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A Finite Element Simulation of Water Quality Measurement in the Open Reservoir

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Abstract : In this research, a mathematical model is used to simulate the pollution in the reservoir. The hydrodynamic model that provides the velocity field and elevation of the water. The dispersion model gives the pollutant concentration as a water quality measurement. The reservoir is connected to the outer water area is considered. In the simulating processes, we combined the Galerkin finite element method and the finite difference method to the system of hydrodynamic model. At each time step the calculated velocity of the hydrodynamic model are input to the second model as field data. The case study in simulation of the current and the pollutant concentration in the open reservoir are presented.

Keywords : tidal current; hydrodynamic model; shallow water equation; convection-diffusion equation; finite element method; finite difference method.
2000 Mathematics Subject Classification : 65M06, 62P12

1 Introduction

The increases in an industrial occupation is the principal reason for the growth of pollution. The methods to detect the amount of pollutant both in the air and water mostly are conducted by a field measurement and a mathematical simulation.

For the shallow water mass transport problems that presented in [2], the method of characteristics has been reported as being applied with success, but it presents in real cases some difficulties. In [5], the finite element method for solving a steady water pollution model is presented. The most of mathematical model require data concerning with velocity of the current at any point in the domain. The hydrodynamic model provides the velocity field and tidal elevation of the water. Those results are data for the dispersion model. In [6], they used the finite difference method to the hydrodynamic model. In [8] and [9], they used the finite element method to simulate the dispersion in the Bay of Santander and the Lumtakong reservoir, respectively. In [7], the finite difference method to simulate the water current and the pollutant dispersion in the regular geometry is presented.

The reservoir has two space dimensions as shown in Fig.1. Averaging the equation over the depth, discarding the term due to friction force and surface wind, it follows that the two-dimensional shallow water is applicable [4]. In this study, the finite element techniques in the treatment of space discretisation will be used and the finite differences in the time domain will be considered.

2 The Hydrodynamic Model

2.1 Shallow water equations

In this section, we will give the basic equations which described the tidal current. We consider an area of the reservoir with rectangular coordinates x, y, z (m). Let the surface of the mean water level designate the reference plane $z = 0$. When the shearing stresses, the surface wind and frictional forces are neglected [4], the equations of continuity and motion become

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (HU)}{\partial x} + \frac{\partial (HV)}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} - fV + g\frac{\partial \zeta}{\partial x} = 0, \tag{2.2}$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + fU + g\frac{\partial \zeta}{\partial y} = 0 \tag{2.3}$$

where $h(x, y)$ is the depth measured from the mean water level to the reservoir bed, $\zeta(x, y, t)$ is the tidal elevation which the elevation from the mean water level to the temporary water surface with the time variable t (s), $H = h+\zeta$ is the total depth of the reservoir, U, V (m/s) are the velocity components of the current in x and y directions averaged over H, g is the gravity acceleration, and f is the Coriolis parameter. These equations are called the shallow water equations.
There are two kinds of boundary conditions that we are considered.

(i) Along the coastal denoted by Γ_V, the normal component of the current velocity is equal to zero:

$$V_n = Un_x + Vn_y = 0 \text{ on } \Gamma_V, \tag{2.4}$$

where n_x, n_y are the components of the external unit normal to the boundary.

(ii) Along the boundary Γ_ζ, where the region is open to the outer water area, the tidal elevation is prescribed.

$$\zeta(t) = A\sin(\frac{2\pi}{T}t) \text{ on } \Gamma_\zeta, \tag{2.5}$$

where A is the amplitude of the monochromatic tidal wave (m), and $T = 12+5/12$ (hours) being its period.

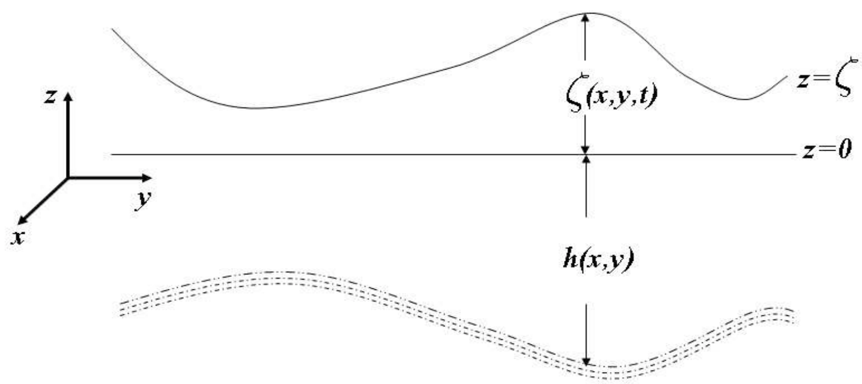

Figure 1: The shallow water system.

The initial condition is calculated from calm reservoir, we suppose that $\zeta = 0, U = V = 0$ at $t = 0$, and in Ω, where Ω is the domain to be analyzed. This technique is called a cold start.

3 Numerical solution of the hydrodynamic model

3.1 The finite element discretisation

We will apply the Galerkin finite element method to the continuity equation (2.1) and the shallow water equations (2.2)-(2.3) for the discretisation. Let ω_ζ be the weighting functions such that $\omega_\zeta(x, y) = 0$ for all $(x, y) \in \Gamma_\zeta$. Consider the weighted residual form of the continuity equation.

$$\iint_\Omega \omega_\zeta \frac{\partial \zeta}{\partial t} d\Omega + \iint_\Omega \omega_\zeta \left(\frac{\partial}{\partial x}(HU) + \frac{\partial}{\partial y}(HV)\right) d\Omega = 0 \tag{3.1}$$

where $d\Omega = dxdy$. Integration by parts of the second term yields that

$$\iint_\Omega \omega_\zeta \frac{\partial \zeta}{\partial t} d\Omega - \iint_\Omega H\left(\frac{\partial \omega_\zeta}{\partial x} U + \frac{\partial \omega_\zeta}{\partial y} V\right) d\Omega = -\int_{\Gamma_v} \omega_\zeta H V_n d\Gamma. \tag{3.2}$$

Let ω_u and ω_v be arbitrary weighting functions. The weighted residual form corresponding to the equations of motion can be written as follows,

$$\iint_\Omega \omega_u \frac{\partial U}{\partial t} d\Omega + \iint_\Omega \omega_u \left(U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y}\right) d\Omega - \iint_\Omega \omega_u f V d\Omega + g \iint_\Omega \omega_u \frac{\partial \zeta}{\partial x} d\Omega = 0, \tag{3.3}$$

and

$$\iint_\Omega \omega_v \frac{\partial V}{\partial t} d\Omega + \iint_\Omega \omega_v \left(U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y}\right) d\Omega - \iint_\Omega \omega_v f U d\Omega + g \iint_\Omega \omega_v \frac{\partial \zeta}{\partial y} d\Omega = 0. \tag{3.4}$$

The domain Ω is divided into triangular elements. Inside each triangle e with its tree vertices (x_j, y_j) as the element nodes 1,2,3, the tidal elevation as well as the mean velocity components are linearly interpolated as follows,

$$\zeta \simeq \sum_{j=1}^{3} \phi_j(x,y)\zeta_j(t) \quad , \quad \omega_\zeta = \sum_{j=1}^{3} \phi_j(x,y)\omega_{\zeta_j}(t), \tag{3.5}$$

$$U \simeq \sum_{j=1}^{3} \phi_j(x,y)U_j(t) \quad , \quad \omega_u = \sum_{j=1}^{3} \phi_j(x,y)\omega_{u_j}(t), \tag{3.6}$$

$$V \simeq \sum_{j=1}^{3} \phi_j(x,y)V_j(t) \quad , \quad \omega_v = \sum_{j=1}^{3} \phi_j(x,y)\omega_{v_j}(t) \tag{3.7}$$

where ϕ_j for all $j = 1, 2, 3$ are linear interpolation functions, given by $\phi_j = \frac{1}{2\Delta^e}(a_j + b_j x + c_j y)$ when Δ^e is the area of the element e, $\Delta^e = \frac{b_2c_3 - b_3c_2}{2}$ where

$$\begin{array}{lll} a_1 = x_2x_3 - x_3x_2, & a_2 = x_3x_1 - x_1x_3, & a_3 = x_1x_2 - x_2x_1, \\ b_1 = y_2 - y_3, & b_2 = y_3 - y_1, & b_3 = y_1 - y_2, \\ c_1 = x_3 - x_2, & c_2 = x_1 - x_3, & c_3 = x_2 - x_1. \end{array} \tag{3.8}$$

The interpolation Eqs.(3.5)-(3.7) are substituted into Eqs.(3.3)-(3.2). The element equations in each triangle becomes

$$\sum_{i=1}^{3} M_{ij}^e \dot{\zeta}_i - \sum_{i=1}^{3}\sum_{k=1}^{3} X_{ikj}^e U_i H_k - \sum_{i=1}^{3}\sum_{k=1}^{3} Y_{ikj}^e V_i H_k = -\sum_{i=1}^{3} R_{ij}^e H_i, \tag{3.9}$$

$$\sum_{i=1}^{3} M_{ij}^e \dot{U}_i + \sum_{i=1}^{3}\sum_{k=1}^{3} X_{jki}^e U_i U_k + \sum_{i=1}^{3}\sum_{k=1}^{3} Y_{jik}^e V_i U_k - \sum_{i=1}^{3}\sum_{k=1}^{3} N_{jik}^e f_i V_k + g\sum_{i=1}^{3} P_{ji}\zeta_i = 0, \tag{3.10}$$

$$\sum_{i=1}^{3} M_{ij}^e \dot{V}_i + \sum_{i=1}^{3}\sum_{k=1}^{3} X_{jik}^e U_i V_k + \sum_{i=1}^{3}\sum_{k=1}^{3} Y_{jik}^e V_i V_k + \sum_{i=1}^{3}\sum_{k=1}^{3} N_{jik}^e f_i U_k + g\sum_{i=1}^{3} Q_{ji}\zeta_i = 0 \tag{3.11}$$

where the matrices are given by

$$M_{ij}^e = \iint_e \phi_i \phi_j d\Omega = \frac{\Delta^e}{12}(1+\delta_{ij}), \tag{3.12}$$

$$X_{jik}^e = \iint_e \phi_j \phi_i \frac{\partial \phi_k}{\partial x} d\Omega = \frac{b_k}{24}(1+\delta_{ji}), \tag{3.13}$$

$$Y_{jik}^e = \iint_e \phi_j \phi_i \frac{\partial \phi_k}{\partial y} d\Omega = \frac{c_k}{24}(1+\delta_{ji}), \tag{3.14}$$

$$N_{jik}^e = \iint_e \phi_j \phi_i \phi_k d\Omega, \tag{3.15}$$

$$P_{ji}^e = \iint_e \phi_j \frac{\partial \phi_i}{\partial x} d\Omega = \frac{b_i}{6}, \tag{3.16}$$

$$Q_{ji}^e = \iint_e \phi_j \frac{\partial \phi_i}{\partial y} d\Omega = \frac{c_i}{6}, \tag{3.17}$$

$$R_{ji}^e = \int_{\Gamma_v^e} \phi_j \phi_i V_n d\Gamma = \frac{\Gamma_v^e}{6}(1+\delta_{ji})V_n \tag{3.18}$$

and $\Gamma_v^e = \Gamma_v \cap \partial e$ where ∂e is a boundary of an element e. After assembling all

Table 1: Matrix N_{jik}^e, each entries are multiplied by $\Delta^e/60$

j	1			2			3		
i, k	1	2	3	1	2	3	1	2	3
1	6	2	2	2	2	1	2	1	2
2	2	2	1	2	6	2	1	2	2
3	2	1	2	1	2	2	2	2	6

the element over the domain, we can obtain the total equation in the form,

$$[M]\{\dot{\zeta}\} - [H(U,V)]\{H\} = -[R]\{H\}, \tag{3.19}$$

$$[M]\{\dot{U}\} + [A(U,V)]\{U\} - [N]\{V\} + g[P]\{\zeta\} = \{0\}, \tag{3.20}$$

$$[M]\{\dot{V}\} + [A(U,V)]\{V\} + [N]\{U\} + g[Q]\{\zeta\} = \{0\}. \tag{3.21}$$

where $[M], [N], [H(U,V)], [A(U,V)], [P], [Q]$ and $[R]$ are assemble matrices of Eqs.(3.12-3.18)

3.2 The time integration of hydrodynamic model

We will solve the equations (3.19)-(3.21) as a non-linear system of first-order ordinary differential equations for unknowns $\{H\}, \{U\}$ and $\{V\}$. We consider discretisation of the total equations with respect to time. We will present the finite difference method for the initial value problem of differential equation,

$$\frac{dy}{dx} = f(x,y) \text{ with } y(x_0) = y_0 \tag{3.22}$$

with a sufficiently smooth function $f(x,y)$. Using the small increment in x and time t, denoted by Δx and Δt respectively, we know that the following two-step explicit scheme is second-order accurate [1],

$$y^{n+1/2} = y^n + \frac{1}{2}\Delta x f(x_n, y^n), \tag{3.23}$$
$$y^{n+1} = y^n + \Delta x f(x_{n+1/2}, y^{n+1/2}) \tag{3.24}$$

for the time step $n = 1, 2, 3, \ldots, T$, where T is the ended time. The scheme is called a modified Euler method [1, 4]. Application of the scheme to the column vector $\{U(t)\}$ is

$$\{U^{n+1/2}\} = \{U^n\} + \frac{1}{2}\Delta t\{\dot{U}^n\}, \tag{3.25}$$
$$\{U^{n+1}\} = \{U^n\} + \Delta t\{\dot{U}^{n+1/2}\}. \tag{3.26}$$

Multiplying the matrix $[M]$ from the left of both Eqs.(3.25)-(3.26), it follows from Eq.(3.20) that

$$[M]\{U^{n+1/2}\} = [M]\{U^n\} + \frac{1}{2}\Delta t\,[M]\{\dot{U}^n\}, \tag{3.27}$$
$$= [M]\{U^n\} - \frac{1}{2}\Delta t([A(U^n, V^n)]\{U^n\} - [N]\{V^n\} + g[P]\{\zeta^n\}), \tag{3.28}$$

and

$$\begin{aligned} [M]\{U^{n+1}\} &= [M]\{U^n\} + \Delta t\,[M]\{\dot{U}^{n+1/2}\}, &(3.29)\\ &= [M]\{U^n\} - \Delta t([A(U^{n+1/2}, V^{n+1/2})]\{U^{n+1/2}\} \\ &\quad -[N]\{V^{n+1/2}\} + g[P]\{\zeta^{n+1/2}\}), &(3.30) \end{aligned}$$

Since the derivation of the equations is based on the explicit modified Euler method, they are implicit because the mass matrix $[M]$ is not diagonal in general. To make the scheme truly explicit, we may replace $[M]$ on the left hand side with its lumped mass matrix $[\bar{M}]$. We also replace the mass matrix on the right hand side with the selectively lumped mass matrix. We then have

$$\left[\tilde{M}\right] = \epsilon\left[\bar{M}\right] + (1-\epsilon)[M] \tag{3.31}$$

with selectively lumping parameter ϵ. In [3] shown that $\epsilon = 0.4$. It follows that

$$\begin{aligned} \left[\bar{M}\right]\{U^{n+1/2}\} &= \left[\tilde{M}\right]\{U^n\} - \frac{1}{2}\Delta t([A(U^n, V^n)]\{U^n\} - [N]\{V^n\} \\ &\quad +g[P]\{\zeta^n\}), &(3.32)\\ \left[\bar{M}\right]\{U^{n+1}\} &= \left[\tilde{M}\right]\{U^n\} - \Delta t([A(U^{n+1/2}, V^{n+1/2})]\{U^{n+1/2}\} - [N]\{V^{n+1/2}\} \\ &\quad +g[P]\{\zeta^{n+1/2}\}). &(3.33) \end{aligned}$$

In the similar way, from Eq.(3.20) and Eq.(3.21), we can obtain

$$\begin{aligned}\left[\bar{M}\right]\{V^{n+1/2}\} &= \left[\tilde{M}\right]\{V^n\} - \frac{1}{2}\Delta t([A(U^n,V^n)]\{V^n\} - [N]\{U^n\} \\ &\quad +g[Q]\{\zeta^n\}), \qquad (3.34)\\ \left[\bar{M}\right]\{V^{n+1}\} &= \left[\tilde{M}\right]\{V^n\} - \Delta t([A(U^{n+1/2},V^{n+1/2})]\{V^{n+1/2}\} - [N]\{U^{n+1/2}\} \\ &\quad +g[Q]\{\zeta^{n+1/2}\}), \qquad (3.35)\end{aligned}$$

and

$$\begin{aligned}\left[\bar{M}\right]\{\zeta^{n+1/2}\} &= \left[\tilde{M}\right]\{\zeta^n\} + \frac{1}{2}\Delta t([H(U^n,V^n)]\{H^n\} - [R]\{H^n\}), \qquad (3.36)\\ \left[\bar{M}\right]\{\zeta^{n+1}\} &= \left[\tilde{M}\right]\{\zeta^n\} + \Delta t([H(U^{n+1/2},V^{n+1/2})]\{H^{n+1/2}\} \\ &\quad -[R]\{H^{n+1/2}\}). \qquad (3.37)\end{aligned}$$

From Eqs.(3.32)-(3.37), values of the unknowns U^{n+1}, V^{n+1} and ζ^{n+1} can be obtained as functions of their values at the previous time step, U^n, V^n and ζ^n. Using this step-by-step procedure and considering the initial condition, the unknown values at each time can be found.

4 The Dispersion Model

4.1 Convection-diffusion equation

In this section we now introduced the convection-diffusion by averaging the equation over the depth, we can obtain

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} - D_x\frac{\partial^2 C}{\partial x^2} - D_y\frac{\partial^2 C}{\partial y^2} + RC - Q = 0, \qquad (4.1)$$

where $C(x,y)$ is the concentration at the point (x,y) in Ω (kg/m^3), u and v are flow velocity in X,Y directions (m/sec), D_x and D_y are diffusion coefficients in X,Y directions (m^2/sec), R is the substance decaying rate (sec^{-1}) and Q is the increasing rate of substance concentration due to a source(kg/m^3sec).

Two types of boundary condition can be considered: S_1 with specified COD concentration and S_2 with specified flux of concentration, and total boundary Γ is $\Gamma = S_1 \cup S_2$. The boundary condition on S_1 and S_2 are

$$\begin{aligned} C &= C_B \text{ on } S_1, \qquad (4.2)\\ \frac{\partial C}{\partial n} &= \frac{\partial C}{\partial x}cos(x,n) + \frac{\partial C}{\partial y}cos(y,n) = T_B \text{ on } S_2. \qquad (4.3)\end{aligned}$$

The initial conditions for all domain points are

$$C_{t=0} = C_0. \qquad (4.4)$$

Application of the Galerkin method and the Green theorem give the weak formulation

$$\iint_\Omega \frac{\partial \widetilde{C}}{\partial t}\theta_i dxdy + \iint_\Omega \Big\{D_x \frac{\partial \theta_i}{\partial x}\frac{\partial \widetilde{C}}{\partial x} + D_y \frac{\partial \theta_i}{\partial y}\frac{\partial \widetilde{C}}{\partial y}\Big\}dxdy + \iint_\Omega \Big\{u\frac{\partial \widetilde{C}}{\partial x} + v\frac{\partial \widetilde{C}}{\partial y}\Big\}\theta_i dxdy$$

$$+\iint_\Omega \Big\{R\widetilde{C} - Q\Big\}\theta_i dxdy = \oint_\Gamma \theta_i\Big\{(-D_y\frac{\partial \widetilde{C}}{\partial y})dx + (D_x\frac{\partial \widetilde{C}}{\partial x})dy\Big\}. \qquad (4.5)$$

The approximation form of solution is

$$\widetilde{C} = \sum_{j=1}^{N} \widetilde{C}_j \theta_j, \qquad (4.6)$$

where the global basis functions are the ones previously defined. The matrix form of Eq.(4.5) is

$$[M]\{\frac{dC}{dt}\} = \{-([K]+[S]+[H])\{C\} + \{F\} + \{B\}\} \qquad (4.7)$$

where square matrices of order $nde \times nde$, for all $i = 1, 2, \ldots, nde$ and $j = 1, 2, \ldots, nde$ and nde is the total number of nodes, which are

$$\begin{aligned}
M_{ij} &= \iint_e \phi_i\phi_j d\Omega, && (4.8)\\
K_{ij} &= \iint_e \Big\{D_x\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} + D_y\frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\Big\}d\Omega, && (4.9)\\
S_{ij} &= \iint_e \Big\{u_j\frac{\partial\phi_j}{\partial x} + v_j\frac{\partial\phi_j}{\partial y}\Big\}\phi_i d\Omega, && (4.10)\\
H_{ij} &= \iint_e \Big\{R(\phi_i\phi_j)\Big\}d\Omega, && (4.11)\\
F_i &= \iint_e \Big\{Q\phi_i\Big\}d\Omega, && (4.12)\\
B_i &= \oint_\Gamma \phi_i\Big\{(-D_y\frac{\partial\widetilde{C}}{\partial y})dx + (D_x\frac{\partial\widetilde{C}}{\partial x})dy\Big\}. && (4.13)
\end{aligned}$$

Matrices are given by

$$M_{ij}^e = \frac{\Delta^e}{24}(1+\delta_{ij}), \qquad (4.14)$$

$$K_{ij}^e = \frac{1}{2\Delta^e}\begin{bmatrix} D_xb_1^2 + D_yc_1^2 & D_xb_1b_2 + D_yc_1c_2 & D_xb_1b_3 + D_yc_1c_3 \\ D_xb_1b_2 + D_yc_1c_2 & D_xb_2^2 + D_yc_2^2 & D_xb_2b_3 + D_yc_2c_3 \\ D_xb_1b_3 + D_yc_1c_3 & D_xb_2b_3 + D_yc_2c_3 & D_xb_3^2 + D_yc_3^2 \end{bmatrix}, (4.15)$$

$$S_{ij}^e = \frac{1}{6}\begin{bmatrix} u_1b_1 + v_1c_1 & u_2b_2 + v_2c_2 & u_3b_3 + v_3c_3 \\ u_1b_1 + v_1c_1 & u_2b_2 + v_2c_2 & u_3b_3 + v_3c_3 \\ u_1b_1 + v_1c_1 & u_2b_2 + v_2c_2 & u_3b_3 + v_3c_3 \end{bmatrix}, \tag{4.16}$$

$$H_{ij}^e = \frac{R\Delta^e}{24}(1 + \delta_{ij}), \tag{4.17}$$

$$F_i^e = \frac{Q\Delta^e}{6}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \tag{4.18}$$

$$B_\alpha^{[b]} = \frac{D_x T_B \ell}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{4.19}$$

where $\{B\}^{[b]}$ arising from boundary elements edge $[b]$ which connecting between two nodes such adjacent on the boundary, ℓ is the length of boundary element, and α are local node number of adjacent nodes on the boundary.

4.2 Time integration of dispersion model

For tim integration, the equation is solved by finite differences. This leads to the following matrix equation

$$\begin{aligned}\left[\bar{M}\right]\{C^{n+1/2}\} &= \left[\tilde{M}\right]\{C^n\} \\ &+\frac{1}{2}\Delta t\Big\{-([K]+[S]+[H])\{C^n\}+\{F\}+\{B\}\Big\},\end{aligned} \tag{4.20}$$

$$\begin{aligned}\left[\bar{M}\right]\{C^{n+1}\} &= \left[\tilde{M}\right]\{C^n\} + \Delta t\{C^n\} \\ &+\Delta t\Big\{-([K]+[S]+[H])\{C^{n+1/2}\}+\{F\}+\{B\}\Big\}.\end{aligned} \tag{4.21}$$

The values of unknowns $\{C^{n+1}\}$ can be obtained as functions of their values at the previous time step $\{C^n\}$. Using this step-by-step procedure, the unknown values at each time can be found.

5 Application to the Bay

Consider the opens bay to the outer water area with an island in Fig.2. The bay is opened to the outer water area from the boundary. The coastal along the bay and island Γ_V is non-absorbing wave. Along the estuary on the boundary Γ_ζ, the tidal elevation is $\zeta(t) = \sin(\frac{2\pi}{T}t)$, where $T = 12$ hours. We take time increment $\Delta t = 80$ sec.

We can use these physical parameter to set up the boundary and initial condition for system of Eq.(2.3). We can then using the finite element mesh, the nodal points, water depth and element connectivity are given in Fig.3, Table 3-4, respectively. The calculated velocity and water elevation from the mean water are shown in the Table 3 and Fig.4.

Assume that there are plants and villages which discharge wastewater into the reservoir along the coastal of the reservoir and that the pollutant concentrations of the wastewater are specified on the Table 2. The physical parameters are: diffusion coefficient in x-direction and y-direction are 2.5 m^2/s, substance decay rate 1.0 sec^{-1} and rate of change of substance concentration due to a source 0.0 $kg/lsec$. We can use these physical parameter to set up the boundary and initial condition for Eq.(4.1) . The calculated pollutant concentrations at 12 hrs. are shown on the Table 6 and Fig.5.

6 Conclusions

In this research a hydrodynamic model and the dispersion model for simulate the water current and the pollutant concentration in the reservoir when the velocity of the current is not uniform, are constructed. For the estuary of the reservoir, we can obtain that the pollutant concentrations are smaller than the concentration of far away points in the reservoir. The totally pollutant concentrations will be increasing when the water has flow backward to the reservoir from the outer sea and decreasing when the water has flow outward to the outer sea. This model can be applied to the real cases for sewage effluent in the reservoir by the industry, which we can change the reliable functions of input of elevation and concentration of inflow pollutant.

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Table 2: The pollutant discharged along the coastal (kg/m^3) at 0 hrs.

Node No.	Concentration	Node No.	Concentration	Node No.	Concentration
1	0.01	40	0.01	101	0.01
5	0.005	41	0.01	102	0.01
6	0.01	62	0.01	103	0.01
12	0.01	63	0.01	104	0.01
13	0.01	70	0.01	105	0.01
14	0.01	71	0.01	106	0.01
15	0.01	79	0.01	107	0.01
16	0.01	80	0.01	108	0.01
17	0.01	90	0.01	109	0.01
28	0.01	91	0.01	110	0.01
29	0.01	100	0.01		

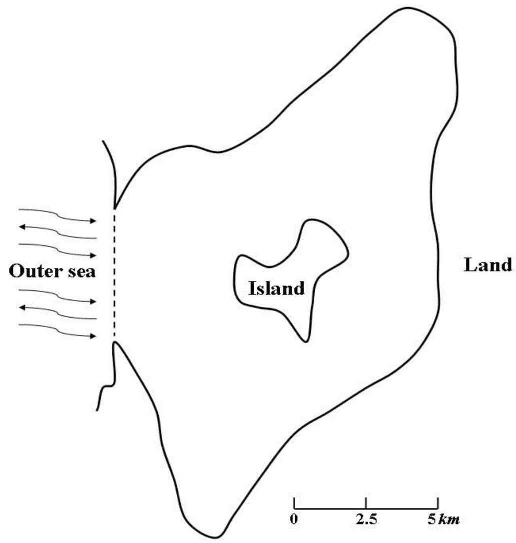

Figure 2: The bay opens to the outer water area with an island.

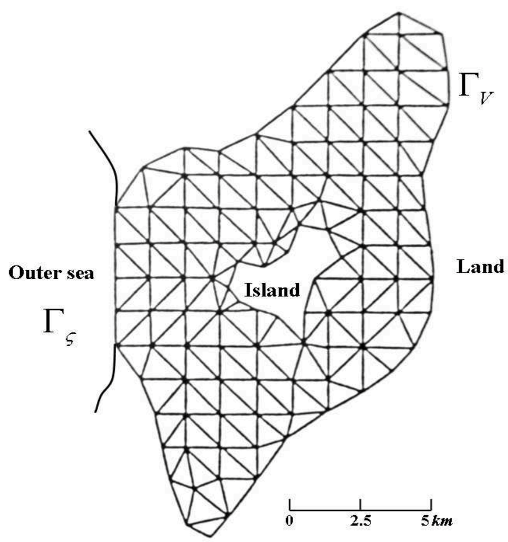

Figure 3: The finite element mesh of the bay.

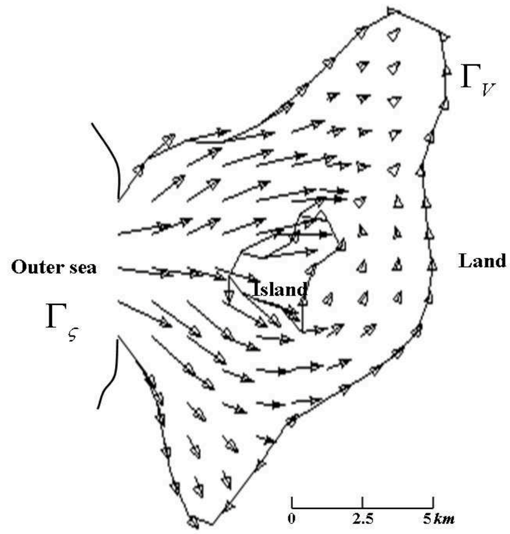

Figure 4: The velocity vectors at nodal points.

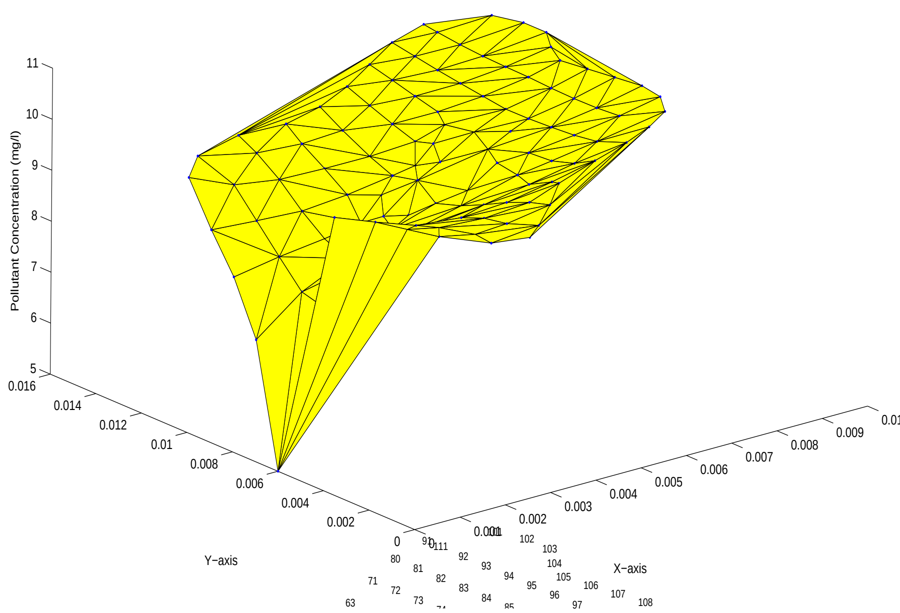

Figure 5: The pollutant concentration (mg/l) at 12 hrs

Table 3: Topological data for nodal coordinates where h is the bay depth

Node No.	(x, y)	$h(x, y)$	Node No.	(x, y)	$h(x, y)$	Node No.	(x, y)	$h(x, y)$
1	(0 , 10)	1	38	(3 , 3)	0.2	75	(5.8 , 10.3)	0.07
2	(0 , 9)	1	39	(3 , 2)	0.08	76	(6 , 7)	0.8
3	(0 , 8)	1	40	(3.2 , 1)	0.01	77	(6 , 6)	1
4	(0 , 7)	1	41	(4 , 12.2)	0.8	78	(6 , 5)	1
5	(0 , 6)	1	42	(4 , 11)	0.6	79	(6 , 4.1)	0.3
6	(0.8 , 11.2)	1	43	(4 , 10)	0.07	80	(7 , 15)	0.5
7	(1 , 10)	1	44	(4 , 9)	0.04	81	(7 , 14)	0.7
8	(1 , 9)	1	45	(3.5 , 8.5)	0.02	82	(7 , 13)	1
9	(1 , 8)	0.7	46	(3.2 , 7.8)	0.02	83	(7 , 12)	0.7
10	(1 , 7)	0.75	47	(3.5 , 7.3)	0.05	84	(7 , 11)	0.4
11	(1 , 6)	0.8	48	(4 , 7.1)	0.07	85	(7 , 10)	0.3
12	(0.7 , 5)	1	49	(4.6 , 6.9)	0.1	86	(7 , 9)	0.4
13	(1.1 , 4)	0.8	50	(5.3 , 6.1)	0.5	87	(7 , 8)	0.9
14	(1.3 , 3)	0.4	51	(5.3 , 7)	1	88	(7 , 7)	1
15	(1.5 , 2)	0.2	52	(5.6 , 8)	0.1	89	(7 , 6)	1
16	(2 , 0.7)	0.01	53	(6.4 , 8.6)	0.07	90	(7 , 4.7)	0.6
17	(2 , 11.8)	0.9	54	(6 , 9.4)	0.03	91	(8 , 15.6)	0.4
18	(2 , 11)	0.9	55	(5.2 , 9.6)	0.02	92	(8 , 14)	0.75
19	(2 , 10)	0.8	56	(4.9 , 8.7)	0.01	93	(8 , 13)	1
20	(2 , 9)	0.4	57	(4.2 , 8.3)	0.01	94	(8 , 12)	0.8
21	(2 , 8)	0.3	58	(4 , 6)	0.4	95	(8 , 11)	0.7
22	(2 , 7)	0.38	59	(4 , 5)	1	96	(8 , 10)	0.8
23	(2 , 6)	0.9	60	(4 , 4)	0.5	97	(8 , 9)	0.9
24	(2 , 5)	1	61	(4 , 3)	0.1	98	(8 , 8)	1
25	(2 , 4)	0.7	62	(4 , 2.1)	0.05	99	(8 , 7)	1
26	(2 , 3)	0.3	63	(5 , 13)	1	100	(7.8 , 5.3)	0.8
27	(2.2 , 1.8)	0.1	64	(5 , 12)	0.7	101	(9.2 , 15)	0.5
28	(2.7 , 0.4)	0	65	(5 , 11)	0.3	102	(9.4 , 14)	0.8
29	(3 , 11.7)	0.8	66	(5 , 10)	0.05	103	(9.4 , 13)	1
30	(3 , 11)	0.6	67	(4.5 , 9)	0.02	104	(9 , 12)	1
31	(3 , 10)	0.3	68	(5 , 5)	1	105	(8.7 , 11)	1
32	(3 , 9)	0.07	69	(5 , 4)	0.4	106	(8.8 , 10)	1
33	(2.8 , 8)	0.05	70	(4.8 , 3.3)	0.08	107	(8.9 , 9)	1
34	(3 , 7)	0.07	71	(6 , 14)	0.8	108	(9 , 8)	1
35	(3 , 6)	0.6	72	(6 , 13)	1	109	(8.9 , 7)	1
36	(3 , 5)	1	73	(6 , 12)	0.8	110	(8.5 , 6)	1
37	(3 , 4)	0.6	74	(6 , 11)	0.2	111	(8 , 15)	0.6

Table 4: Topological data for element connectivity

Element	Nodes			Element	Nodes			Element	Nodes		
number	i_1	i_2	i_3	number	i_1	i_2	i_3	number	i_1	i_2	i_3
1	1	2	8	59	32	44	31	117	77	78	89
2	2	3	9	60	45	44	32	118	68	79	78
3	3	4	9	61	33	45	32	119	68	69	79
4	4	5	10	62	33	46	45	120	70	79	69
5	1	7	6	63	34	46	33	121	79	90	78
6	1	8	7	64	34	47	46	122	78	90	89
7	2	9	8	65	34	48	47	123	71	81	80
8	4	10	9	66	34	58	48	124	71	82	81
9	5	11	10	67	35	58	34	125	71	72	82
10	5	12	11	68	35	59	58	126	72	83	82
11	6	18	17	69	35	36	59	127	72	73	83
12	6	7	18	70	36	60	59	128	73	84	83
13	7	19	18	71	36	37	60	129	73	74	84
14	7	20	19	72	37	61	60	130	74	85	84
15	7	8	20	73	37	38	61	131	75	85	74
16	8	21	20	74	38	62	61	132	75	54	85
17	9	21	8	75	38	39	62	133	54	86	85
18	10	21	9	76	39	40	62	134	54	53	86
19	10	22	21	77	41	64	63	135	53	87	86
20	10	11	22	78	41	65	64	136	80	111	91
21	11	23	22	79	42	65	41	137	80	92	111
22	11	24	23	80	42	66	65	138	81	92	80
23	12	24	11	81	43	66	42	139	81	93	92
24	12	13	24	82	43	67	66	140	82	93	81
25	13	25	24	83	43	44	67	141	82	94	93
26	14	25	13	84	66	67	55	142	83	94	82
27	14	26	25	85	44	57	67	143	83	95	94
28	14	15	26	86	67	56	55	144	84	95	83
29	15	27	26	87	45	57	44	145	84	96	95
30	15	16	27	88	67	57	56	146	85	96	84
31	17	30	29	89	48	58	49	147	85	97	96
32	17	18	30	90	49	58	50	148	86	97	85
33	18	31	30	91	58	68	50	149	86	98	97
34	18	19	31	92	58	59	68	150	87	98	86
35	19	32	31	93	59	69	68	151	87	88	98
36	19	20	32	94	59	60	69	152	88	99	98
37	20	33	32	95	60	70	69	153	88	89	99
38	20	21	33	96	60	61	70	154	89	110	99
39	21	22	33	97	61	62	70	155	89	100	110
40	22	34	33	98	63	72	71	156	90	100	89
41	22	23	34	99	63	73	72	157	111	101	91
42	23	35	34	100	63	64	73	158	111	102	101
43	23	36	35	101	64	74	73	159	92	102	111
44	23	24	36	102	64	65	74	160	92	103	102
45	24	37	36	103	65	75	74	161	93	103	92
46	25	37	24	104	65	66	75	162	93	104	103
47	25	38	37	105	66	55	75	163	94	104	93
48	26	38	25	106	55	54	75	164	94	105	104
49	26	39	38	107	52	87	53	165	95	105	94
50	27	39	26	108	52	76	87	166	95	106	105
51	27	40	39	109	51	76	52	167	96	106	95
52	16	40	27	110	50	76	51	168	96	107	106
53	16	28	40	111	76	88	87	169	97	107	96
54	29	42	41	112	76	89	88	170	97	108	107
55	30	42	29	113	50	77	76	171	98	108	97
56	30	43	42	114	77	89	76	172	98	99	108
57	31	43	30	115	50	78	77	173	99	109	108
58	31	44	43	116	50	68	78	174	99	110	109

Table 5: The velocity of water (m/s) and elevation (m) at 12 hrs.

Node No.	U	V	ζ	Node No.	U	V	ζ	Node No.	U	V	ζ
1	0.15	0.22	0.00	38	0.09	-0.09	-0.03	75	0.16	-0.01	-0.08
2	0.30	0.06	0.00	39	0.04	-0.11	-0.05	76	0.03	0.05	-0.06
3	0.31	-0.04	0.00	40	-0.03	-0.04	-0.07	77	0.10	0.07	-0.06
4	0.31	-0.14	0.00	41	0.21	0.14	-0.04	78	0.14	0.06	-0.04
5	0.21	-0.31	0.00	42	0.29	0.07	-0.03	79	0.14	0.09	-0.06
6	0.17	0.15	-0.05	43	0.33	0.07	-0.03	80	0.03	0.02	-0.05
7	0.18	0.14	-0.07	44	0.34	0.07	-0.02	81	0.04	0.02	-0.04
8	0.23	0.09	-0.05	45	0.25	0.12	-0.02	82	0.05	0.02	-0.05
9	0.28	-0.04	-0.01	46	0.00	-0.17	-0.03	83	0.05	0.01	-0.05
10	0.28	-0.20	-0.02	47	0.23	-0.20	-0.08	84	0.04	0.01	-0.06
11	0.24	-0.23	-0.07	48	0.23	-0.09	-0.09	85	0.02	0.02	-0.08
12	0.12	-0.21	-0.05	49	0.12	-0.09	-0.07	86	-0.01	0.06	-0.06
13	0.04	-0.14	-0.01	50	0.14	0.02	-0.06	87	0.01	0.05	-0.05
14	0.03	-0.14	-0.02	51	0.01	0.05	-0.11	88	0.03	0.06	-0.06
15	0.04	-0.14	-0.03	52	0.02	0.03	-0.06	89	0.08	0.07	-0.05
16	0.06	-0.08	-0.07	53	0.02	0.06	-0.06	90	0.12	0.08	-0.06
17	0.24	0.05	0.01	54	0.00	0.00	-0.09	91	0.01	0.00	-0.05
18	0.19	0.11	-0.02	55	0.16	0.10	-0.08	92	0.02	0.02	-0.05
19	0.18	0.12	-0.02	56	0.06	0.08	-0.09	93	0.02	0.01	-0.05
20	0.19	0.09	0.01	57	0.30	0.04	-0.06	94	0.02	0.01	-0.05
21	0.22	-0.07	0.03	58	0.21	-0.03	-0.05	95	0.01	0.02	-0.05
22	0.26	-0.22	-0.02	59	0.19	-0.02	-0.04	96	0.00	0.03	-0.05
23	0.22	-0.16	-0.06	60	0.16	-0.01	-0.03	97	-0.01	0.05	-0.05
24	0.15	-0.12	-0.05	61	0.11	-0.03	-0.03	98	0.01	0.06	-0.05
25	0.10	-0.10	-0.02	62	-0.01	-0.02	-0.05	99	0.03	0.06	-0.05
26	0.07	-0.13	-0.02	63	0.09	0.09	-0.04	100	0.08	0.07	-0.08
27	0.04	-0.12	-0.05	64	0.18	0.07	-0.05	101	0.00	0.00	-0.05
28	0.00	0.00	-0.10	65	0.25	0.06	-0.01	102	0.00	0.01	-0.05
29	0.30	0.06	-0.03	66	0.31	0.01	-0.03	103	0.00	0.02	-0.05
30	0.29	0.10	-0.03	67	0.33	0.00	-0.06	104	0.01	0.02	-0.05
31	0.27	0.11	0.01	68	0.18	0.03	-0.03	105	0.00	0.02	-0.05
32	0.25	0.09	0.01	69	0.17	0.02	-0.04	106	0.00	0.03	-0.05
33	0.26	-0.07	0.04	70	0.07	0.07	-0.04	107	0.00	0.05	-0.05
34	0.26	-0.15	-0.04	71	0.04	0.04	-0.04	108	0.00	0.06	-0.05
35	0.21	-0.09	-0.04	72	0.07	0.03	-0.04	109	0.02	0.07	-0.05
36	0.18	-0.07	-0.04	73	0.10	0.02	-0.04	110	0.04	0.06	-0.05
37	0.14	-0.06	-0.03	74	0.14	0.01	-0.06	111	0.01	0.01	-0.05

Table 6: The calculated pollutant concentrations (kg/m^3) at 12 hrs.

Node No.	Concentration	Node No.	Concentration	Node No.	Concentration
1	0.01000	38	0.00181	75	0.00011
2	0.00188	39	0.00413	76	0.00023
3	0.00056	40	0.01000	77	0.00093
4	0.00104	41	0.01000	78	0.00322
5	0.00500	42	0.00163	79	0.01000
6	0.01000	43	0.00027	80	0.01000
7	0.00277	44	0.00003	81	0.00378
8	0.00046	45	0.00004	82	0.00106
9	0.00013	46	0.00002	83	0.00050
10	0.00038	47	0.00003	84	0.00055
11	0.00213	48	0.00004	85	0.00053
12	0.01000	49	0.00008	86	0.00053
13	0.01000	50	0.00034	87	0.00065
14	0.01000	51	0.00019	88	0.00080
15	0.01000	52	0.00012	89	0.00163
16	0.01000	53	0.00036	90	0.01000
17	0.01000	54	0.00015	91	0.01000
18	0.00366	55	0.00005	92	0.00181
19	0.00121	56	0.00003	93	0.00130
20	0.00020	57	0.00002	94	0.00213
21	0.00004	58	0.00009	95	0.00305
22	0.00008	59	0.00035	96	0.00275
23	0.00052	60	0.00180	97	0.00242
24	0.00098	61	0.00482	98	0.00252
25	0.00244	62	0.01000	99	0.00282
26	0.00368	63	0.01000	100	0.01000
27	0.00499	64	0.00333	101	0.01000
28	0.01000	65	0.00034	102	0.01000
29	0.01000	66	0.00004	103	0.01000
30	0.00314	67	0.00003	104	0.01000
31	0.00063	68	0.00108	105	0.01000
32	0.00010	69	0.00490	106	0.01000
33	0.00002	70	0.01000	107	0.01000
34	0.00002	71	0.01000	108	0.01000
35	0.00014	72	0.00364	109	0.01000
36	0.00029	73	0.00085	110	0.01000
37	0.00082	74	0.00015	111	0.00523