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A Note on the General Split Common Fixed Point Problem in Hilbert Spaces

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Abstract This paper examines a general form of the split common fixed point problem in which a finite family of bounded linear operators is involved. We propose viscosity approximation methods with choosing two different types of stepsizes (one depends on operator norms and the other is selected in a self-adaptive way) for classes of attracting quasi-nonexpansive mappings and demicontractive mappings, respectively. Using the Landweber technique and some properties of the attracting quasi-nonexpansiveness, strong convergence results of the proposed methods are established in Hilbert spaces. Our results presented in this paper generalize many existing results in the literature.

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1. Introduction

The first instance of the split inverse problems, which was introduced by Censor and Elfving [1] in 1994, is the so-called *split feasibility problem* (SFP). The number of research works on the SFP has been continuously increasing (see [2–8] and the references therein) because its model can be applied in other mathematical problems and many real-world problems, such as in constrained least-squares problems, in linear programming problems, in intensity-modulated therapy, in signal/image restoration, in pattern recognition, in data prediction, etc., see [1, 4, 9–12] for instance. The SFP is a problem of finding a point of a closed convex subset of one space, whose image under a bounded linear transformation belongs to a closed convex subset of the image space.

Specifically, we suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two real Hilbert spaces and hence the SFP is formulated as finding a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.1}$$

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where $C \subseteq \mathcal{X}$ and $Q \subseteq \mathcal{Y}$ are nonempty closed convex subsets and and $A : \mathcal{X} \to \mathcal{Y}$ is bounded linear operator. Byrne [2] employed a projected Landweber method [13] called the *CQ-algorithm* for solving the SFP (1.1) as follows: Choosing $x^1 \in \mathcal{X}$,

$$x^{k+1} = P_C\left(x^k + \gamma A^*\left(P_Q - I\right)Ax^k\right), \quad k \in \mathbb{N}, \tag{1.2}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, P_C and P_Q are metric projections onto C and Q, respectively, I denotes the identity mapping and A^* stands for the adjoint operator of A. In the case that $\|A\|$ in the SFP (1.1) is difficult to calculate, López et al. [4] presented an alternative way to select the stepsize

$$\gamma^k := \lambda^k \frac{\|\left(P_Q - I\right)Ax^k\|^2}{\|A^*\left(P_Q - I\right)Ax^k\|^2}, \tag{1.3}$$

where $\lambda^k \in (0, 2)$, for replacing the parameter γ in (1.2). One can see that the stepsize γ^k (1.3) does not depend on $\|A\|$ (indeed, it depends on x^k). The CQ-algorithm (1.2) with the stepsizes selected as (1.3) is called a *self-adaptive CQ-algorithm*.

A natural generalization of the SFP is to extand from one closed convex subset to a finite family of such subsets or/and from one bounded linear operator to a finite family of such linear tranformations. Another generalization is to extand from closed convex subsets (emphasized only metric projections) to fixed point sets of any mappings. In this paper, we pay attention to a general form of the above mentions, called the *general split common fixed point problem* (GSCFPP), which was introduced by Gibali [14]. The GSCFPP is formulated as finding a point

$$x \in \bigcap_{i=1}^{m} \operatorname{Fix}\left(S_i\right) \quad \text{such that} \quad A_j x \in \operatorname{Fix}\left(T_j\right), \quad \forall j \in \{1, 2, \ldots, n\}, \tag{1.4}$$

where $S_i : \mathcal{X} \to \mathcal{X}$ $(i = 1, 2, \ldots, m)$ and $T_j : \mathcal{Y} \to \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ are mappings with nonempty fixed point sets $\operatorname{Fix}(S_i)$ and $\operatorname{Fix}(T_j)$, respectively and $A_j : \mathcal{X} \to \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ are bounded linear operators. Note that if $A_j = A$ for all j, then (1.4) is called the well-known *split common fixed point problem* (SCFPP) [15]. Especially, if $S_i := P_{C_i}$ and $T_j := P_{Q_j}$, where C_i and Q_j are closed convex subsets of $\mathcal{X}$ and $\mathcal{Y}$, respectively, then (1.4) is reduced to the *constrained multiple-sets split feasibility problem* (CMSSCFP) [16] and if further $A_j = A$ for all j, the problem becomes the *multiple-sets split feasibility problem* (MSSFP) [10]. Recently, these problems have been widely studied by many authors, see [6, 8, 17–27] for instance (also see [8] for application in compressed sensing with using two different bounded linear operators). By making use of the product space technique, Censor and Segal [15] presented a parallel algorithm for the SCFPP for two finite families of directed mappings. Gibali [14] also used the same technique for formulating a parallel algorithm for the GSCFPP for two finite families of demicontractive mappings.

The aim of this paper is to formulate some algorithms for the GSCFPP for two finite families of some nonlinear mappings and prove their strong convergence results in a simple way by using a technique different from that in [14] and [15]. In Section 3, we present viscosity approximation methods in two cases of stepsize selection−one depends on operator norms $\|A_j\|$ and the other is selected in the adaptive way similar to (1.3)−for solving the GSCFPP (1.4) for classes of attracting quasi-nonexpansive mappings and demicontractive mappings, respectively. We obtain strong convergence theorems of our proposed methods by using the Landweber technique and some properties of the attracting quasi-nonexpansiveness, which are provided in Section 2. Finally, in Section 4, we give some consequences of the main results for the constrained multiple-sets split feasibility problem and the split common null point problem.

2. Attracting Quasi-Nonexpansiveness and Landweber-Type Operators

This section reviews and collects the concepts and some properties of attracting quasi-nonexpansive mappings and Landweber-type operators that will be useful tools for designing our iterative methods and proving their convergence results for the GSCFPP in Section 3.

Throughout the section, we assume that $\mathcal{X}$ and $\mathcal{Y}$ are real Hilbert spaces equipped with their inner products $\langle\cdot,\cdot\rangle$ and the induced norms $\|\cdot\|$. Let $S:\mathcal{X}\to\mathcal{X}$ be a mapping. An element $z\in\mathcal{X}$ is called a *fixed point* of S if $z=Sz$. Denote by $\mathrm{Fix}(S)$ the set of all fixed points of S. The mapping

$$S_\beta := I+\beta(S-I),$$

where $\beta\geq 0$, is called a *β-relaxation* of S. Note that $\mathrm{Fix}\,(S_\beta)=\mathrm{Fix}(S)$ for $\beta\neq 0$. We say that S satisfies the *demi-closedness* (DC) *principle* if for any sequence $\{x^k\}\subset\mathcal{X}$, there holds the following implication:

$$x^k \rightharpoonup z \ \text{ and } \ (S-I)x^k\to 0 \implies z\in\mathrm{Fix}(S),$$

where the notations $\rightharpoonup$ and $\to$ stand for weak and strong convergence, respectively. A mapping S with the nonexpansiveness always satisfies the DC principle (see [28, Lemma 2]). Obviously, if S satisfies the DC principle, then S_β also satisfies the DC principle for $\beta\neq 0$.

Definition 2.1. A mapping $S:\mathcal{X}\to\mathcal{X}$ having a fixed point is said to be *attracting quasi-nonexpansive* (AQNE) if there exists $\rho>0$ such that

$$\|Sx-z\|^2\leq\|x-z\|^2-\rho\|Sx-x\|^2,\quad \forall x\in\mathcal{X},\forall z\in\mathrm{Fix}(S). \tag{2.1}$$

S satisfying (2.1) is also said to be *ρ-attracting quasi-nonexpansive* (ρ-AQNE) (see [29]). If $\rho=1$, then we call S a *directed* mapping.

The notion of mappings satisfying (2.1) was first introduced by Bruck [30] in metric spaces and it is referred to [31–33] for the other names (also see [15, 34, 35] in the case that $\rho=1$). Note that the class of AQNE mappings generalizes the so-called classes of firmly nonexpansive mappings and metric projections. There is an AQNE mapping which is not firmly nonexpansive, see Example 2.2.8 in [32]. The following is a characterization of AQNE mappings as follows.

Proposition 2.2. *A mapping $S:\mathcal{X}\to\mathcal{X}$ is ρ-AQNE, where $\rho>0$ if and only if*

$$\|Sx-x\|^2\leq\frac{2}{1+\rho}\langle Sx-x,z-x\rangle,\quad\forall x\in\mathcal{X},\forall z\in\mathrm{Fix}(S).$$

We give a relationship between an AQNE mapping and its relaxation that is a slight generalization of Theorem 2.1.39 in [32] for the directed mapping case.

Proposition 2.3. *Let $S:\mathcal{X}\to\mathcal{X}$ be a mapping having a fixed point and let $\rho>0$, $\beta\in(0,1+\rho)$. Then, S is ρ-AQNE if and only if S_β is $\left(\frac{1+\rho-\beta}{\beta}\right)$-AQNE.*

Proof. Let $x \in \mathcal{X}$ and $z \in \mathrm{Fix}(S) = \mathrm{Fix}(S_\beta)$. Consider the following expressions:

$$\begin{aligned}
&\|S_\beta x - z\|^2 - \|x - z\|^2 + \left(\frac{1+\rho-\beta}{\beta}\right)\|S_\beta x - x\|^2 \\
&= \|x + \beta(Sx - x) - z\|^2 - \|x - z\|^2 + \beta(1+\rho-\beta)\|Sx - x\|^2 \\
&= \|(x - z) + \beta(Sx - x)\|^2 - \|x - z\|^2 - \beta^2\|Sx - x\|^2 + \beta(1+\rho)\|Sx - x\|^2 \\
&= 2\langle x - z, \beta(Sx - x)\rangle + \beta(1+\rho)\|Sx - x\|^2 \\
&= \beta(1+\rho)\left(\|Sx - x\|^2 - \frac{2}{1+\rho}\langle Sx - x, z - x\rangle\right).
\end{aligned}$$

Since $\beta(1+\rho) > 0$, the assertion follows readily from the above expressions together with Proposition 2.2 and Definition 2.1. ∎

The next proposition shows that the class of AQNE mappings respects the covex combination and the composition.

Proposition 2.4 ([22, 32]). *Let $\{S_i : \mathcal{X} \to \mathcal{X}\}_{i=1}^m$ be a finite family of ρ_i-AQNE mappings, where $\rho_i > 0$ such that $\bigcap_{i=1}^m \mathrm{Fix}(S_i) \neq \emptyset$. Suppose that $S := \sum_{i=1}^m \omega_i S_i$, where $\omega_i > 0$ and $\sum_{i=1}^m \omega_i = 1$ and $S' := S_m S_{m-1} \dots S_1$. Then:*

(i) ([34]) $\mathrm{Fix}(S) = \mathrm{Fix}(S') = \bigcap_{i=1}^m \mathrm{Fix}(S_i)$.

(ii) *The mapping S is ρ-AQNE, where $\rho = \left(\sum_{i=1}^m \frac{\omega_i}{1+\rho_i}\right)^{-1} - 1$.*

(iii) *The mapping S' is ρ'-AQNE, where $\rho' = \left(\sum_{i=1}^m \frac{1}{\rho_i}\right)^{-1}$.*

(iv) *If every S_i satisfies the DC principle, then both S and S' also satisfy the DC principle.*

Now let us focus on the notion of a Landweber-type operator [23] which is more general than the classical Landweber operator [2, 13] for solving the linear equations and the split feasibility problem.

Definition 2.5 ([23]). Let $A : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator with $\|A\| > 0$ and $T : \mathcal{Y} \to \mathcal{Y}$ an AQNE mapping. An operator $V : \mathcal{X} \to \mathcal{X}$ defined by

$$V := I + \frac{1}{\|A\|^2} A^*(T - I)A \tag{2.2}$$

is called a *Landweber-type* operator related to T.

Define $W : \mathcal{X} \to \mathcal{X}$ by

$$Wx = \begin{cases} x + \frac{\|(T-I)Ax\|^2}{\|A^*(T-I)Ax\|^2} A^*(T - I)Ax, & \text{if } Ax \notin \mathrm{Fix}(T), \\ x, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that W is an extrapolation of the Landweber-type operator V defined by (2.2), see [23] for more details. Some important properties of a Landweber-type operator and its extrapolation are shown below.

Proposition 2.6 ([22, 23]). *Let $A : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator with $\|A\| > 0$ and $T : \mathcal{Y} \to \mathcal{Y}$ a ρ-AQNE mapping, where $\rho > 0$ such that $\mathrm{im}(A) \cap \mathrm{Fix}(T) \neq \emptyset$. Further, let V be a Landweber-type operator defined by (2.2) and W an extrapolation of V defined by (2.3). Then:*

(i) $\text{Fix}(V) = \text{Fix}(W) = A^{-1}(\text{Fix}(T))$.
(ii) *The operators V and W are ρ-AQNE.*
(iii) *If T satisfies the DC principle, then both V and W also satisfy the DC principle.*

We end this section with a strong convergence result for finding a fixed point of an AQNE mapping under the *viscosity* approximation scheme [36]. Recall that a mapping $F : \mathcal{X} \to \mathcal{X}$ is called a *contraction* with respect to $C \subseteq \mathcal{X}$ if there exists $\sigma \in [0, 1)$ such that $\|Fx - Fy\| \leq \sigma\|x - y\|$ for all $x \in \mathcal{X}$ and for all $y \in C$.

Theorem 2.7 ([24, 37]). *Let $S : \mathcal{X} \to \mathcal{X}$ be an AQNE mapping satisfying the DC principle. Let $F : \mathcal{X} \to \mathcal{X}$ be a contraction with respect to $\text{Fix}(S)$. Suppose that $\{x^k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{X}$ generated iteratively by $x^1 \in \mathcal{X}$ and*

$$x^{k+1} = \alpha^k F x^k + (1 - \alpha^k) S x^k, \quad k \in \mathbb{N},$$

where $\{\alpha^k\}_{k=1}^{\infty}$ is a real sequence in $(0, 1)$ such that $\alpha^k \to 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$. Then, $\{x^k\}_{k=1}^{\infty}$ converges strongly to a point $z \in \text{Fix}(S)$, where z is the unique fixed point of a contraction $P_{\text{Fix}(S)} F$.

In fact, Theorem 2.7 holds for a strongly quasi-nonexpansive mapping S (see [37, Corollary 3.5]); however, it was proved that every AQNE mapping is strongly quasi-nonexpansive (see [25, Proposition 2.6]).

3. Strong Convergence Results for the GSCFPP

To solve the GSCFPP, we first introduce an iteration formula based on the viscosity approximation and prove two strong convergence theorems of the proposed formula with choosing two different types of stepsizes which depend (or not) on operator norms for two finite families of AQNE mappings. Subsequently, we modify our proposed formula to extend the results to the larger class of demiconteactive mappings.

In the sequel, we suppose that

- $\mathcal{X}$ and $\mathcal{Y}$ are two real Hilbert spaces,
- $\{S_i : \mathcal{X} \to \mathcal{X}\}_{i=1}^{m}$ and $\{T_j : \mathcal{Y} \to \mathcal{Y}\}_{j=1}^{n}$ are two families of nonlinear mappings,
- $\{A_j : \mathcal{X} \to \mathcal{Y}\}_{j=1}^{n}$ is a family of bounded linear operators with $\|A_j\| > 0$ and accompanying their adjoint operators A_j^*,
- the solution set of the GSCFPP (1.4), denoted by Γ, is nonempty.

3.1. Attracting Quasi-Nonexpansive Mappings

Let us take a look at the following viscosity iteration formula: Let $F : \mathcal{X} \to \mathcal{X}$ be a contraction with respect to Γ. Pick a starting point $x^1 \in \mathcal{X}$ arbitrarily. For $k \in \mathbb{N}$, let

$$x^{k+1} = \alpha^k F x^k + \left(1 - \alpha^k\right) \frac{1}{m} \sum_{i=1}^{m} S_i \left(x^k + \frac{1}{n} \sum_{j=1}^{n} \gamma_j^k A_j^* \left(T_j - I\right) A_j x^k \right), \tag{3.1}$$

where $\{\gamma_j^k\}_{k=1}^{\infty}$ $(j = 1, 2, \ldots, n)$ are sequences of positive real numbers and $\{\alpha^k\}_{k=1}^{\infty}$ is a sequence of real numbers in $(0, 1)$.

Theorem 3.1. *If S_i $(i = 1, 2, \ldots, m)$ are ρ_i-AQNE mappings, where $\rho_i > 0$ and T_j $(j = 1, 2, \ldots, n)$ are $\tilde{\rho}_j$-AQNE mappings, where $\tilde{\rho}_j > 0$ such that both satisfy the DC principle, then any sequence $\{x^k\}_{k=1}^{\infty}$ generated by (3.1), where the parameter sequences $\{\gamma_j^k\}_{k=1}^{\infty}$ $(j = 1, 2, \ldots, n)$ are chosen as:*

$$\gamma_j^k := \gamma_j \in \left(0, \frac{1 + \tilde{\rho}_j}{\|A_j\|^2}\right), \tag{3.2}$$

converges strongly to a point $x^* \in \Gamma$, provided that $\alpha^k \to 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$.

Proof. Let V_j be a Landweber-type operator related to T_j, that is,

$$V_j = I + \frac{1}{\|A_j\|^2} A_j^* \left(T_j - I\right) A_j.$$

By Proposition 2.6(ii)&(iii), we obtain that V_j is $\tilde{\rho}_j$-AQNE and satisfies the DC principle. Let $\lambda_j = \gamma_j \|A_j\|^2 \in (0, 1 + \tilde{\rho}_j)$. By Proposition 2.3, the λ_j-relaxation of V_j,

$$(V_j)_{\lambda_j} = I + \frac{\lambda_j}{\|A_j\|^2} A_j^* \left(T_j - I\right) A_j = I + \gamma_j A_j^* \left(T_j - I\right) A_j,$$

is τ_j-AQNE, where $\tau_j = \frac{1+\tilde{\rho}_j - \lambda_j}{\lambda_j}$ and also satisfies the DC principle. Set

$$S := \frac{1}{m} \sum_{i=1}^{m} S_i \quad \text{and} \quad V := \frac{1}{n} \sum_{j=1}^{n} (V_j)_{\lambda_j}.$$

By Proposition 2.4(ii)&(iv), S and V are AQNE with coefficients $\rho = m\big(\sum_{i=1}^{m} \frac{1}{1+\rho_i}\big)^{-1} - 1$ and $\tilde{\rho} = n\big(\sum_{j=1}^{n} \frac{1}{1+\tau_i}\big)^{-1} - 1$, respectively and both satisfy the DC principle. By Proposition 2.4(iii)&(iv), we have SV is $\big(\frac{\rho\tilde{\rho}}{\rho+\tilde{\rho}}\big)$-AQNE and satisfies the DC principle. Using Proposition 2.4(i) and Proposition 2.6(i) yields

$$\begin{aligned}
\emptyset \neq \Gamma &= \bigcap_{i=1}^{m} \operatorname{Fix}(S_i) \cap \bigcap_{j=1}^{n} A_j^{-1}\left(\operatorname{Fix}(T_j)\right) \\
&= \bigcap_{i=1}^{m} \operatorname{Fix}(S_i) \cap \bigcap_{j=1}^{n} \operatorname{Fix}(V_j) \\
&= \bigcap_{i=1}^{m} \operatorname{Fix}(S_i) \cap \bigcap_{j=1}^{n} \operatorname{Fix}\left((V_j)_{\lambda_j}\right) \\
&= \operatorname{Fix}(S) \cap \operatorname{Fix}(V) \\
&= \operatorname{Fix}(SV).
\end{aligned}$$

Let $\{x^k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{X}$ defined by (3.1). Now its iteration formula becomes

$$x^{k+1} = \alpha^k F x^k + \left(1 - \alpha^k\right) SV x^k, \quad k \in \mathbb{N}.$$

As $k \to \infty$, we conclude from Theorem 2.7 that $x^k \to x^* \in \Gamma$, where x^* is the unique fixed point of $P_\Gamma F$. ■

Theorem 3.2. *If S_i $(i = 1, 2, \dots, m)$ are ρ_i-AQNE mappings, where $\rho_i > 0$ and T_j $(j = 1, 2, \dots, n)$ are $\tilde{\rho}_j$-AQNE mappings, where $\tilde{\rho}_j > 0$ such that both satisfy the DC principle, then any sequence $\{x^k\}_{k=1}^{\infty}$ generated by (3.1), where the stepsize sequences $\{\gamma_j^k\}_{k=1}^{\infty}$ $(j = 1, 2, \dots, n)$ are selected as:*

$$\gamma_j^k := \begin{cases} \lambda_j \frac{\|(T_j - I)A_j x^k\|^2}{\|A_j^*(T_j - I)A_j x^k\|^2}, & \text{if } A_j x^k \notin \operatorname{Fix}(T_j), \\ 1, & \text{otherwise}, \end{cases} \quad \text{and} \quad \lambda_j \in (0, 1 + \tilde{\rho}_j), \tag{3.3}$$

converges strongly to a point $x^* \in \Gamma$, provided that $\alpha^k \to 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$.

Proof. Let $W_j : \mathcal{X} \to \mathcal{X}$ be an extrapolation of the Landweber-type operator defined by

$$W_j x = \begin{cases} x + \frac{\|(T_j - I)A_j x\|^2}{\|A_j^*(T_j - I)A_j x\|^2} A_j^* (T_j - I) A_j x, & \text{if } A_j x \notin \operatorname{Fix}(T_j), \\ x, & \text{otherwise.} \end{cases}$$

By Proposition 2.6(ii)&(iii), we obtain that W_j is $\tilde{\rho}_j$-AQNE and satisfies the DC principle. Using Proposition 2.3, the relaxation $(W_j)_{\lambda_j}$ defined by

$$(W_j)_{\lambda_j} x = \begin{cases} x + \lambda_j \frac{\|(T_j - I)A_j x\|^2}{\|A_j^*(T_j - I)A_j x\|^2} A_j^* (T_j - I) A_j x, & \text{if } A_j x \notin \operatorname{Fix}(T_j), \\ x, & \text{otherwise,} \end{cases}$$

is τ_j-AQNE, where $\tau_j = \frac{1 + \tilde{\rho}_j - \lambda_j}{\lambda_j}$ and also satisfies the DC principle. Set

$$S := \frac{1}{m} \sum_{i=1}^{m} S_i \quad \text{and} \quad W := \frac{1}{n} \sum_{j=1}^{n} (W_j)_{\lambda_j}.$$

By Proposition 2.4(ii)&(iv), S and W are AQNE with coefficients $\rho = m\left(\sum_{i=1}^{m} \frac{1}{1+\rho_i}\right)^{-1} - 1$ and $\tilde{\rho} = n\left(\sum_{j=1}^{n} \frac{1}{1+\tau_i}\right)^{-1} - 1$, respectively and both satisfy the DC principle. By Proposition 2.4(iii)&(iv), we have SW is $\left(\frac{\rho\tilde{\rho}}{\rho+\tilde{\rho}}\right)$-AQNE and satisfies the DC principle. By using Proposition 2.4(i) and Proposition 2.6(i), one can show that $\operatorname{Fix}(SW) = \Gamma$. Now the iteration formula (3.1) can be in the form:

$$x^{k+1} = \alpha^k F x^k + \left(1 - \alpha^k\right) SW x^k, \quad k \in \mathbb{N}.$$

These together with employing Theorem 2.7 yield $x^k \to x^* \in \Gamma$. ∎

Remark 3.3. It is worth noting on Theorems 3.1 and 3.2 that:

(i) The stepsize γ_j^k defined in (3.2) requires to compute the norm of A_j. Meanwhile, that defined in (3.3) does not depend on any operator norms. It seems that the iteration (3.1) with the stepsizes (3.2) is simple and convenient for use if we know $\|A_j\|$ for all j; however, we should pay attention to the choice of the stepsizes (3.3) in the case that $\|A_j\|$ is difficult to estimate for some j.

(ii) If $\rho_i = 1 = \tilde{\rho}_j$, then we obtain two strong convergence results for the GSCFPP for the class of directed mappings.

(iii) The limit x^* of $\{x^k\}$ has the form as: $x^* = P_\Gamma F x^*$. In particular, if $F \equiv 0$ is constant, then x^* is the minimum norm solution of (1.4).

3.2. Demicontractive Mappings

Recall that a mapping $S : \mathcal{X} \to \mathcal{X}$ having a fixed point is said to be *demicontractive* ([38, 39]) if there exists $\mu \in [0, 1)$ such that

$$\|Sx - z\|^2 \leq \|x - z\|^2 + \mu\|Sx - x\|^2, \quad \forall x \in \mathcal{X}, \forall z \in \operatorname{Fix}(S). \tag{3.4}$$

A mapping S satisfying (3.4) with $\mu = 0$ is said to be *quasi-nonexpansive*.

The following is a relationship between a demicontractive mapping and its relaxation.

Proposition 3.4 ([25]). *Let $S : \mathcal{X} \to \mathcal{X}$ be a mapping having a fixed point and $\mu \in [0, 1)$, $\beta \in (0, 1 - \mu)$. Then, S is μ-demicontractive if and only if S_β is $\left(\frac{1-\mu-\beta}{\beta}\right)$-AQNE.*

Using the above useful property, we slightly modify the iteration (3.1) and obtain convergence results for the GSCFPP for the class of demicontractive mappings by Theorems 3.1 and 3.2.

Let $F : \mathcal{X} \to \mathcal{X}$ be a contraction with respect to Γ. Pick a starting point $x^1 \in \mathcal{X}$ arbitrarily. For $k \in \mathbb{N}$, let

$$x^{k+1} = \alpha^k F x^k + \left(1 - \alpha^k\right) \frac{1}{m} \sum_{i=1}^{m} (S_i)_{\beta_i} \left(x^k + \frac{1}{n} \sum_{j=1}^{n} \gamma_j^k A_j^* \left(T_j - I\right) A_j x^k \right), \tag{3.5}$$

where $\{\gamma_j^k\}_{k=1}^{\infty}$ $(j = 1, 2, \ldots, n)$ are sequences of positive real numbers, β_i $(i = 1, 2, ..., m)$ are positive real numbers and $\{\alpha^k\}_{k=1}^{\infty}$ is a sequence of real numbers in $(0, 1)$.

Corollary 3.5. *If S_i $(i = 1, 2, \ldots, m)$ are μ_i-demicontractive mappings, where $0 \leq \mu_i < 1$ and T_j $(j = 1, 2, \ldots, n)$ are $\tilde{\mu}_j$-demicontractive mappings, where $0 \leq \tilde{\mu}_j < 1$ such that both satisfy the DC principle, then any sequence $\{x^k\}_{k=1}^{\infty}$ generated by (3.5), where the parameter sequences $\{\gamma_j^k\}_{k=1}^{\infty}$ $(j = 1, 2, \ldots, n)$ are chosen as:*

$$\gamma_j^k := \gamma_j \in \left(0, \frac{1 - \tilde{\mu}_j}{\|A_j\|^2}\right), \tag{3.6}$$

converges strongly to a point $x^* \in \Gamma$ (i.e., $x^* = P_\Gamma F x^*$), provided that $\beta_i \in (0, 1 - \mu_i)$; $\alpha^k \to 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$.

Proof. Let $\lambda_j = \gamma_j \|A_j\|^2 \in (0, 1 - \tilde{\mu}_j)$. By Proposition 3.4, both $(S_i)_{\beta_i}$ and $(T_j)_{\lambda_j}$ are AQNE with coefficients $\rho_i = \frac{1-\mu_i-\beta_i}{\beta_i}$ and $\tilde{\rho}_j = \frac{1-\tilde{\mu}_j-\lambda_j}{\lambda_j}$, respectively. Also, they satisfy the DC principle with $\operatorname{Fix}\left((S_i)_{\beta_i}\right) = \operatorname{Fix}\left(S_i\right)$ and $\operatorname{Fix}\left((T_j)_{\lambda_j}\right) = \operatorname{Fix}\left(T_j\right)$. Since

$$(T_j)_{\lambda_j} - I = \lambda_j (T_j - I) = \gamma_j \|A_j\|^2 \left(T_j - I\right),$$

the iteration (3.5) with choosing the stepsizes (3.6) can be rewritten in the form:

$$x^{k+1} = \alpha^k F x^k + \left(1 - \alpha^k\right) \frac{1}{m} \sum_{i=1}^{m} (S_i)_{\beta_i} \left(x^k + \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\|A_j\|^2} A_j^* \left((T_j)_{\lambda_j} - I\right) A_j x^k \right).$$

Since $0 < \frac{1}{\|A_j\|^2} < \frac{1+\tilde{\rho}_j}{\|A_j\|^2}$, the result is obtained directly by Theorem 3.1. ∎

Corollary 3.6. *If S_i $(i = 1, 2, \ldots, m)$ are μ_i-demicontractive mappings, where $0 \le \mu_i < 1$ and T_j $(j = 1, 2, \ldots, n)$ are $\tilde{\mu}_j$-demicontractive mappings, where $0 \le \tilde{\mu_j} < 1$ such that both satisfy the DC principle, then any sequence $\{x^k\}_{k=1}^{\infty}$ generated by (3.5), where the stepsize sequences $\{\gamma_j^k\}_{k=1}^{\infty}$ $(j = 1, 2, \ldots, n)$ are selected as:*

$$\gamma_j^k := \begin{cases} \lambda_j \frac{\|(T_j - I)A_j x^k\|^2}{\|A_j^*(T_j - I)A_j x^k\|^2}, & \text{if } A_j x^k \notin \mathrm{Fix}(T_j), \\ 1, & \text{otherwise}, \end{cases} \quad \text{and} \quad \lambda_j \in (0, 1 - \tilde{\mu}_j), \tag{3.7}$$

converges strongly to a point $x^* \in \Gamma$ (i.e., $x^* = P_\Gamma F x^*$), provided that $\beta_i \in (0, 1 - \mu_i)$; $\alpha^k \to 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$.

Proof. Using Proposition 3.4, we obtain that $(S_i)_{\beta_i}$ and $(T_j)_{\lambda_j}$ are AQNE with coefficients $\rho_i = \frac{1-\mu_i-\beta_i}{\beta_i}$ and $\tilde{\rho}_j = \frac{1-\tilde{\mu}_j-\lambda_j}{\lambda_j}$, respectively such that both satisfy the DC principle with $\mathrm{Fix}\,((S_i)_{\beta_i}) = \mathrm{Fix}\,(S_i)$ and $\mathrm{Fix}\,\big((T_j)_{\lambda_j}\big) = \mathrm{Fix}\,(T_j)$. Since

$$(T_j)_{\lambda_j} - I = \lambda_j (T_j - I),$$

the iteration (3.5) with selecting the stepsizes (3.7) can be rewritten in the form:

$$x^{k+1} = \alpha^k F x^k + (1 - \alpha^k) \frac{1}{m} \sum_{i=1}^{m} (S_i)_{\beta_i} \left(x^k + \frac{1}{n} \sum_{j=1}^{n} \delta_j^k A_j^* \left((T_j)_{\lambda_j} - I \right) A_j x^k \right),$$

where

$$\delta_j^k := \begin{cases} \frac{\|((T_j)_{\lambda_j} - I)A_j x^k\|^2}{\|A_j^*((T_j)_{\lambda_j} - I)A_j x^k\|^2}, & \text{if } A_j x^k \notin \mathrm{Fix}\,\big((T_j)_{\lambda_j}\big), \\ 1, & \text{otherwise}. \end{cases}$$

The result is obtained immediately by Theorem 3.2. ∎

Remark 3.7. It is worth mentioning that:

(i) Applying Theorems 3.1 and 3.2, one can get iterative methods and their strong convergence reults in the case that either $\{S_i\}$ or $\{T_j\}$ is a finite family of demicontractive mappings by means of relaxing the class of demicontractive mappings.

(ii) If $\mu_i = 0 = \tilde{\mu}_j$ in Corollaries 3.5 and 3.6, then we get two strong convergence results for the GSCFPP for the class of quasi-nonexpansive mappings.

4. Other Split Inverse Problems

This section devotes to some consequences of Theorems 3.1 and 3.2 for the constrained multiple-sets split feasibility problem and the split common null point problem, respectively.

4.1. Constrained Multiple-Sets Split Feasibility Problem

Let $\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces. The *constrained multiple-sets split feasibility problem* (CMSSFP) [16] is formulated as finding a point

$$x \in \bigcap_{i=1}^{m} C_i \quad \text{such that} \quad A_j x \in Q_j, \quad \forall j \in \{1, 2, \ldots, n\}, \tag{4.1}$$

where $C_i \subseteq \mathcal{X}$ $(i = 1, 2, \ldots, m)$ and $Q_j \subseteq \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ are nonempty closed convex subsets and $A_j : \mathcal{X} \to \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ are bounded linear operators.

The CMSSFP (4.1) is a special case of the GSCFPP (1.4); namely that if we take $S_i := P_{C_i}$ and $T_j := P_{Q_j}$ are metric projections, then $\operatorname{Fix}(S_i) = C_i$ and $\operatorname{Fix}(T_j) = Q_j$. Since every metric projection is firmly nonexpansive (i.e., 1-AQNE) and always satisfies the DC principle, then the following convergence result for the CMSSFP (4.1) follows from Theorems 3.1 and 3.2.

Corollary 4.1. *Let $C_i \subseteq \mathcal{X}$ $(i = 1, 2, \ldots, m)$ and $Q_j \subseteq \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ be nonempty closed convex subsets and let $A_j : \mathcal{X} \to \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ be bounded linear operators. Assume that the solution set of (4.1), denoted by Ω, is nonempty and let $F : \mathcal{X} \to \mathcal{X}$ be a contraction with respect to Ω. Let $\{x^k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{X}$ defined by $x^1 \in \mathcal{X}$ and*

$$x^{k+1} = \alpha^k F x^k + (1 - \alpha^k) \frac{1}{m} \sum_{i=1}^{m} P_{C_i}\left(x^k + \frac{1}{n} \sum_{j=1}^{n} \gamma_j^k A_j^* (P_{Q_j} - I) A_j x^k \right), \quad k \in \mathbb{N}, \tag{4.2}$$

where $\{\gamma_j^k\}_{k=1}^{\infty} \subset (0, \infty)$ $(j = 1, 2, \ldots, n)$ and $\{\alpha^k\}_{k=1}^{\infty} \subset (0, 1)$ satisfying that $\alpha^k \to 0$ and $\sum_{k=1}^{\infty} \alpha^k = \infty$. If either $\gamma_j^k := \gamma_j \in \left(0, \frac{2}{\|A_j\|^2}\right)$ or

$$\gamma_j^k := \begin{cases} \lambda_j \dfrac{\left\|(P_{Q_j} - I) A_j x^k\right\|^2}{\left\|A_j^*(P_{Q_j} - I) A_j x^k\right\|^2}, & \textit{if } A_j x^k \notin Q_j, \\ 1, & \textit{otherwise}, \end{cases} \quad \textit{and} \quad \lambda_j \in (0, 2), \tag{4.3}$$

then $\{x^k\}_{k=1}^{\infty}$ generated by (4.2) converges strongly to a point $x^* \in \Omega$, where $x^* = P_\Omega F x^*$.

4.2. Split Common Null Point Problem

Let $\mathcal{X}$ and $\mathcal{Y}$ be real Hilbert spaces. Given set-valued mappings $G_i : \mathcal{X} \to 2^{\mathcal{X}}$ $(i = 1, 2, \ldots, m)$ and $H_j : \mathcal{Y} \to 2^{\mathcal{Y}}$ $(j = 1, 2, \ldots, n)$, and bounded linear operators $A_j : \mathcal{X} \to \mathcal{Y}$ $(j = 1, 2, \ldots, n)$, the *split common null point problem* (SCNPP) [18] is to find a point

$$x \in \mathcal{X} \text{ such that } 0 \in \bigcap_{i=1}^{m} G_i(x) \text{ and } 0 \in \bigcap_{j=1}^{n} H_j(A_j x). \tag{4.4}$$

The SCNPP (4.4) has often been discussed when G_i and H_j are maximal monotone. Recall that a set-valued mapping $G : \mathcal{X} \to 2^{\mathcal{X}}$ is called *maximal monotone* if G is *monotone*, i.e.,

$$\langle x - y, v - w \rangle \geq 0, \quad \forall x, y \in \operatorname{dom}(G), \forall v \in Gx, w \in Gy,$$

where $\operatorname{dom}(G) = \{x \in \mathcal{X} : Gx \neq \emptyset\}$, and the graph of G,

$$\operatorname{gph}(G) := \{(x, v) \in \mathcal{X} \times \mathcal{X} : v \in Gx\},$$

is not properly contained in the graph of any other monotone mapping.

A *resolvent* of $G : \mathcal{X} \to 2^{\mathcal{X}}$ with $\xi > 0$ is defined by

$$J_\xi^G := (I + \xi G)^{-1}.$$

It is well known ([18, 40]) that if G is maximal monotone, then $J_\xi^G : \mathcal{X} \to \operatorname{dom}(G)$ is a firmly nonexpansive single-valued mapping (i.e., 1-AQNE) and it also satisfies the DC principle. Moreover, $0 \in Gx$ if and only if $x \in \operatorname{Fix}(J_\xi^G)$.

Using the above properties, we obtain the following convergence result for the SCNPP (4.4) from Theorems 3.1 and 3.2.

Corollary 4.2. *Let $G_i : \mathcal{X} \to 2^{\mathcal{X}}$ $(i = 1, 2, \ldots, m)$ and $H_j : \mathcal{Y} \to 2^{\mathcal{Y}}$ $(j = 1, 2, \ldots, n)$ be maximal monotone mappings with their resolvents $J_\xi^{G_i}$ and $J_\xi^{H_j}$, where $\xi > 0$. Let $A_j : \mathcal{X} \to \mathcal{Y}$ $(j = 1, 2, \ldots, n)$ be bounded linear operators. Assume that the solution set of (4.4), denoted by Δ, is nonempty and let $F : \mathcal{X} \to \mathcal{X}$ be a contraction with respect to Δ. Let $\{x^k\}_{k=1}^\infty$ be a sequence in $\mathcal{X}$ defined by $x^1 \in \mathcal{X}$ and*

$$x^{k+1} = \alpha^k F x^k + (1 - \alpha^k) \frac{1}{m} \sum_{i=1}^m J_\xi^{G_i} \left(x^k + \frac{1}{n} \sum_{j=1}^n \gamma_j^k A_j^* \left(J_\xi^{H_j} - I \right) A_j x^k \right), \quad k \in \mathbb{N}, \quad (4.5)$$

where $\{\gamma_j^k\}_{k=1}^\infty \subset (0, \infty)$ $(j = 1, 2, \ldots, n)$ and $\{\alpha^k\}_{k=1}^\infty \subset (0, 1)$ satisfying that $\alpha^k \to 0$ and $\sum_{k=1}^\infty \alpha^k = \infty$. If either $\gamma_j^k := \gamma_j \in \left(0, \frac{2}{\|A_j\|^2}\right)$ or

$$\gamma_j^k := \begin{cases} \lambda_j \dfrac{\left\| \left(J_\xi^{H_j} - I \right) A_j x^k \right\|^2}{\left\| A_j^* \left(J_\xi^{H_j} - I \right) A_j x^k \right\|^2}, & \text{if } 0 \notin H_j \left(A_j x^k \right), \\ 1, & \text{otherwise}, \end{cases} \quad \text{and} \quad \lambda_j \in (0, 2), \quad (4.6)$$

then $\{x^k\}_{k=1}^\infty$ generated by (4.5) converges strongly to a point $x^* \in \Delta$, where $x^* = P_\Delta F x^*$.

5. Concluding Remarks

In this paper, we discuss the general split common fixed point problem (GSCFPP) in Hilbert spaces, which was introduced in [14]. Strongly convergent algorithms for solving the GSCFPP for the classes of attracting quasi-nonexpansive mappings and demicontractive mappings are presented in two cases of stepsize selection−one depends on the norms of bounded linear operators and the other is independent of any operator norms. The proofs of our main convergence results are concise and straightforward due to some useful properties of attracting quasi-nonexpansive mappings and Landweber-type operators (thanks to [22–25, 29, 32, 33, 37]). Our Theorems 3.1 and 3.2 and Corollaries 3.5, 3.6, 4.1, and 4.2 extend some existing results in [4, 6, 18–21, 24–26] from one mapping/operator to the finite family of such mappings/operators under the same/similar viscosity approximation scheme.

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