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Endomorphism Monoid of C_{2n+1} Book Graphs

J. Thomkeaw and S. Arworn

Abstract : The main purpose of this paper is to prove that the odd cycle book graphs $B_{2n+1}(m)$ are endo-regular for all positive integers m, n and to find the cardinality of the monoid of all endomorphisms on those graphs.

Keywords : Cycle book graph, graph endomorphism, regular endomorphism, endo-regular, cardinality of monoid.
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1 Introduction

The special algebraic properties of groups are that every group has identity element, say e, and every element a has the unique inverse, say a^{-1} which $aa^{-1} = a^{-1}a = e$, therefore, $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

For semigroups or monoids S, we may study a weaker algebraic property of them, pseudo inverse. An $x \in S$ is a pseudo inverse of a if $axa = a$. We say that $a \in S$ is regular if a has a pseudo inverse and the semigroup S is regular if every element in S has pseudo inverse.

For a graph G, we say that G is endo-regular if the monoid $End(G)$ of all endomorphisms on G with the composition is regular.

E. Wilkeit [9], found out the characterization of endo-regular connected bipartite graphs, N. Pipattanajinda [8], characterized regular, completely regular of path and of cycle endomorphisms.

There are some published papers to find the cardinality of the monoid of endomorphisms on graphs, for instance [1], Sr. Arworn found the cardinality of $End(P_n)$, [7] N. Pipattanajinda found the cardinality of $End(C_n)$. From [3] Sr. Arworn and P. Wojtylak and [2], Sr. Arworn and Y. Kim, and [10] W. Wannasit, they found three algorithms for the cardinality of $Hom(P_n, P_m)$, the classes of all homomorphisms from path P_n to path P_m.

For this paper, we considered cycle book graphs. We prove that every odd cycle book graphs are endo-regular and we find out the cardinality of their endomorphism monoids.

2 Definitions and Basic Knowledge

In this section we collect the information that will be needed for an understanding of the other sections. Although, the details are included in some cases, many of the fundamental principles of graph are merely stated without proof.

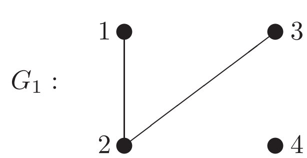
Definition 2.1. *For any graph G, we denote $V(G)$ and $E(G)$ be the vertex set and edge set of the graph G, respectively, where $V(G) \neq \varnothing$ and $E(G) \subseteq \{\{x, y\}|\ x \neq y \text{ in } V(G)\}$. A graph G_1 is called a subgraph of a graph G_2 if $V(G_1) \subseteq V(G_2)$ and $E(G_1) \subseteq E(G_2)$.*

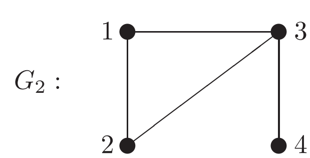
Definition 2.2. *The graph with vertex set $\{0, 1, ..., n-1\}$, such that $n \geq 3$ and edge set $\{\{i, i+1\}|i = 0, 1, ..., n-1\}$ (with addition modulo n)is called the cycle C_n. Therefore, C_n has n vertices and n edges. We call C_n odd or even cycle if n is odd or even, respectively.*

Definition 2.3. *A (graph) homomorphism of a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e. $\{a, b\} \in E(G)$ implies $\{f(a), f(b)\} \in E(H)$ for all $a, b \in V(G)$, we may write $f : G \to H$. If $G = H$, we call f an endomorphism on G. If $f : G \to H$ is a bijective homomorphism and $f^{-1} : H \to G$ is also a homomorphism, we call f an isomorphism, and say that the graph G is isomorphic to the graph H.*

Note that For any graph G, $End(G)$ the class of all endomorphisms on the graph G always forms a monoid with binary operation, composition.

Example 2.4. *Examples of homomorphism and endomorphism.*





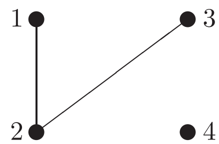
Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}$. *Then f is a homomorphism of the graph G_1 to the graph G_2 but not a homomorphism of G_2 to G_1.*
Let $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}$. *Then $g \in End(G_1)$ but $g \notin End(G_2)$.*

Definition 2.5. *An endomorphism, $f \in End(G)$ is regular if there exists $g \in End(G)$ such that $fgf = f$ and we call g a pseudo inverse to f. We say $End(G)$ is regular if f is regular for all $f \in End(G)$. In this case we like to say that the graph G is $endo-regular$.*

Note 1. Every injective endomorphism is regular.
2. If $End(G)$ is a group, then $End(G)$ is regular.
3. $End(C_{2n+1})$ is a group for all positive integer n.

Example 2.6. *Consider a regular endomorphism f on a graph G.*

$G:$



Let $f \in End(G)$ such that $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}$ and $g_1, g_2 \in End(G)$ such that

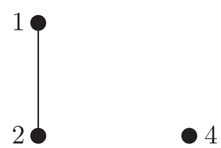
$g_1 = f$, $g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 4 \end{pmatrix}$,

$$\begin{aligned}
\textit{Thus } fg_1f &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}, \\
\textit{and } fg_2f &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 4 \end{pmatrix}.
\end{aligned}$$

Therefore, $fg_1f = f$, $fg_2f = f$ then f is regular and both g_1, g_2 are pseudo inverses of f.

Definition 2.7. *Let f be an endomorphism on a graph G. A subgraph I_f of G is the endomorphic image of G under f if $V(I_f) = f(V(G))$ and $\forall a, b \in V(G)$, $\{f(a), f(b)\} \in E(I_f)$ if and only if there exists $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$. By ρ_f we denote the equivalence relation on $V(G)$ induced by f, if $\forall a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$.*

Example 2.8. *Example of endomorphic image I_f and equivalence relation ρ_f*
Let G and f be as in the Example 2.6,
then $\rho_f = \{(1,1),(2,2),(3,3),(4,4),(1,3),(3,1)\}$ and
$V(I_f) = \{1, 2, 4\}$ which $f^{-1}(1) = \{2\}, f^{-1}(2) = \{1, 3\}, f^{-1}(4) = \{4\}$,
and the graph I_f is the graph below.



The next theorem, W. Li characterized regular endomorphisms on arbitrary graphs.

Theorem 2.9. [6] *Let G be a graph and let $f \in End(G)$. Then f is regular if and only if there exist idempotent endomorphisms $g, h \in End(G)$ such that $\rho_g = \rho_f$ and $I_f = I_h$.*

Theorem 2.10, E. Wilkeit characterized endo-regular connected bipartite graphs and Theorem 2.11, N. Pipattanajinda characterized endo-regular even cycles.

Theorem 2.10. [9] *Let G be a connected bipartite graph. Then G is endo-regular if and only if G is one of the following graphs:*
1. completely bipartite graph $K_{m,n}$, (including K_1, K_2, cycle C_4 and tree T with $d(T) = 2$),
2. tree T with $d(T) = 3$,
3. cycle C_6 and C_8,
4. path with 5 vertices, i.e. P_5,
where $d(T)$ is the diameter of the tree T.

Theorem 2.11. [8] *For any even cycle C_{2n} is an endo-regular if and only if $n \leq 4$.*

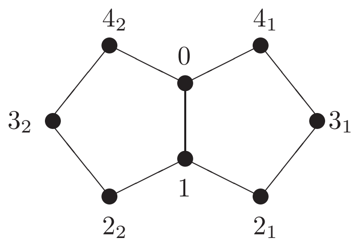
Therefore, for even cycles, only C_4, C_6, C_8 are endo-regular.

Definition 2.12 (Cycle C_n Book Graphs)**.** *For each $i = 1, 2, ..., m$, let G_i be a graph which isomorphic to a cycle C_n with the following vertex set and edge set*
$V(G_i) = \{0_i, 1_i, 2_i, ..., (n-1)_i\}$,
$E(G_i) = \{\{x_i, (x+1)_i\} | \ x = 0, 1, 2, ..., n-1\}$

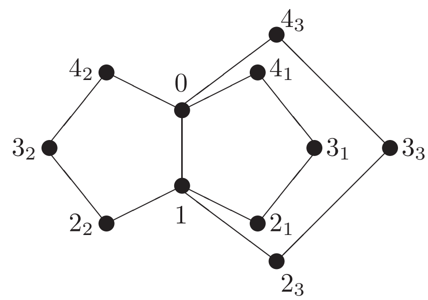
where $0_i = 0$, $1_i = 1$ *for all* $i = 1, ..., m$ *and* $+$ *is the addition modulo* n.
Let $B_n(m)$ *be the* C_n *book graph of* m *pages with*

the vertex set $V(B_n(m)) = \bigcup_{i=1}^{m} V(G_i)$ *and the edge set* $E(B_n(m)) = \bigcup_{i=1}^{m} E(G_i)$.

Example 2.13. *Examples of* $B_5(2)$ *and* $B_5(3)$



$B_5(2)$



$B_5(3)$

Example 2.14. *An example of a regular endomorphism on* $B_5(2)$.

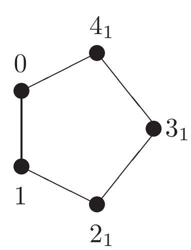
Let $f = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & 4_1 & 2_2 & 3_2 & 4_2 \\ 1 & 2_1 & 3_1 & 4_1 & 0 & 3_1 & 4_1 & 0 \end{pmatrix}$,
$g = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & 4_1 & 2_2 & 3_2 & 4_2 \\ 0 & 1 & 2_1 & 3_1 & 4_1 & 2_1 & 3_1 & 4_1 \end{pmatrix}$.

Then $f, g \in End(B_5(2))$, g *is an idempotent such that*

$V(I_f) = V(I_g) = \{0, 1, 2_1, 3_1, 4_1\}$,

$f^{-1}(0) = \{4_1, 4_2\}, f^{-1}(1) = \{0\}, f^{-1}(2_1) = \{1\}, f^{-1}(3_1) = \{2_1, 2_2\}, f^{-1}(4_1) = \{3_1, 3_2\}$,
$g^{-1}(0) = \{0\}, g^{-1}(1) = \{1\}, g^{-1}(2_1) = \{2_1, 2_2\}, g^{-1}(3_1) = \{3_1, 3_2\}, g^{-1}(4_1) = \{4_1, 4_2\}$,

and $I_f = I_g$:



And

$$\rho_f = \rho_g = \{(0,0), (1,1), (2_1, 2_1), (3_1, 3_1), (4_1, 4_1), (4_1, 4_2), (4_2, 4_1), (2_1, 2_2), (2_2, 2_1), (3_1, 3_2), (3_2, 3_1)\}.$$

Therefore, by Theorem 2.9 (using $h = g$*),* f *is regular.*

3 Regularity of the monoid $End(B_{2n+1}(m))$

This section, we prove that the odd cycle book graphs, $End(B_{2n+1}(m))$ are endo-regular.

Since the graph endomorphisms preserve edges, it is easy to see that

Lemma 3.1. *For any positive integers m, n, $f \in End(B_{2n+1}(m))$ if and only if f is in one of the following forms :*

1.1 $f(0) = 0, f(1) = 1$, and for each $i = 1, 2, ..., m$ there exists $j_i = 1, ..., m$ such that $f(x_i) = x_{j_i}$ for all $x = 2, 3, 4, ..., 2n$.

1.2 $f(0) = 1, f(1) = 0$, and for each $i = 1, 2, ..., m$ there exists $j_i = 1, ..., m$ such that $f(x_i) = (2n - x + 2)_{j_i}$ for all $x = 2, 3, 4, ..., 2n$.

1.3 There exist $a \in \{1, 2, 3, ..., 2n\}$ and $j \in \{1, ..., m\}$ such that $f(x_i) = (a + x)_j$ for all $i = 1, 2, ..., m$ and for all $x = 0, 1, 2, ..., 2n$.

1.4 There exist $a \in \{0, 2, 3, ..., 2n\}$ and $j \in \{1, ..., m\}$ such that $f(x_i) = (a - x)_j$ for all $i = 1, 2, ..., m$ and for all $x = 0, 1, 2, ..., 2n$.

Example 3.2. *Example of endomorphisms on $End(B_5(5))$:*

$$\text{Let } f_1 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & 4_1 & 2_2 & 3_2 & 4_2 & 2_3 & 3_3 & 4_3 & 2_4 & 3_4 & 4_4 & 2_5 & 3_5 & 4_5 \\ 0 & 1 & 2_4 & 3_4 & 4_4 & 2_3 & 3_3 & 4_3 & 2_3 & 3_3 & 4_3 & 2_1 & 3_1 & 4_1 & 2_4 & 3_4 & 4_4 \end{pmatrix},$$

$$f_2 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & 4_1 & 2_2 & 3_2 & 4_2 & 2_3 & 3_3 & 4_3 & 2_4 & 3_4 & 4_4 & 2_5 & 3_5 & 4_5 \\ 1 & 0 & 4_4 & 3_4 & 2_4 & 4_3 & 3_3 & 2_3 & 4_3 & 3_3 & 2_3 & 4_1 & 3_1 & 2_1 & 4_4 & 3_4 & 2_4 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & 4_1 & 2_2 & 3_2 & 4_2 & 2_3 & 3_3 & 4_3 & 2_4 & 3_4 & 4_4 & 2_5 & 3_5 & 4_5 \\ 2_4 & 3_4 & 4_4 & 0 & 1 & 4_4 & 0 & 1 & 4_4 & 0 & 1 & 4_4 & 0 & 1 & 4_4 & 0 & 1 \end{pmatrix},$$

$$f_4 = \begin{pmatrix} 0 & 1 & 2_1 & 3_1 & 4_1 & 2_2 & 3_2 & 4_2 & 2_3 & 3_3 & 4_3 & 2_4 & 3_4 & 4_4 & 2_5 & 3_5 & 4_5 \\ 4_4 & 3_4 & 2_4 & 1 & 0 & 2_4 & 1 & 0 & 2_4 & 1 & 0 & 2_4 & 1 & 0 & 2_4 & 1 & 0 \end{pmatrix}.$$

Then f_1, f_2, f_3, f_4 are in the form 1.1, 1.2, 1.3, 1.4 in Lemma 3.1, respectively.

From Lemma 3.1, then the forms of endomorphic image of odd cycle book graphs $B_{2n+1}(m)$ are as the following lemma.

Lemma 3.3. *If $f \in End(B_{2n+1}(m))$, then I_f is in one of the following forms:*

1. If f is in the form 1.1 or 1.2 in Lemma 3.1 then I_f is a C_{2n+1} book graph of k pages for some positive integer $k \leq m$.

2. If f is in the form 1.3 or 1.4 in Lemma 3.1 then I_f is isomorphic to $B_{2n+1}(1)$, a C_{2n+1} book graph of one pages.

Lemma 3.4. *Let $f \in End(B_{2n+1}(m))$. If there exists $x \neq 0, 1$ and $f(x_i) = f(x_j)$ then $(x_i, x_j) \in \rho_f$ for all $x = 0, 1, 2, ..., 2n$..*

Since $End(B_{2n+1}(1))$ is isomorphic to cycle C_{2n+1} and $End(C_{2n+1})$ is a group,

Lemma 3.5. *For any positive integer n, $End(B_{2n+1}(1))$ is regular.*

Theorem 3.6. *For any integer $m \geq 2$, $End(B_{2n+1}(m))$ is regular.*

Proof: *Let $f \in End(B_{2n+1}(m))$.*

Case 1 *If f is in the form 1.1 or 1.2 in Lemma 3.1. Let $J_f = \{j_i | i = 1, 2, ..., m\}$ and let $g, h \in End(B_{2n+1}(m))$ be such that $g(0) = 0, g(1) = 1$, and for all $x \in \{2, 3, 4...., 2n\}$,*

$$g(x_i) = \begin{cases} x_i & : \quad i \in J_f \\ x_j, \ for \ some \ j \in J_f & : \quad i \notin J_f \end{cases}$$

And $h(0) = 0, h(1) = 1$, and for $i_1, i_2 \in \{2, ..., m\}$, if $i_1 \leq i_2$ and $j_{i_1} = j_{i_2}$, we let $h(x_{i_1}) = h(x_{i_2}) = x_{i_1}$ for all $x = 2, 3, 4, ..., 2n$.

Then g, h are an idempotent, $I_f = I_g$, where $V(I_f) = V(I_g) = \{x_j | j \in J_f$ and $x = 0, 1, ..., 2n\}$, and $\rho_f = \rho_h$.

Therefore, by Theorem 2.9, f is regular.

Case 2 *If f is in the form 1.3 or 1.4 in Lemma 3.1,*
We let $h = g \in End(B_{2n+1}(m))$ be such that
$g(x_i) = x_j$ for all $i = 1, 2, ..., m$ and for all $x = 0, 1, 2, ..., 2n$.
Then g, h are an idempotent, $I_f = I_g$ and $\rho_f = \rho_h$. Therefore, f is regular.

□

4 Cardinality of the monoid $End(B_{2n+1}(m))$

This section, we calculate for the cardinality of the monoid $End(B_{2n+1}(m))$. There are some previous results about these, for instance [1], Sr. Arworn found the cardinality of $End(P_n)$, [7] N. Pipattanajinda found the cardinality of $End(C_n)$, [2],[3], and [10] Sr. Arworn and P. Wojtylak , Y. Kim, W. Wannasit found three different algorithms for the cardinality of $Hom(P_n, P_m)$, the classes of all homomorphisms from path P_n to path P_m.

Theorem 4.1. *For any positive integer m, n,*

$$|End(B_{2n+1}(m))| = 2[m^m + 2nm]$$

.

Proof: *Let $f \in End(B_{2n+1}(m))$. From Lemma 3.1,*

Case 1 *f is in the form 1.1,*

$f(0) = 0, f(1) = 1$, and for each $i = 1, 2, ..., m$ there exists $j_i = 1, ..., m$ such that $f(x_i) = x_{j_i}$ for all $x = 2, 3, 4, ..., 2n$.

Therefore, $f \in End(B_{2n+1}(m))$ of this form if and only if f is a mapping which fixed $0, 1$ and for each $i = 1, 2, 3, ..., m$, select any $j \in \{1, 2, 3, ..., m\}$ and let $f(x_i) =$

x_j. *Then the number of endomorphisms in this form is* $\underbrace{m \times m \times m \times \ldots \times m}_{m\ term} = m^m$.

Case 2 *f is in the form 1.2,*

$f(0) = 1, f(1) = 0$, for each $i = 1, 2, ..., m$ there exists $j_i = 1, ..., m$ such that $f(x_i) = (2n - x + 2)_{j_i}$ for all $x = 2, 3, 4, ..., 2n$.

Similar to the case 1, $f \in End(B_{2n+1}(m))$ of this form if and only if f is a mapping which $f(0) = 1, f(1) = 0$ and for each $i = 1, 2, 3, ..., m$, select any $j \in \{1, 2, 3, ..., m\}$ and let $f(x_i) = (2n - x + 2)_j$. Then the number of endomorphisms in this form is $\underbrace{m \times m \times m \times \ldots \times m}_{m\ term} = m^m$.

Case 3 *f is in the form 1.3,*

there exists $a \in \{1, 2, 3, ..., 2n\}$ and $j \in \{1, ..., m\}$ such that

$f(x_i) = (a + x)_j$ for all $i = 1, 2, ..., m$ and for all $x = 0, 1, 2, ..., 2n$.

For this case, we concentrate only how many ways we can map 0

to, the other points will follow the form of the mapping. Then the number of endomorphisms in this form is $\sum_{j=1}^{m}(2n) = (2n)m$.

Case 4 *f is in the form 1.4,*

there exists $a = 0, 2, 3, ..., 2n$ and $j = 1, ..., m$ such that

$f(x_i) = (a - x)_j$ for all $i = 1, 2, ..., m$ and for all $x = 0, 1, 2, ..., 2n$.

As in the case 3, the number of endomorphisms in this form is

$\sum_{j=1}^{m}(2n) = (2n)m$.

Therefore, $|End(B_{2n+1}(m))| = m^m + m^m + 2nm + 2nm = 2[m^m + 2nm]$.

□

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Jakkrit Thomkeaw, Srichan Arworn
Department of Mathematics, Faculty of Sciences
Chiang Mai University,
Chiang Mai 50200
Thailand
e-mail: `chackerr@hotmail.com, srichan28@yahoo.com`