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Weak Convergence Theorem of Noor Iterative Scheme for Nonself I-Nonexpansive Mapping

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Abstract : In this paper, we prove the weak convergence of a modified Noor iteration for nonself I-nonexpansive mapping in a Banach space which satisfies Opial's condition. Our result extends and improves these announced by Kumam, Kumethog and Jewwaiworn [Weak convergence theorem of Three-step Noor iteration scheme for I-nonexpansive mappings in Banach spaces, Applied Mathematical Science, Vol.2, 2008, no.59, 2915-2920] from self maps into nonself maps. And Kiziltunc and Ozdemir [On convergence theomrem for Nonself I-nonexpansive mapping in Banach spaces, Applied Mathematical Science, Vol.1, 2007, no.48, 2379-2383].

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1 Introduction

Let $E := (E, \|.\|)$ be a real Banach space, K be a nonempty convex subset of E, and T be a self mapping of K. The Mann iteration [9] is defined as $x_1 \in K$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1. \tag{1.1}$$

The Ishikawa iteration [5] is defined as $x_1 \in K$ and

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1. \end{cases} \tag{1.2}$$

The Noor iteration [8] is defined as $x_1 \in K$ and

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n = (1-\beta_n)x_n + \beta_n T z_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$.

In the above taking $\beta_n = 0$ in (1.2) and taking $\beta_n = 0, \gamma_n = 0$ in (1.3) we obtain iteration (1.1).

In 1975, Baillon [1] first introduced nonlinear ergodic theorem for general nonexpansive mapping in a Hilbert space H: if K is a closed and convex subset of H and T has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of T. It was also shown by Pazy [11] that if H is a real Hilbert space and $(\frac{1}{n})\sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

In 1941, Tricomi introduced the concept of a quasi-nonexpansive mapping for real functions. Later Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [6] in metric spaces which we adapt to a normed space as the following: T is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \leq \|x - f\| \tag{1.4}$$

for all $x \in K$ and $f \in F(T)$.

Recall that a Banach space E is said to satisfy Opial's condition [10] if, for each sequence $\{x_n\}$ in E, the condition $x_n \to x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \tag{1.5}$$

for all $y \in E$ with $y \neq x$. It is well known from [10] that all l_p spaces for $1 < p < \infty$ have this property. However, the l_p spaces do not, unless $p = 2$.

The following definitions and statements are needed for the proof of our theorem.

Let K be a closed convex bounded subset of uniformly convex Banach spaces E and T self-mapping of E. Then T is called nonexpansive on K if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.6}$$

for all $x, y \in K$. Let $F(T) := \{x \in K : Tx = x\}$ be denote the set of fixed points of a mapping T.

Let K be a subset of a normed space E and T and I self-mappings of K. Then T is called I-nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\| \tag{1.7}$$

for all $x, y \in K$ [14].

A mapping T is called I-quasi-nonexpansive on K if

$$\|Tx - f\| \leq \|Ix - f\| \tag{1.8}$$

for all $x, y \in K$ and $f \in F(T) \cap F(I)$.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A set K is optimal if each point outside K can be moved to be closer to all points of K. Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive. There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, Petryshyn and Williamson [12] studied the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive mapping. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, Ghosh and Debnath [4] considered the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces. Later Temir and Gul [15] proved the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space. In [16], the convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In [13], Rhoades and Temir considered T and I self-mappings of K, where T is I-nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I. More precisely, they proved the following theorems.

Theorem (Rhoades and Temir [13]): *Let K be a closed convex bounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Noor iterates converges weakly to common fixed point of $F(T) \cap F(I)$.*

In the above theorem, T remains self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain K of T is a proper subset of E and T maps K into E then, the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction $P : E \to K$ in the recursion formula (1.1) as follows: $x_1 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n PTx_n, \qquad n \geq 1.$$

In [7], Kiziltunc and Ozdemir considered T and I are nonself mapping of K where T is an I-nonexpensive mapping. They established the weak convergence of the sequence of the modified Ishikawa iterative scheme to a common fixed point of T and I.

$$\begin{cases} y_n = P((1-\beta_n)x_n + \beta_n T x_n) \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T y_n), \quad n \geq 1. \end{cases} \tag{1.9}$$

In this paper, we consider T and I are nonself mappings of K, where T is an I-nonexpansive mappings. We prove the weak convergence of the sequence of modified Noor iterative scheme to a common fixed point of $F(T) \cap F(I)$.

2 Main Results

In this section, we prove the weak convergence theorem.

Theorem 2.1. *Let K be a closed convex bounded subset of a uniformly convex Banach space E which satisfies Opial's condition, and let T, I nonself mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Noor iterates defined by $x_1 \in K$,*

$$\begin{cases} z_n = P((1-\gamma_n)x_n + \gamma_n T x_n) \\ y_n = P((1-\beta_n)x_n + \beta_n T z_n) \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T y_n), \quad n \geq 1, \end{cases} \tag{2.1}$$

converges weakly to common fixed point of $F(T) \cap F(I)$.

Proof. If $F(T) \cap F(I)$ is nonempty and a singleton, then the proof is complete. We will assume that $F(T) \cap F(I)$ is nonempty and that $F(T) \cap F(I)$ is not a singleton.

$$\begin{aligned} \|x_{n+1} - f\| &= \|P((1-\alpha_n)x_n + \alpha_n T y_n) - f\| \\ &= \|(1-\alpha_n)x_n + \alpha_n T y_n - (1-\alpha_n+\alpha_n)f)\| \\ &= \|(1-\alpha_n)(x_n - f) + \alpha_n(T y_n - f)\| \\ &\leq (1-\alpha_n)\|x_n - f\| + \alpha_n \|T y_n - f\| \\ &\leq (1-\alpha_n)\|x_n - f\| + \alpha_n \|y_n - f\| \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \|y_n - f\| &= \|P((1-\beta_n)x_n + \beta_n T z_n) - f\| \\ &= \|(1-\beta_n)(x_n - f) + \beta_n(T z_n - f)\| \\ &\leq (1-\beta_n)\|x_n - f\| + \beta_n \|T z_n - f\| \\ &\leq (1-\beta_n)\|x_n - f\| + \beta_n \|z_n - f\| \end{aligned} \tag{2.3}$$

and also, we get

$$\begin{aligned} \|z_n - f\| &= \|P((1-\gamma_n)x_n + \gamma_n T x_n) - f\| \\ &= \|(1-\gamma_n)(x_n - f) + \gamma_n(T x_n - f)\| \\ &\leq (1-\gamma_n)\|x_n - f\| + \gamma_n \|T x_n - f\| \\ &\leq (1-\gamma_n)\|x_n - f\| + \gamma_n \|x_n - f\| \\ &= \|x_n - f\|. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\begin{aligned} \|y_n - f\| &\leq (1-\beta_n)\|x_n - f\| + \beta_n\|z_n - f\| \\ &\leq (1-\beta_n)\|x_n - f\| + \beta_n\|x_n - f\| \\ &= \|x_n - f\|. \end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.2), we have

$$\begin{aligned} \|x_{n+1} - f\| &\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|y_n - f\| \\ &\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|x_n - f\| \\ &= \|x_n - f\|. \end{aligned} \tag{2.6}$$

Thus, for $\alpha_n \neq 0, \beta_n \neq 0$ and $\gamma_n \neq 0, \{\|x_n - f\|\}$ is a nonincreasing sequence. Then, $\lim_{n\to\infty} \|x_n - f\|$ exists.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and I. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in K. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converge weakly to f and q, respectively. We will show that $f = q$. Suppose that E satisfies Opial's condition and that $f \neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\} \to f$ and $\{x_{m_k}\} \to q$, respectively. Since $\lim_{n\to\infty} \|x_n - f\|$ exists for any $f \in F(T) \cap F(I)$, by Opial's condition, we conclude that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - f\| &= \lim_{k\to\infty} \|x_{n_k} - f\| < \lim_{k\to\infty} \|x_{n_k} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| = \lim_{j\to\infty} \|x_{m_j} - q\| \\ &< \lim_{j\to\infty} \|x_{m_j} - f\| = \lim_{n\to\infty} \|x_n - f\|. \end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T)\cap F(I)$. This completes the proof. □

Corollary 2.2. (Kumam et al. [8, Theorem 2.1]) *Let K be a closed convex bounded subset of a uniformly convex Banach space X, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-quasi-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of three-step Noor iterative scheme defined by (1.3) converges weakly to common fixed point of $F(T) \cap F(I)$.*

Corollary 2.3. (Kiziltunc and Ozdemir [7, Theorem 2.1]) *Let K be a closed convex bounded subset of a uniformly convex Banach space E, which satisfies Opial's condition, and let T, I nonself mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_1 \in K$, the sequence $\{x_n\}$ of modified Ishikawa iterates defined by (1.9) converges weakly to common fixed point of $F(T) \cap F(I)$.*

Theorem 2.4. *Let K be a closed convex bounded subset of a uniformly convex Banach space E, which satisfies Opial's condition, and let T, I nonself mappings* *of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_1 \in K$, the sequence $\{x_n\}$ of Mann converges weakly to common fixed point of $F(T) \cap F(I)$.*

Proof. Putting $\gamma_n = 0$ and $\beta_n = 0$ in Theorem 2.1, we obtain the desired result. □

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