of primary and semiprimary Γ-hyperideals in Γ-semihypergroups are introduced and few results are proved. It is shown that if $rad.(I)$ is a maximal Γ-hyperideal of S then I is a semiprimary Γ-hyperideal of S. The equivalence between prime, primary and semiprimary Γ-hyperideals is established. Finally it is proved that a Γ-semihypergroup S is semiprimary if and only if prime Γ-hyperideals of S forms a chain under set inclusion.

Definition 3.1. [11] A proper Γ-hyperideal P of a Γ-semihypergroup S is said to be a *prime* Γ-hyperideal if for every Γ-hyperideal I, J of S $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If a Γ-semihypergroup S is commutative, then a proper Γ-hyperideal P is *prime* if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.

Definition 3.2. [12] Let S be a Γ-semihypergroup. The prime radical $P(S)$ (or $rad.(S)$) of S is the intersection of all prime ideals of S. If I is a Γ-hyperideal of S, the prime radical $P(I)$ (or $rad.(I)$) of I is the intersection of all prime ideals containing I.

Definition 3.3. A Γ-hyperideal I of a Γ-semihypergroup S is said to be a *left primary* Γ-*hyperideal* if

a. $rad.(I)$ is a prime Γ-hyperideal.
b. For any two Γ-hyperideals A, B of S such that $A\Gamma B \subseteq I$ and $B \nsubseteq I$ implies that $A \subseteq rad.(I)$.

Definition 3.4. A Γ-hyperideal I of a Γ-semihypergroup S is said to be a *right primary* Γ-*hyperideal* if

a. $rad.(I)$ is a prime Γ-hyperideal.
b. For any two Γ-hyperideals A, B of S such that $A\Gamma B \subseteq I$ and $A \nsubseteq I$ implies that $B \subseteq rad.(I)$.

A Γ-hyperideal I is said to be a *primary* Γ-*hyperideal* if I is both left primary and right primary Γ-hyperideal of S.

Remark 3.5. A left primary Γ-hyperideal of a Γ-semihypergroup S need not be a right primary and vice-versa.

Example 3.6. Let $S = \{x, y, z\}$ and $\Gamma = \{\alpha, \beta\}$. Define the hyperoperation $\circ$ on S as follows:

$\circ$	x	y	z
x	x	x	x
y	x	x	x
z	x	$\{x, y\}$	$\{x, z\}$

Now define a map $S \times \Gamma \times S \to \wp^*(S)$ as $a\gamma b = a \circ b$ for every $a, b \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup whose ideals are $\{x\}, \{x, y\}$ and $\{x, y, z\}$ of which $\{x\}$ is a left primary Γ-hyperideal but not right primary whereas $\{x, y\}$ is a primary Γ-hyperideal of S.
It is now left for readers to find example of a right primary Γ-hyperideal which is not left primary.

Remark 3.7. A prime Γ-hyperideal is primary.

In the next two results, the equivalent conditions of statements (b) of Definition 3.3 and Definition 3.4 are given.

Theorem 3.8. *A Γ-hyperideal I of a Γ-semihypergroup S satisfies condition (b) of Definition 3.3 if and only if for non-empty subsets A and B of S, $\langle A\rangle\Gamma\langle B\rangle \subseteq I$ and $B \nsubseteq I$ implies that $A \subseteq rad.(I)$.*

Proof. Suppose that I satisfies condition (b) of Definition 3.3 and A, B are two non-empty subsets of S such that $\langle A\rangle\ \Gamma\ \langle B\rangle \subseteq I$ with $B \nsubseteq I$. This implies that $\langle B\rangle \nsubseteq I$ hence by hypothesis $\langle A\rangle \subseteq rad.(I)$. Therefore $A \subseteq rad.(I)$.
Conversely let J and K be two Γ-hyperideals of S such that $J\Gamma K \subseteq I$ and $K \nsubseteq I$. If $J \nsubseteq rad.(I)$ then there exists an element $a \in J$ such that $a \notin rad.(I)$. Also $K \nsubseteq I$ implies there is an element $b \in K$ such that $b \notin I$. Now $\langle a\rangle\Gamma\langle b\rangle \subseteq J\Gamma K \subseteq I$ and $b \notin I$ implies $a \in rad.(I)$, a contradiction. Hence $J \subseteq rad.(I)$. ∎

Theorem 3.9. *A Γ-hyperideal I of a Γ-semihypergroup S satisfies condition (b) of Definition 3.4 if and only if for non-empty subsets A and B of S, $\langle A\rangle\Gamma\langle B\rangle \subseteq I$ and $A \nsubseteq I$ implies that $B \subseteq rad.(I)$.*

Proof. Similar to the proof of Theorem 3.8. ∎

Definition 3.10. A Γ-hyperideal I of a Γ-semihypergroup S is said to be a *semiprimary* Γ-*hyperideal* if $rad.(I)$ is a prime Γ-hyperideal of S.

In general, a Γ-hyperideal need not be semiprimary. Following is the result which gives a condition under which Γ-hyperideal becomes semiprimary.

Theorem 3.11. *Let S be a Γ-semihypergroup and I be a Γ-hyperideal of S. If $rad.(I)$ is a maximal Γ-hyperideal of S then I is semiprimary Γ-hyperideal of S.*

Proof. If $rad.(I)$ is a maximal Γ-hyperideal of S then it must be a proper subset of S. Hence there exists a proper prime Γ-hyperideal P of S containing I. Now $rad.(I) \subseteq P \subseteq S$ and $rad.(I)$ is maximal hence $P = rad.(I)$. Thus $rad.(I)$ is a prime Γ-hyperideal. ∎

Definition 3.12. [17] A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a partially semiprime Γ-hyperideal if for a non-empty subset A of S, $A\Gamma S\Gamma A \subseteq I$ implies that $A \subseteq I$.

Theorem 3.13. [18] *A* Γ*-hyperideal* Q *of a* Γ*-semihypergroup* S *is partially semiprime if and only if* $rad.(Q) = Q$*.*

In the following reuslt the equivalence between a prime Γ-hyperideal, primary Γ-hyperideal and a semiprimary Γ-hyperideal is established given that the Γ-hyperideal under consideration is partially semiprime.

Proposition 3.14. *Let* I *be a partially semiprime* Γ*-hyperideal of a* Γ*-semihypergroup* S*. If any one of the following statements are true then so are the others.*

a. I *is a prime* Γ*-hyperideal.*
b. I *is a primary* Γ*-hyperideal.*
c. I *is a semiprimary* Γ*-hyperideal.*

Proof. Straightforward. ■

Definition 3.15. If every Γ-hyperideal of a Γ-semihypergroup S is semiprimary then S is said to be a *semiprimary* Γ-semihypergroup.

Following is the characterization of semiprimary Γ-semihypergroups.

Theorem 3.16. *A* Γ*-semihypergroup* S *is semiprimary if and only if prime* Γ*-hyperideals of* S *form a chain under set inclusion.*

Proof. Suppose that S is a semiprimary Γ-semihypergroup and I, J are two prime Γ-hyperideals of S. As $rad.(I \cap J) = rad.(I) \cap rad.(J) = I \cap J$, Theorem 3.13 implies that $I \cap J$ is a partially semiprime Γ-hyperideal of S. Since S is a semiprimary Γ-semihypergroup, by Proposition 3.14 $I \cap J$ is a prime Γ-hyperideal of S. If neither $I \subseteq J$ nor $J \subseteq I$, then there would exist $a \in I \setminus J$ and $b \in J \setminus I$ such that $\langle a\rangle\Gamma\langle b\rangle \subseteq I\Gamma J \subseteq I\Gamma S \subseteq I$ and $\langle a\rangle\Gamma\langle b\rangle \subseteq I\Gamma J \subseteq S\Gamma J \subseteq J$. Thus $\langle a\rangle\Gamma\langle b\rangle \subseteq I \cap J$ implies $a \in \langle a\rangle \subseteq I \cap J$ or $b \in \langle b\rangle \subseteq I \cap J$, a contradiction. Hence either $I \subseteq J$ or $J \subseteq I$. It means that prime Γ-hyperideals of S form a chain under set inclusion. Conversely assume that prime Γ-hyperideals of S form a chain under set inclusion. Let I be a Γ-hyperideal of S and $rad.(I) = \bigcap_{\alpha\in\Delta} P_\alpha$ where P_α is a prime Γ-hyperideal of S containing I for each α. By assumption $P_{\alpha_0} \subseteq P_{\alpha_1} \subseteq P_{\alpha_2} \cdots\cdots$. This implies that $rad.(I) = P_{\alpha_k}$ for some $\alpha_k \in \Delta$. Thus $rad.(I)$ is a prime Γ-hyperideal of S. Therefore I is a semiprimary Γ-hyperideal of S and S is a semiprimary Γ-semihypergroup. ■

4. Regular Γ-Semihypergroups

This section deals with regular sets in Γ-semihypergroups, regular elements and regular Γ-hypergroups with several examples. It is proved that every regular Γ-hypergroup is a Γ-hypergroup but not conversely. The notion of inverse subset is defined and it is proved that a regular set has an inverse set. Characterization for an element a to be regular and characterization of regular Γ-semihypergroups are established.

Definition 4.1. A subset A of a Γ-semihypergroup S is said to be regular if $A \subseteq A\Gamma_1 B\Gamma_2 A$ for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq S$.

A singleton set $\{a\}$ or simply an element a of a Γ-semihypergroup S is said to be regular if $a \in a\Gamma_1 B\Gamma_2 a$ for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq S$ where $a\Gamma_1 B\Gamma_2 a = \{x \mid x \in a\alpha b\beta a, \alpha \in \Gamma_1, b \in B, \beta \in \Gamma_2\}$.

A Γ-semihypergroup S is said to be *regular* if every element of S is regular.

Example 4.2. Consider the following sets:

$$S = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mid x, y, z, w \in \mathbb{R} \right\}$$

$$\Gamma = \{z | z \in \mathbb{Z}\}$$

and

$$A_\alpha = \left\{ \begin{bmatrix} \alpha x & 0 \\ 0 & \alpha w \end{bmatrix} \mid x, w \in \mathbb{R}, \alpha \in \Gamma \right\}$$

Define hyperoperation $S \times \Gamma \times S \to P^*(S)$ as $M\alpha N \mapsto MA_\alpha N$ for all $M, N \in S$ and $\alpha \in \Gamma$. Because matrix multiplication is associative we have $M\alpha(N\beta P) = (M\alpha N)\beta P$ and hence S is Γ-semihypergroup. Also $M \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} M^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} M = M$ hence $M \in M\alpha M^{-1}\alpha M$ and the invertible matrix M is a regular element of S with $\alpha = 1 \in \Gamma$. The collection of all invertible matrices from the set S forms a regular set of S.

Definition 4.3. An element a of a Γ-semihypergroup S is said to be a *left* (*right*) *identity* of S if $s \in a\alpha s$ ($s \in s\alpha a$) for all $s \in S$ and $\alpha \in \Gamma$.

An element a of a Γ-semihypergroup is said to be a *two sided identity* or simply an *identity* if a is both left and right identity, i.e. $s \in a\alpha s \cap s\alpha a$ for all $s \in S$ and $\alpha \in \Gamma$. We denote identity element of a Γ- semihypergroup by 1 or e.

Example 4.4. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$. Define a hyperoperation $\circ$ on S as follows:

$\circ$	a	b	c	d
a	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	a	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
c	a	b	c	d
d	a	b	c	d

Define a mapping $S \times \Gamma \times S \to \wp^*(S)$ by $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup. Observe that each element of S is left identity of S but S does not have a right identity.

Example 4.5. Let $S = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ defined as follows:

α	x	y
x	$\{x, y\}$	$\{x, y\}$
y	$\{x, y\}$	$\{x, y\}$

β	x	y
x	$\{x\}$	$\{y\}$
y	$\{y\}$	$\{x\}$

Then S is a Γ-semihypergroup and x is a two sided identity of S.

Definition 4.6. An element b of a Γ-semihypergroup S is said to be an *α-inverse* of an element a if there exists an identity element e of S such that $e \in a\alpha b \cap b\alpha a$.

Element b is said to be a Γ-*inverse* of an element a if for every $\alpha \in \Gamma$ there exists an identity e of S such that $e \in a\alpha b \cap b\alpha a$, that is $e \in a\Gamma b \cap b\Gamma a$. We denote Γ-inverse of a by a^{-1}.

Definition 4.7. A Γ-semihypergroup S is said to be a *regular* Γ-*hypergroup* or simply r-Γ-*hypergroup* if it has an identity e and for each $x \in S$ there exists a Γ-inverse x^{-1} of x in S.

Example 4.8. Consider a set of two elements $S = \{a, b\}$ and set of hyperoperations $\Gamma = \{\alpha, \beta\}$ defined as follows:

α	a	b
a	a	b
b	$\{a, b\}$	$\{a, b\}$

β	a	b
a	a	b
b	b	a

Then a is two sided identity , inverse of a is a and inverse of b is b hence S is a regular Γ-hypergroup.

Example 4.9. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$ define the hyperoperations as follows.

α	a	b	c	d
a	$\{a, b\}$	$\{b, c\}$	$\{c, d\}$	$\{a, d\}$
b	$\{b, c\}$	$\{c, d\}$	$\{a, d\}$	$\{a, b\}$
c	$\{c, d\}$	$\{a, d\}$	$\{a, b\}$	$\{b, c\}$
d	$\{a, d\}$	$\{a, b\}$	$\{b, c\}$	$\{c, d\}$

β	a	b	c	d
a	$\{b, c\}$	$\{c, d\}$	$\{a, d\}$	$\{a, b\}$
b	$\{c, d\}$	$\{a, d\}$	$\{a, b\}$	$\{b, c\}$
c	$\{a, d\}$	$\{a, b\}$	$\{b, c\}$	$\{c, d\}$
d	$\{a, b\}$	$\{b, c\}$	$\{c, d\}$	$\{a, d\}$

Here d is a Γ-identity $a^{-1} = c, b^{-1} = b, c^{-1} = a, d^{-1} = d$ and S is a regular Γ-hypergroup.

Example 4.10. Let $S = \{0, 1\}$ and $\Gamma = \{\alpha, \beta\}$ and the operations are defined as follows:

α	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{0, 1}

β	0	1
0	0	1
1	1	0

Then S is a regular Γ-hypergroup.

Proposition 4.11. *Every regular* Γ*-hypergroup is a* Γ*-hypergroup.*

Proof. Straightforward. ∎

Converse of above Proposition need not be true.

Example 4.12. Let $S = \{0, 1\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ defined as follows:

α	0	1
0	{1}	{0}
1	{0}	{1}

β	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{0, 1}

γ	0	1
0	{0}	{1}
1	{1}	{0}

Here S is a Γ-hypergroup but it does not have Γ-identity hence it is not a regular Γ-hypergroup.

Definition 4.13. An element a of Γ-semihypergroup S is said to be an *α-idempotent* if $a \in a\alpha a$. An element a of Γ- semihypergroup S is said to be a *Γ-idempotent* or simply *idempotent* if $a \in a\alpha a$ for all $\alpha \in \Gamma$ i.e. $a \in a\Gamma a$.

Definition 4.14. A Γ-semihypergroup S is said to be an *idempotent Γ-semihypergroup* if every element in S is a Γ-idempotent.

Definition 4.15. A non-empty subset A of a Γ-semihypergroup S is said to be a *Γ-idempotent* subset of S if $A \subseteq A\Gamma A$.

Example 4.16. In Example 4.5, x is α-idempotent as well as β-idempotent hence x is Γ-idempotent whereas y is only α-idempotent. In Example 4.4 each element is Γ-idempotent, hence S there is a Γ-semihypergroup. In Example 4.12 S has neither a left(right) identity nor a Γ-idempotent element.

Example 4.17. Let $S = \{0, 1\}$ and $\Gamma = \{\alpha, \beta\}$ be defined as follows:

α	0	1
0	$\{0\}$	$\{0, 1\}$
1	$\{0,1\}$	$\{0,1\}$

β	0	1
0	$\{0, 1\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{0, 1\}$

S has two two-sided identities and both of them are Γ-idempotent.

Proposition 4.18. *Let S be a Γ-semihypergroup and A be an idempotent subset in S then $A\Gamma A$ is a Γ-subsemihypergroup of S if and only if $A\Gamma A\Gamma A = A\Gamma A$.*

Proof. Let A be an idempotent subset of a Γ-semihypergroup S. If $A\Gamma A$ is a Γ-subsemihypergroup of S then for $x \in A\Gamma A$ we have, $A\Gamma x \subseteq A\Gamma(A\Gamma A) \subseteq (A\Gamma A)\Gamma(A\Gamma A) \subseteq (A\Gamma A)$. Hence $A\Gamma A\Gamma A \subseteq A\Gamma A$ and $A \subseteq A\Gamma A$ implies that $A\Gamma A \subseteq A\Gamma A\Gamma A$. Therefore $A\Gamma A\Gamma A = A\Gamma A$. Conversely, assume that $A\Gamma A\Gamma A = A\Gamma A$ and let $x, y \in A\Gamma A$. Then $x\Gamma y \subseteq (A\Gamma A)\Gamma(A\Gamma A) = (A\Gamma A\Gamma A)\Gamma A = (A\Gamma A)\Gamma A = A\Gamma A\Gamma A = A\Gamma A$. ∎

Definition 4.19. Let A be a non-empty subset of a Γ-semihypergroup S. A subset B of S is said to an *inverse* subset of A in S if $A \subseteq A\Gamma_1 B\Gamma_2 A$ and $B \subseteq B\Gamma_2 A\Gamma_1 B$.

In the following result it is proved that every regular subset of a Γ-semihypergroup has an inverse subset.

Proposition 4.20. *Let S be a Γ-semihypergroup. If A is regular then it has an inverse subset.*

Proof. Let a subset A of a Γ-semihypergroup S be regular then there exist subsets Γ_1, Γ_2 of Γ and $B \subseteq S$ such that $A \subseteq A\Gamma_1 B\Gamma_2 A$. Let $E = B\Gamma_2 A\Gamma_1 B$, then $A \subseteq A\Gamma_1 B\Gamma_2 A \subseteq A\Gamma_1 B\Gamma_2(A\Gamma_1 B\Gamma_2 A) = A\Gamma_1 E\Gamma_2 A$. Also $E = B\Gamma_2 A\Gamma_1 B \subseteq B\Gamma_2(A\Gamma_1 B\Gamma_2 A)\Gamma_1 B = (B\Gamma_2 A\Gamma_1 B)\Gamma_2 A\Gamma_1 B \subseteq (B\Gamma_2 A\Gamma_1 B)\Gamma_2 A\ \Gamma_1(B\Gamma_2 A\Gamma_1 B) = E\Gamma_2 A\Gamma_1 E$. Thus E is an inverse subset of A in S. ∎

Following is the characterization of a regular element in the Γ- semihypergroup using the notion of idempotent set.

Proposition 4.21. *In a Γ-semihypergroup S an element a is regular if and only if there exists an idempotent set $A \subseteq S$ such that $a \in a\Gamma A$ and $S\Gamma a = S\Gamma A$.*

Proof. Let a be a regular element of a Γ-semihypergroup S, that is $a \in a\alpha b\beta a$ for some $\alpha, \beta \in \Gamma$ and $b \in S$. Consider the set $A = b\Gamma a$. As $A \subseteq b\Gamma a \subseteq (b\Gamma a)\Gamma(b\Gamma a) = A\Gamma A$, A is an idempotent subset of S. Also $a \in a\alpha b\beta a \subseteq a\Gamma(b\Gamma a) = a\Gamma A$. Now $S\Gamma a \subseteq S\Gamma a\Gamma A \subseteq S\Gamma A \subseteq S\Gamma b\Gamma a \subseteq S\Gamma a$. That is, $S\Gamma a \subseteq S\Gamma A \subseteq S\Gamma a$ therefore $S\Gamma a = S\Gamma A$. Conversely, assume that S has an idempotent set A and an element a such that $a \in a\Gamma A$ and $S\Gamma a = S\Gamma A$. As $a \in a\Gamma A \subseteq a\Gamma A\Gamma A \subseteq a\Gamma S\Gamma A = a\Gamma S\Gamma a$., implies that there exist $\alpha, \beta \in \Gamma$ and $b \in S$ such that $a \in a\alpha b\beta a$. Hence a is regular. ■

In general, in a Γ-semihypergroup S, Γ-hyperideal of a Γ-hyperideal need not be a Γ-hyperideal of S. We have proved in the next result that this holds when S is a regular Γ-semihypergroup.

Theorem 4.22. *Let S be a Γ-semihypergroup and I be a Γ-hyperideal of S. If S is regular then I is regular and any Γ-hyperideal J of I is a Γ-hyperideal of S.*

Proof. Let S be a regular Γ-semihypergroup and I be a Γ-hyperideal of S. Consider $A = \{a\} \subseteq I$, by Proposition 4.20 A has an inverse set B in S such that $a \in A\Gamma_1 B\Gamma_2 A$ and $B \subseteq B\Gamma_2 A\Gamma_1 B \subseteq S\Gamma I\Gamma_1 B \subseteq I$. That is $a \in a\Gamma_1 B\Gamma_2 a$ where $B \subseteq I$ and $\Gamma_1, \Gamma_2 \subseteq \Gamma$. Therefore $I \subseteq I\Gamma_1 B\Gamma_2 I, B \subseteq I$, hence I is regular.
Let J be a Γ-hyperideal of $I, a \in J, s \in S$ then $a\Gamma S \subseteq I$. For $x \in a\Gamma S$ there exist $y \in I, \alpha, \beta \in \Gamma$ such that $x \in x\alpha y\beta x$. Let D be the collection of such $y's$ then $a\Gamma S \subseteq (a\Gamma S)\alpha D\beta(a\Gamma S) \subseteq a\Gamma(S\Gamma I)\Gamma I \subseteq a\Gamma I \subseteq J\Gamma I \subseteq J$. That is, for all $a \in J, a\Gamma S \subseteq J$. Hence $J\Gamma S \subseteq J$. Similarly, $S\Gamma a \subseteq I$ and $S\Gamma J \subseteq J$. Therefore J is Γ-hyperideal of S. ■

Now we present the extremely useful characterization of regular Γ- semihypergroups in terms of one-sided Γ-hyperideals.

Theorem 4.23. *S is a regular Γ-semihypergroup if and only if for any left Γ-hyperideal I and for any right Γ-hyperideal J of S, $I \cap J = J\Gamma I$.*

Proof. Let I and J be the left and right Γ-hyperideals of a regular Γ- semihypergroup S respectively. Then $J\Gamma I \subseteq J \cap I$. Also for $x \in I \cap J$ there exist $\alpha \in \Gamma, y \in S$ and $\beta \in \Gamma$ such that $x \in x\alpha y\beta x \subseteq J\Gamma I$. That is, $I \cap J \subseteq J\Gamma I$ hence $I \cap J = J\Gamma I$. Conversely, assume that for any left Γ-hyperideal I and for any right Γ-hyperideal J of S, $I \cap J = J\Gamma I$. Let $a \in S$ and set $A = S\Gamma a$ then A is a left Γ-hyperideal of S. Let $I = \{a\} \cup A$ then $S\Gamma I \subseteq S\Gamma(\{a\} \cup A) \subseteq S\Gamma\{a\} \cup S\Gamma A = A \cup S\Gamma A \subseteq A \subseteq I$. Therefore I is a left Γ-hyperideal of S containing $\{a\}$. Let $B = a\Gamma S$ then B is a right Γ-hyperideal of S and $J = B \cup \{a\}$ is a right Γ-hyperideal of S containing $\{a\}$. Now $I = I \cap S = S\Gamma I \subseteq A$. Similarly $J = S \cap J = J\Gamma S \subseteq B$. Thus $a \in I \cap J \subseteq A \cap B \subseteq B\Gamma A \subseteq a\Gamma S\Gamma a$. Hence there exists an $\alpha \in \Gamma, s \in S, \beta \in \Gamma$ such that $a \in a\alpha s\beta a$. It means that an element $a \in S$ is a regular element of S. Since a was arbitrary we conclude that S is regular. ■

In the next result it is proved that every one-sided Γ-hyperideal in a regular Γ-semihypergroup S are idempotent.

Theorem 4.24. *Let I be a one sided Γ-hyperideal in a regular Γ-semihyp-ergroup S. Then $I = I\Gamma I$ and hence all one sided Γ-hyperideals in a Γ-semihypergroup S are idempotent.*

Proof. Let I be a left Γ-hyperideal of a regular Γ-semihypergroup S and $x \in I$. Then $x \in S$ and S is regular, hence there exist $\alpha \in \Gamma, y \in S, \beta \in \Gamma$ such that $x \in x\alpha y\beta x = x\alpha(y\beta x) \subseteq x\alpha I \subseteq I\Gamma I$. That is, $x \in I\Gamma I$ for all $x \in I$ so $I \subseteq I\Gamma I$. Since I is a left Γ-hyperideal of S we see that $I\Gamma I \subseteq I$. Therefore $I\Gamma I = I$. Similarly, for a right

Γ-hyperideal J of S we have $J = J\Gamma J$. Hence all one sided Γ-hyperideals I of S are idempotent. ∎

Lemma 4.25. [11] *Let S be a Γ-semihypergroup. If A is a non-empty subset of S, then $\langle A\rangle = A \cup A\Gamma S \cup S\Gamma A \cup S\Gamma A\Gamma S$.*

Following is the result which states the usefulness of idempotent subsets of a Γ-semihypergroup S.

Theorem 4.26. *Let S be a regular Γ-semihypergroup. Then every principal Γ-hyperideal of S is generated by an idempotent subset of S.*

Proof. Let S be a regular Γ-semihypergroup and $\langle x\rangle$ be a principal Γ- hyperideal of S. As $x \in S$ and S is regular, there exist $\alpha \in \Gamma, y \in S, \beta \in \Gamma$ such that $x \in x\alpha y\beta x$. Let $x\alpha y = A$ then $x \in A\beta x \subseteq A\Gamma S \subseteq \langle A\rangle$ by Lemma 4.25 and $A = x\alpha y \subseteq x\Gamma S \subseteq \langle x\rangle$. Thus $x \in \langle A\rangle$ and $A \subseteq \langle x\rangle$, this implies that $\langle A\rangle = \langle x\rangle$. Moreover, $A = x\alpha y \subseteq (x\alpha y\beta x)\alpha y = (x\alpha y)\beta(x\alpha y) \subseteq A\Gamma A$ therefore A is idempotent. ∎

Definition 4.27. A Γ-semihypergroup S is said to be a *left regular* (*right* regular) if for any element $x \in S$ there exist $\alpha, \beta \in \Gamma$ and $y \in S$ such that $x \in y\alpha x\beta x (x \in x\alpha x\beta y)$. S is said to be *intra regular* if for every element $x \in S$ there exists $\alpha, \beta, \gamma \in \Gamma$ and $y, z \in S$ such that $x \in y\alpha x\beta x\gamma z$.

Remark 4.28. A left (right) regular Γ-semihyperroup is intra regular.

Theorem 4.29. *Let S be a Γ-semihypergroup. Then the following statements hold:*

(1) S is left regular if and only if for every left Γ-hyperideal I of S, $a\Gamma a \subseteq I$ implies $a \in I$ for all $a \in S$.

(2) S is right regular if and only if for every right Γ-hyperideal I of S, $a\Gamma a \subseteq I$ implies $a \in I$ for all $a \in S$.

(3) S is intra regular if and only for every two sided Γ-hyperideal I of S , $a\Gamma a \subseteq I$ implies $a \in I$ for all $a \in S$.

Proof. (1) Let I be a left Γ-hyperideal of a left regular Γ-semihyperg-roup S. Suppose that for $a \in S$, $a\Gamma a \subseteq I$. There exists $x \in S$ such that $a \in x\alpha a\beta a \subseteq I$ implies that $a \in I$. To prove the converse let $a \in S$. Then $S\Gamma a$ is a left Γ-hyperideal and $a\Gamma a \subseteq S\Gamma a$ implies that $a \in S\Gamma a$, by assumption. Similarly, $S\Gamma S\Gamma a$ is a left Γ-hyperideal of S. As $a\Gamma a \subseteq S\Gamma a\Gamma a$, $a \in S\Gamma a\Gamma a$. Hence there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \in x\alpha a\beta a$. Therefor S is left regular.

(2) On the similar lines as that of proof (1).

(3) Let K be a two sided Γ-hyperideal of an intra regular Γ-semihype-rgroup S and for $a \in S$, $a\Gamma a \subseteq K$. There exist $\alpha, \beta, \gamma \in S, x, y \in S$ such that $a \in x\alpha a\beta a\gamma y \subseteq S\Gamma(a\Gamma a)\Gamma S \subseteq S\Gamma K\Gamma S \subseteq K$. Therefore $a \in K$. Conversely, assume that for any two sided Γ-hyperideal K of S and for $a \in S$, $a\Gamma a \subseteq K$ implies $a \in K$. We know that $S\Gamma a\Gamma a\Gamma S$ is a two sided Γ-hyperideal of S, we have for $a \in S, a\Gamma a \subseteq S\Gamma a$ and $a\Gamma a \subseteq a\Gamma S$. Therefore $(a\Gamma a)\Gamma(a\Gamma a) \subseteq S\Gamma a\Gamma a\Gamma S$. It means that for $x \in a\Gamma a, x\Gamma x \subseteq S\Gamma a\Gamma a\Gamma S$. As $S\Gamma a\Gamma a\Gamma S$ is a two sided Γ-hyperideal, we have $x \in S\Gamma a\Gamma a\Gamma S$ for all $x \in a\Gamma a$. Hence $a\Gamma a \subseteq S\Gamma a\Gamma a\Gamma S$ and $a \in S\Gamma a\Gamma a\Gamma S$. It implies that there exist $\alpha, \beta, \gamma \in \Gamma$, and $x, y \in S$ such that $a \in x\alpha a\beta a\gamma y$ for all $a \in S$. Therefore S is intra regular.

∎

5. Conclusion

In this paper the notions of primary and semiprimary Γ-hyperideals are introduced and few results are proved. We defined and studied regular set in Γ-semihypergroup and regular Γ-hypergroup with several examples. The notion of inverse subset in a Γ-semihypergroup is defined and it is proved that a regular set has an inverse set. One can extend this study to various ordered and unordered hyperstructures like hyperring and semihyperring and find inverse of regular subsets considering several examples. Characterization of an element a to be regular and and characterization of regular Γ-semihypergroups are established.

Acknowledgment

The authors are thankful to the Department of Science and Technology for providing research facilities under the FIST programme (Project No.: SR/FIST-415/2018).

References

[1] F. Marty, Sur une generalization de la notion de groupe, 8^{th} Congress Math. Scandinaves, Stockholm (1934) 45–49.

[2] P. Corsini, Hypergraphs and hypergroups, Algebra Universalis 35 (4) (1996) 548–555.

[3] P. Corsini, Hypergroups associated with HX-groups, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat. 25 (2017).

[4] P. Corsini, L. Leoreanu, Applications of Hyperstructure Theory, Advances in Mathematics, Dordrecht: Kluwer Academic Publishers, 2003.

[5] P. Corsini, Prolegomena of Hypergroup Theory, 2^{nd} ed., Aviani Editore Publisher, Italy 1993.

[6] B. Davvaz, V. Leoreanu-Fotea, Hyperring Theory and Applications, Florida, USA: International Academic Press, 2007.

[7] M.K. Sen, N.K. Saha, On Γ-semigroup I, Bulletin of the Calcutta Mathematical Society 78 (1986) 180–186.

[8] S. Chattopadhyay, A semihypergroup associated with a Γ-semigroup, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat. 56 (2010) 209–225.

[9] S.M. Anvariyeh, S. Mirvakili, B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, Carpathian J. Math. 26 (1) (2010) 11–23.

[10] D. Heidari, B. Davvaz, Γ-hypergroups and Γ-semihypergroups associated to binary relations, Iranian Journal of Science & Technology A2: (2011) 69–80.

[11] D. Heidari, S.O. Dehkordi, B. Davvaz, Γ-semihypergroups and their properties, U.P.B. Sci. Bull. (Series A) 72 (1) (2010).

[12] K. Hila, B. Davvaz, J. Dine, Study on the structure of Γ-semihypergroups, Communication in Algebra 40 (2012) 2932–2948.

[13] S. Hoskova-Mayerova, A. Maturo, On some applications of algebraic hyperstructures for the management of teaching and relationships in schools, Italian Journal of Pure and Applied Mathematics 41 (2019) 584–592.

[14] B. Davvaz, Semihypergroup Theory, Academic Press Elsevier, 2016.

[15] A. Anjaneyulu, On primary semigroups, Czechoslovak Mathematical Journal 30 (3) (1980) 382–386.

[16] A. Anjaneyulu, Primary ideals in semigroups, Semigroup Forum 20 (1980) 129–144.

[17] S.J. Ansari, K.F. Pawar, On certain Γ-hyperideals in Γ-semihypergroups, Journal of Hyperstructures 9 (2) (2020) 34–51.

[18] K.F. Pawar, S.J. Ansari, On systems, maximal Γ-hyperideals and complete prime Γ-radicals in Γ-semihypergroups, Journal of Hyperstructures 8 (2) (2019) 135–149.