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Parameter Estimation for Weighted Two-Parameter Exponential Distribution

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Abstract Herein, estimation techniques for the two parameters of the weighted two-parameter exponential distribution (WTED) are presented. To this end, various methods such as maximum likelihood estimation (MLE), method of moment (MOM), jackknife of MLE (JMLE), and jackknife of MOM (JMOM) were utilized. In a simulation study, the performance of the proposed methods were compared based on their mean square error estimates. The results show that JMLE, MLE, and MOM provide the most suitable estimators for the WTED in cases where $\theta > \beta$, $\theta = \beta$, and $\theta < \beta$, respectively.

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1. Introduction

Statistical inference based on parameter estimation is classified into two categories: interval estimation and point estimation, both of which employ statistical distributions to estimate parameters. Most data in real-world problems are not normally distributed, and so distributions such as the weighted exponential distribution are used in these situations [1]. It has been widely employed in various research fields pertaining to reliability, biomedicine, and ecology, among others [2].

As examples of applying the WED, Shakhatreh [2] analyzed two real datasets, one of which comprised right-censored data, Gupta and Kundu [1] used it on a dataset comprising the marks in mathematics assignments for students, and Hosmer and Lemeshow [3] analyzed the survival times in months of 100 patients infected with HIV.

Many researchers have attempted to estimate distribution parameters using various techniques. Of these, Azzalini [4] provided estimators for the skewness and shape parameters of normal distributions. In this approach, the density function of random variable

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Z is defined as

$$\phi(z;\lambda) = 2\phi(z)\Phi(\lambda z), \quad (-\infty < z < \infty) \tag{1.1}$$

where ϕ and Φ are the standard normal density and distribution function, respectively. Then it can be obtained that Z is a skew-normal random variable with parameter λ.

Other researchers have also developed estimators for the parameters of symmetric distributions [5], [6]. Concept of Azzalini was extended to exponential distributions by Gupta and Kundu [1], who defined its shape parameter as follows:

Definition 1.1. Let X be a random sample drawn from weighted two-parameter exponential distribution (WTED) [1], with the shape and scale parameters as $\theta > 0$ and $\beta > 0$, respectively, if the probability density function (PDF) of X is

$$f(x;\theta,\beta) \quad = \begin{cases} \frac{1+\theta}{\theta}\beta e^{-\beta x}(1-e^{-\beta\theta x}), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

This model can be obtained as a hidden truncation model as it was observed by Arnold and Beaver [5] in case of skew-normal distribution. Suppose Z and Y are two dependent random variables with the joint PDF as given below for $\beta > 0$;

$$f_{Z,Y}(z,y) = \beta^2 z e^{-z(1+y)}; \;\; z > 0, \;\; y > 0. \tag{1.3}$$

Subsequently, the maximum likelihood estimation for the unknown parameter was calculated, and its asymptotic distribution was considered.

Following that, Mezaal [7] used the numerical technique to compare the properties of estimators for the parameters of the weighted two-parameter exponential distribution (WTED) using three approaches: the jackknife method, the method of moments (MOM), and maximum likelihood estimation (MLE). Mean squared error (MSE) estimation uncovered that the Jackknife method was the most suitable technique for estimating the two parameters.

Therefore, the aim of this study is to propose estimators for the parameters of WTED based on MLE, MOM, JMLE, and JMOM. Since all of the estimators are mathematically intractable, the Newton-Raphson method was applied to calculate the estimator of WTED parameters. This method is different from the numerical technique proposed by Mezaal [7]. The following condition is considered to obtain the estimator:

$$\left|\hat{\theta}_i - \hat{\theta}_{i-1}\right| + \left|\hat{\beta}_i - \hat{\beta}_{i-1}\right| < \varepsilon,$$

where $i = 1, 2, \ldots$. The criteria for evaluating the performance of the proposed method is the estimation of MSE.

The rest of this paper is organized as follows. Some basic definitions and useful statistical properties of WTED are provided in Section 2, while the details of the MLE, MOM, JMLE, and JMOM methods are presented in Section 3. The simulation study to test the methods and the results thereof are contained in Section 4. An example using real data is used to further compare the efficacies of the four estimation methods for WTED is provided in Section 5. Finally, conclusions from the study are detailed in Section 6.

2. Weighted Two-Parameter Exponential Distribution

First, cumulative distribution function (CDF), and other statistical properties are derived, followed by estimation formulas for (θ) and (β) based on the four methods.

The CDF for WTED can be expressed as

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= \int_0^x f(t)\,dt \\ &= \tfrac{1+\theta}{\theta} \int_0^x \beta e^{-\beta t}(1-e^{-\beta\theta t})\,dt \\ &= \tfrac{1+\theta}{\theta}\{\int_0^x \beta e^{-\beta t}\,dt - \int_0^x \beta e^{-\beta t(1+\theta)}\,dt\} \\ &= \tfrac{1+\theta}{\theta}\{(1-e^{-\beta x}) - \tfrac{1}{1+\theta}(1-e^{-\beta x(1+\theta)})\}. \end{aligned}$$

Thus,

$$\begin{aligned} F_X(x) &= \tfrac{1+\theta}{\theta}\tfrac{1}{1+\theta}\{(1+\theta)(1-e^{-\beta x}) - (1-e^{-\beta x(1+\theta)}))\} \\ &= \tfrac{1}{\theta}\{e^{-\beta x(1+\theta)} + \theta - e^{-\beta x}(1+\theta)\}. \end{aligned} \tag{2.1}$$

2.1. The Statistical Properties of WTED

The moment generating function is given by:

$$\begin{aligned} M_X(t) &= E(e^{tx}) = \int_0^\infty e^{tx} f(x;\beta,\theta)\,dx \\ &= (\tfrac{1+\theta}{\theta})\beta \int_0^\infty e^{tx} e^{-\beta x}(1-e^{-\beta\theta x})\,dx \\ &= (\tfrac{1+\theta}{\theta})\beta\{\int_0^\infty e^{-x(\beta-t)}dx - \int_0^\infty e^{-x(\beta-t+\beta\theta)}\,dx\}. \end{aligned}$$

So

$$M_X(t) = \frac{1+\theta}{\theta}\beta[\frac{\beta\theta}{(\beta-t)(\beta-t+\beta\theta)}]. \tag{2.2}$$

The expected value of X can be obtained by the following step. Let

$$M_X'(t) = (1+\theta)\beta^2\{(\beta-t)^{-1}(\beta-t+\beta\theta)^{-2} + (\beta-t+\beta\theta)^{-1}(\beta-t)^{-2}\}.$$

Then

$$\begin{aligned} M_X'(0) &= (1+\theta)\beta^2[\beta^{-1}(\beta+\beta\theta)^{-2} + (\beta+\beta\theta)\beta^{-2}] \\ &= (1+\theta)\beta^2[\tfrac{1}{\beta(\beta+\beta\theta)^2} + \tfrac{1}{(\beta+\beta\theta)\beta^2}] \\ &= \tfrac{(1+\theta)(2\beta+\beta\theta)}{(\beta+\beta\theta)^2} \\ &= \tfrac{\beta(1+\theta)(2+\theta)}{\beta^2(1+\theta)^2} \\ &= \tfrac{(2+\theta)}{\beta(1+\theta)}. \end{aligned}$$

Therefore, the expected value of X can be easily obtained as

$$E(X) = \frac{(2+\theta)}{\beta(1+\theta)}. \tag{2.3}$$

Consider the variance of X. Let

$$\begin{aligned} M''_X(t) &= (1+\theta)\beta^2[\{2(\beta-t)^{-1}(\beta-t+\beta\theta)^{-3}+(\beta-t+\beta\theta)^{-2}(\beta-t)^{-2}\} \\ &\quad +\{2(\beta-t+\beta\theta)^{-1}(\beta-t)^{-3}+(\beta-t)^{-2}(\beta-t+\beta\theta)^{-2}\}]. \end{aligned}$$

Then

$$\begin{aligned} M''_X(0) &= (1+\theta)\beta^2[\tfrac{2}{\beta(\beta+\beta\theta)^3}+\tfrac{1}{\beta^2(\beta+\beta\theta)^2}+\tfrac{2}{\beta^3(\beta+\beta\theta)}+\tfrac{1}{\beta^2(\beta+\beta\theta)^2}] \\ &= (1+\theta)[\tfrac{2}{\beta^2(1+\theta)^3}+\tfrac{1}{\beta^2(1+\theta)^2}+\tfrac{2}{\beta^2(1+\theta)}+\tfrac{1}{\beta^2(1+\theta)^2}] \\ &= \tfrac{2}{\beta^2}[\tfrac{3+3\theta+\theta^2}{(1+\theta)^2}]. \end{aligned}$$

Subsequently,

$$E(X^2) = \frac{2(3+3\theta+\theta^2)}{\beta^2(1+\theta)^2}. \tag{2.4}$$

Therefore, the variance of X is given by

$$\begin{aligned} V(X) &= E(X^2)-(E(X))^2 \\ &= \tfrac{2(3+3\theta+\theta^2)}{\beta^2(1+\theta)^2}-(\tfrac{(2+\theta)}{\beta(1+\theta)})^2 \\ &= \tfrac{2+2\theta}{\beta^2(1+\theta)^2} \\ &= \tfrac{2}{\beta^2(1+\theta)}. \end{aligned} \tag{2.5}$$

3. Materials and Methods

3.1. Maximum Likelihood Method

The log likelihood function for WTED is given by:

$$L = \prod_{i=1}^{n} f(x_i,\beta,\theta) = (1+\theta)^n\theta^{-n}\beta^n e^{-\beta\sum_{i=1}^n x_i}\prod_{i=1}^{n}(1-e^{-\beta\theta x_i}), \tag{3.1}$$

$$\log L = n\log(1+\theta) - n\log\theta + n\log\beta - \beta\sum_{i=1}^{n}x_i + \log[\prod_{i=1}^{n}(1-e^{-\beta\theta x_i})]. \tag{3.2}$$

Then, it can be obtained that

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{1+\theta} - \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\beta x_i e^{-\beta\theta x_i}}{1-e^{-\beta\theta x_i}} = 0.$$

This implies

$$\frac{n}{\theta} = \frac{n}{1+\theta} + \sum_{i=1}^{n}\frac{\beta x_i e^{-\beta\theta x_i}}{1-e^{-\beta\theta x_i}}.$$

So

$$\theta = \frac{n}{\frac{n}{1+\theta}+\sum_{i=1}^{n}\frac{\beta x_i e^{-\beta\theta x_i}}{1-e^{-\beta\theta x_i}}}. \tag{3.3}$$

Similarly, it can be obtained that

$$\frac{\partial \log L}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\frac{-\theta x_i e^{-\beta\theta x_i}}{1-e^{-\beta\theta x_i}} = 0.$$

Hence

$$\frac{n}{\beta}=\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{\theta x_i e^{-\beta\theta x_i}}{1-e^{-\beta\theta x_i}},$$

and

$$\beta=\frac{n}{\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{\theta x_i e^{-\beta\theta x_i}}{1-e^{-\beta\theta x_i}}}. \tag{3.4}$$

Then the estimators $\hat{\theta}_{MLE}$ and $\hat{\beta}_{MLE}$ can be obtained from

$$\hat{\theta}_{MLE}=\frac{n}{\frac{n}{1+\hat{\theta}_{MLE}}+\sum_{i=1}^{n}\frac{\hat{\beta}_{MLE}x_i e^{-\hat{\beta}_{MLE}\hat{\theta}_{MLE}x_i}}{1-e^{-\hat{\beta}_{MLE}\hat{\theta}_{MLE}x_i}}}, \tag{3.5}$$

and

$$\hat{\beta}_{MLE}=\frac{n}{\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{\hat{\theta}_{MLE}x_i e^{-\hat{\beta}_{MLE}\hat{\theta}_{MLE}x_i}}{1-e^{-\hat{\beta}_{MLE}\hat{\theta}_{MLE}x_i}}}. \tag{3.6}$$

3.2. Method of Moments

Let $\mu'_k=E(X^k)$ and $M'_k=\frac{\sum_{i=1}^{n}X_i^k}{n}$ represent the k-moment of the population and the k-moment of the sample, respectively. The k-moment of the sample is well known to be an approximation of the k-moment of the population, that is, $\mu'_k\approx M'_k$, where $k=1,2,\ldots$ (see Casella [8]).

Consider the k-moment with $k=1$ and 2, which are obtained from the equation (2.3) and (2.4) that

$$\mu'_1=E(X)=\frac{2+\theta}{\beta(1+\theta)}\approx\frac{\sum_{i=1}^{n}X_i}{n}, \tag{3.7}$$

and

$$\mu'_2=E(X^2)=\frac{2(3+3\theta+\theta^2)}{\beta^2(1+\theta)^2}\approx\frac{\sum_{i=1}^{n}X_i^2}{n}. \tag{3.8}$$

From equation (3.7), it can be obtained that

$$\beta(1+\theta)\approx\frac{2+\theta}{\overline{X}}. \tag{3.9}$$

From equation (3.8), it can be obtained that

$$\frac{\sum_{i=1}^{n}X_i^2}{n}\beta^2(1+\theta)^2\approx 2(3+3\theta+\theta^2). \tag{3.10}$$

By substituting equation (3.9) in equation (3.10), it can be obtained that

$$\frac{\sum_{i=1}^{n}X_i^2}{n}\cdot\frac{2+\theta}{\overline{X}}\beta(1+\theta)\approx 2(3+3\theta+\theta^2). \tag{3.11}$$

Then

$$\frac{\beta\sum_{i=1}^{n}X_i^2}{n}\approx\frac{2\overline{X}(3+3\theta+\theta^2)}{2+3\theta+\theta^2}, \tag{3.12}$$

and so

$$\frac{\beta\sum_{i=1}^{n}X_i^2}{n}\approx 2\overline{X}(1+\frac{1}{2+3\theta+\theta^2}). \tag{3.13}$$

Hence

$$\hat{\beta}_{MOM} = \frac{2n\overline{X}}{\sum_{i=1}^{n} X_i^2}\left(1 + \frac{1}{2 + 3\hat{\theta}_{MOM} + \hat{\theta}_{MOM}^2}\right), \tag{3.14}$$

and

$$\hat{\theta}_{MOM} = \frac{\hat{\beta}_{MOM}\overline{X} - 2}{1 - \hat{\beta}_{MOM}\overline{X}}, \tag{3.15}$$

where $1 < \hat{\beta}_{MOM}\overline{X} < 2$.

3.3. Jackknife Method

In 1956, Quenouille [9] proposed the Jackknife method for estimating parameters to decrease bias of estimator. The Jackknife method is a sampling method which generates an additional sample from a single arbitrary sample without replacement (Resampling without replacement).

This method requires eliminating a single value at position i, where $i = 1, 2, ..., n$, to generate an additional sample of size $n - 1$. The cut-off values will be put back into the original sample before generating the next new sample. By repeating this n times, the data will be obtained and used to estimate parameters.

3.3.1. Jackknife of Maximum Likelihood

If $\hat{\theta}_{MLE}(j)$ and $\hat{\beta}_{MLE}(j)$ are the maximum likelihood estimator after excluding j value, then the bias-corrected jackknife estimators for θ defined as follows:

$$\hat{\theta}_{J\_MLE} = n\hat{\theta}_{MLE} - (n-1)\frac{\sum_{j=1}^{n} \hat{\theta}_{MLE}(j)}{n}, \tag{3.16}$$

and the bias-corrected jackknife estimator for β is defined as

$$\hat{\beta}_{J\_MLE} = n\hat{\beta}_{MLE} - (n-1)\frac{\sum_{j=1}^{n} \hat{\beta}_{MLE}(j)}{n}. \tag{3.17}$$

3.3.2. Jackknife of Method of Moments

If $\hat{\theta}_{MOM}(j)$ and $\hat{\beta}_{MOM}(j)$ are the moment of method estimators after excluding j value, then the jackknife estimator for θ defined as follows:

$$\hat{\theta}_{J\_MOM} = n\hat{\theta}_{MOM} - (n-1)\frac{\sum_{j=1}^{n} \hat{\theta}_{MOM}(j)}{n}, \tag{3.18}$$

and the jackknife estimator for β is defined as

$$\hat{\beta}_{J\_MOM} = n\hat{\beta}_{MOM} - (n-1)\frac{\sum_{j=1}^{n} \hat{\beta}_{MOM}(j)}{n}. \tag{3.19}$$

The above resulted equations have no explicit solutions. The Newton–Raphson of MLE, MOM, JMLE, and JMOM method can be used to obtain the solution under the following condition:

$$\left|\hat{\theta}_i - \hat{\theta}_{i-1}\right| + \left|\hat{\beta}_i - \hat{\beta}_{i-1}\right| < \varepsilon,$$

where $i = 1, 2, \ldots$.

4. Simulation Study and Results

The simulation study was conducted to provide a comparative analysis of the efficacies of the four estimation methods for the parameters of the WTED. To this end, the values of parameters (θ, β) were set as (0.3, 0.3), (0.3, 0.6), (0.3, 0.9), (0.3, 1.5), (0.6, 0.3), (0.9, 0.3), (1.5, 0.3), (0.6, 0.6), (0.9, 0.9), or (1.5, 1.5), the sample size (n) was set as 30, 50, 100, and 300 and the specific error (ϵ) was set as 10^{-5}. The instructions for the simulation were written in the R language (version 4.2.3). Moreover, 10,000 iterations were conducted for each scenario. The $M\hat{S}E(\hat{\theta}, \hat{\beta})$, the criterion for evaluating the performances of the proposed methods, was calculated as follows:

$$M\hat{S}E(\hat{\theta}, \hat{\beta}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2 + (\hat{\beta}_i - \beta)^2.$$

The method with the minimum $M\hat{S}E(\hat{\theta}, \hat{\beta})$ for a particular scenario was chosen as the best performing one.

The results of each experiment are shown in Table 1 - Table 3.

From the simulation results reported in Table 1 - Table 3, it can be observed that JMLE provided the lowest $M\hat{S}E(\hat{\theta}, \hat{\beta})$ values when $\theta > \beta$ for all of the sample size. Meanwhile, when $\theta = \beta$, MLE provided the lowest $M\hat{S}E(\hat{\theta}, \hat{\beta})$ values except for (0.3, 0.3) and n = 300, for which MOM provided the lowest value. Finally, MOM provided the lowest $M\hat{S}E(\hat{\theta}, \hat{\beta})$ values when $\theta < \beta$ for all of the sample sizes.

Table 1. $M\hat{S}E(\hat{\theta}, \hat{\beta})$ of WTED for $(\theta = \beta)$

$\theta = \beta$	**n**	**MLE**	**MOM**	**JMLE**	**JMOM**
0.3	30	**0.001752**	0.006635	0.049452	0.208724
	50	**0.000993**	0.004230	0.048382	0.232898
	100	**0.000891**	0.002414	0.034133	0.209154
	300	0.001502	**0.000860**	0.023298	0.326737
0.6	30	**0.051199**	0.106753	0.521408	0.792553
	50	**0.074092**	0.096603	0.312291	0.533632
	100	**0.024194**	0.090398	0.235900	0.486868
	300	**0.014311**	0.083461	0.153173	0.345446
0.9	30	**0.027168**	0.400452	1.257637	0.402331
	50	**0.014770**	0.377873	0.909477	0.378850
	100	**0.012776**	0.363326	0.723228	0.363368
	300	**0.001915**	0.347842	0.559257	0.348042
1.5	30	**0.949533**	1.630906	1.067826	1.636527
	50	**0.179544**	0.418188	0.198408	0.405825
	100	**0.089984**	0.151520	0.096760	0.151246
	300	**0.029772**	0.146888	0.030998	0.146937

TABLE 2. $M\hat{S}E(\hat{\theta}, \hat{\beta})$ of WTED for $(\theta < \beta)$

θ	β	**n**	**MLE**	**MOM**	**JMLE**	**JMOM**
0.3	0.6	30	0.080939	**0.011551**	0.527006	0.376078
		50	0.092989	**0.007308**	0.529633	0.390024
		100	0.070987	**0.004014**	0.434396	0.343377
		300	0.052375	**0.001335**	0.362341	0.501630
	0.9	30	0.325985	**0.019743**	1.383439	0.654968
		50	0.365213	**0.012437**	1.451409	0.651927
		100	0.298262	**0.006681**	1.309858	0.567091
		300	0.245109	**0.002128**	1.102416	0.793116
	1.5	30	1.285960	**0.046031**	4.768370	1.562189
		50	1.408427	**0.029000**	4.741966	1.523376
		100	1.225951	**0.015246**	4.180342	1.410147
		300	1.007223	**0.004625**	4.393253	2.778765

TABLE 3. $M\hat{S}E(\hat{\theta}, \hat{\beta})$ of WTED for $(\theta > \beta)$

β	θ	**n**	**MLE**	**MOM**	**JMLE**	**JMOM**
0.3	0.6	30	0.084390	0.097595	**0.027526**	0.312930
		50	0.076125	0.090312	**0.015638**	0.337233
		100	0.087171	0.086234	**0.012708**	0.318703
		300	0.099032	0.081046	**0.014098**	0.427024
	0.9	30	0.322163	0.358740	**0.140374**	0.587625
		50	0.305024	0.346017	**0.115282**	0.612200
		100	0.329448	0.339303	**0.126611**	0.599118
		300	0.354893	0.330072	**0.143642**	0.697653
	1.5	30	1.282350	1.389510	**0.798006**	1.639089
		50	1.244456	1.364423	**0.742012**	1.662322
		100	1.301840	1.351494	**0.794438**	1.662297
		300	1.360526	1.333125	**0.850731**	1.730585

5. Applications

In this part of the study, a real data set of Mathematics score [1] was analyzed to compare the performance of the proposed method for estimating the parameters of WTED.

Dataset 5.1. A real dataset of mathematics scores for 48 students in the slow-paced course in 2003 in India was obtained from Gupta and Kundu [1]
29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, 31.

For this dataset, $n = 48$ with a mean of $X = 25.89$ and a standard deviation of $s = 18.60$. The distribution of this data was tested using the Kolmogorov-Smirnov test, which revealed that WTED was a good fit $(D = 0.125, p = 0.8363)$ where D is maximum deviation from Kolmogorov-Smirnov test. From the Dataset 5.1, the MLE of θ and β

$(\hat{\theta}, \hat{\beta})$ based on Gupta and Kundu [1] are 0.2919 and 0.0685, respectively. It is obvious that $\hat{\theta} > \hat{\beta}$. By applying the proposed methods which are MLE, MOM, JMLE, and JMOM based on Newton-Raphson, the results in Table 4 indicated that $\hat{\theta} > \hat{\beta}$ in every method. This result is in good agreement with Gupta and Kundu [1]. In the simulation study, JMLE outperforms when $\theta > \beta$. Therefore, the suitable estimator for this dataset is JMLE.

TABLE 4. The estimation of θ and β

	Gupta and Kundu	**MLE**	**MOM**	**JMLE**	**JMOM**
$\hat{\theta}$	0.2919	0.34235	0.26486	0.31794	0.28107
$\hat{\beta}$	0.0685	0.06738	0.06899	0.06862	0.06810

The JMLE covered most of the PDF of the WTED of the data as shown in Figure 1.

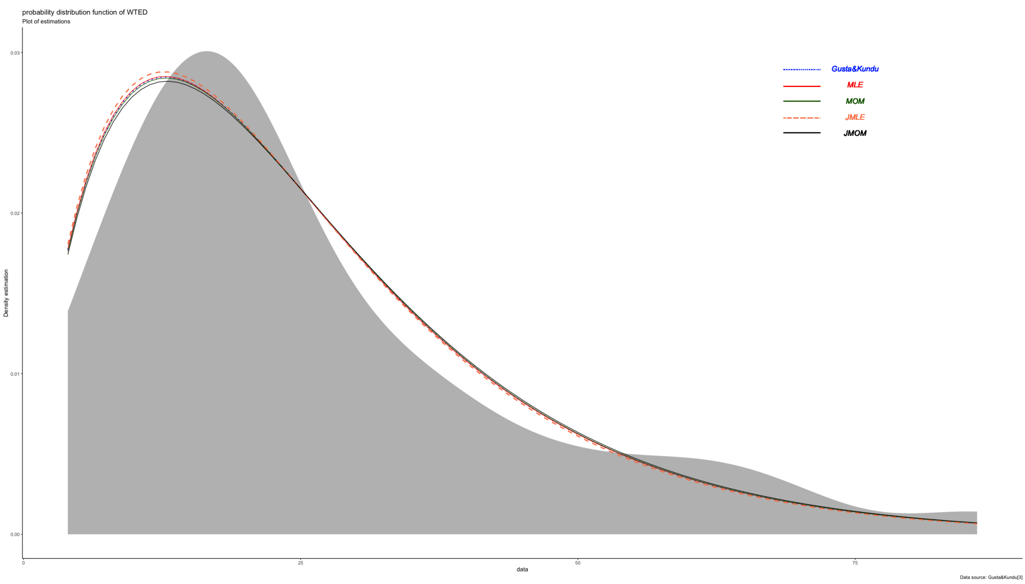

FIGURE 1. A graph of the estimate probability density function of MLE, MOM, JMLE, and JMOM for data set from [1]

Figure 1 shows that JMLE covers most of the PDF of the data. In comparison, the MLE, MOM, and Gupta and Kundu [1] estimation methods provided similar coverage of the PDF, while JMOM covered the least.

6. Conclusion

Parameters θ and β for the WTED were estimated by utilizing MLE, MOM, JMLE, and JMOM. Their efficacies were compared by using a Monte Carlo simulation study based on their $M\hat{S}E(\hat{\theta}, \hat{\beta})$ values. Parameters θ and β were considered simultaneously, for which the non-closed form problem was solved using the Newton-Raphson approach. The results indicate that when $\theta > \beta$, the JMLE is the most suitable for parameter estimation since it provided the lowest $M\hat{S}E(\hat{\theta}, \hat{\beta})$ values in this scenario. Similarly, MLE and MOM are the most suitable when $\theta = \beta$ and $\theta < \beta$, respectively. Furthermore, the parameters of a real dataset that followed WTED were estimated using the four estimation methods. The results are in accordance with Gupta and Kundu [1].

References

[1] R.D. Gupta, D. Kundu, A new class of weighted exponential distributions, Stat. 43 (6) (2009) 621–634.

[2] M.K. Shakhatreh, A two-parameter of weighted exponential distributions, Statist. Probab. Lett. 82 (2) (2012) 252–261.

[3] D.W. Hosmer, S. Lemeshow, Regression Modelling of Time to Event Data, New York: Wiley, 1999.

[4] A. Azzalini, A class of distributions which includes the normal ones, Scand. J. Stat. (1985) 171–178.

[5] B.C. Arnold, R.J. Beaver, The skew-Cauchy distribution, Statist. Probab. Lett. 49 (3) (2000) 285–290.

[6] M.G. Genton, Skew-elliptical distributions and their applications: A journey beyond normality, CRC Press, 2004.

[7] A.P.D.A.R. Mezaal, On the weighted two-parameter exponential distribution, J. Xian Univ. Archit. Technol. 12 (4) (2020) 6013–6020.

[8] G. Casella, R.L. Berger, Statistical Inference Pacific Grove, CA: Duxbury, 2002.

[9] M.H. Quenouille, Problems in plane sampling, Ann. Math. Stat. 20 (3) (1949) 355–375.