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Powers of Some Special Upper Triangular Matrices

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Abstract Given a real upper triangular matrix,

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ 0 & a_2 & \cdots & a_2 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}, \text{ where } n \text{ is a positive integer.}$$

We obtain a general form for the m^{th} power A_n^m of the real upper triangular matrices A_n, where m is any positive integer. Moreover, we demonstrate the other formulae of the m^{th} power A_n^m by using Cornelius' result [E.F. Cornelius, Identities for complete homogeneous symmetric polynomials, JP Journal of Algebra, Number Theory and Applications 21 (1) (2011) 109–116]. Furthermore, we also demonstrate a simple formula for the trace of the m^{th} power A_n^m of the real upper triangular matrix A_n.

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1. Introduction

For a square matrix A and a positive integer m, we define the m^{th} power of A by repeating matrix multiplication; that is, $A^m = A \times A \times \cdots \times A$, where there are m copies of matrix A on the right-hand side. Matrix powers can be calculated explicitly using the rule of matrix multiplication. However, this can be time-consuming for high powers, as the multiplication must be performed many times. A triangular matrix is a special kind of square matrix. There are two basic forms of triangular matrices: lower triangular matrices, which have all zero entries above the main diagonal, and upper triangular matrices, which have all zero entries below the main diagonal. Triangular matrices are easier to solve than other types of matrices and are therefore very important in numerical

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analysis. The calculation of the matrix power of a triangular matrix can be done more efficiently than for other types of matrices. The calculation of matrix powers occurs in many different mathematical frameworks, such as combinatorial sequences, linear differential equations, and statistics. Triangular matrices have attracted the interest of many authors in these fields, who have developed methods and algorithms for calculating their powers, see [1, 8–10].

Trace of an $n \times n$ matrix $A = [a_{ij}]$ is defined to be the sum of the elements on the diagonal of A, that is

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

The trace of powers of matrices arises in several fields of mathematics, such as matrix theory and numerical algebra. The discussion about trace has been widely studied by several researchers before.

Datta et al. [4] obtained an algorithm for the trace of the power of a squared matrix, $Tr\left(A^k\right)$, where k is an integer and A is a Hassenberg matrix with a codiagonal unit. Chu [3] demonstrated a symbolic calculation of the trace of powers of tridiagonal matrices. Pahade and Jha [6] explained that the trace of the positive integer power of a real 2×2 matrix is an equation of the general trace form of the matrices. Recently, Rahmawati et al. [7] discussed the general formula for the power matrix A_n^m with positive integer m, where A_n is an $n \times n$ matrix of real number entries where each entry has the same value in a row. This matrix is formulated as follows:

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ a_2 & a_2 & \cdots & a_2 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} \\ a_n & a_n & \cdots & a_n & a_n \end{bmatrix}$$

In this paper, we derive a general expression for the entries of $n \times n$ upper triangular matrix A_n^m for positive integer m, where A_n is a $n \times n$ upper triangular matrix:

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ 0 & a_2 & \cdots & a_2 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

Moreover, we find the general form of the trace of positive power A_n^m.

2. Preliminaries

In this section, we introduce some basic definitions, notation and well-known results which we will use in the sequel.

The complete homogeneous symmetric polynomial of degree k in n variables $x_1, \ldots, x_n$, written $h_k(x_1, \ldots, x_n)$ for $k = 0, 1, 2, \ldots$, is defined by

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{\substack{l_1 + l_2 + \cdots + l_n = k \\ l_j \geqslant 0}} x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n},$$

when an exponent is zero, the corresponding power variable is taken to be 1.

The first few of these polynomials are

$$\begin{aligned}
h_0(x_1,x_2,\ldots,x_n) &= 1,\\
h_1(x_1,x_2,\ldots,x_n) &= x_1+x_2+\cdots+x_n,\\
h_2(x_1,x_2,\ldots,x_n) &= \sum_{1\le i\le j\le n} x_ix_j,\\
h_3(x_1,x_2,\ldots,x_n) &= \sum_{1\le i\le j\le k\le n} x_ix_jx_k.
\end{aligned}$$

For each nonnegative integer k, there exists a unique complete homogeneous symmetric polynomial of degree k in n variables.

In particular, any complete homogeneous symmetric polynomial of degree k in n variables, $h_k(x_1,x_2,\ldots,x_n)$ can be expressed as linear combination of complete homogeneous symmetric polynomial of degree $k-1$, $h_{k-1}(x_1,x_2,\ldots,x_n), h_{k-1}(x_2,\ldots,x_n),\ldots,h_{k-1}(x_1)$.

For example,

$$\begin{aligned}
h_3(x_1,x_2,x_3) &= x_1^3+x_2^3+x_3^3+x_1^2x_2+x_1^2x_3+x_1x_2^2+x_2^2x_3+x_1x_3^2+x_2x_3^2+x_1x_2x_3,\\
&= x_3(x_3^2+x_1^2+x_2^2+x_1x_3+x_2x_3+x_1x_2)+x_2(x_2^2+x_1^2+x_1x_2)+x_1(x_1^2)\\
&= x_3h_2(x_1,x_2,x_3)+x_2h_2(x_1,x_2)+x_1h_2(x_1).
\end{aligned}$$

In general, we have the following lemma.

Lemma 2.1. *For each positive integer k, we have*

$$h_k(x_1,x_2,\ldots,x_n) = x_nh_{k-1}(x_1,x_2,\ldots,x_n)+x_{n-1}h_{k-1}(x_1,x_2,\ldots,x_{n-1})+\cdots+x_1h_{k-1}(x_1).$$

Proof. Let A be the set of all distinct monomial expression $x_1^{t_1}x_2^{t_2}\cdots x_n^{t_n}$, where $t_1,t_2,\ldots,t_n$ are nonnegative integer and $\sum_{i=1}^{n} t_i = k$. Then the sum of all elements in the set A is $h_k(x_1,x_2,\ldots,x_n)$.

Next, we define the following sets:

B_1 is the subset of A containing $x_1^{t_1}$, where $t_1\ge 1$,

B_2 is the subset of A containing $x_1^{t_1}x_2^{t_2}$, where $t_2\ge 1$,

B_3 is the subset of A containing $x_1^{t_1}x_2^{t_2}x_3^{t_3}$, where $t_3\ge 1$,

$\vdots$

B_k is the subset of A containing $x_1^{t_1}x_2^{t_2}\cdots x_k^{t_k}$, where $t_k\ge 1$,

$\vdots$

B_n is the subset of A containing $x_1^{t_1}x_2^{t_2}\cdots x_n^{t_n}$, where $t_n\ge 1$.

Then $B_1\cap B_2\cap\cdots\cap B_n=\varnothing$ and $A=B_1\cup B_2\cup\cdots\cup B_n$.

The sum of all elements in the set B_1 is $x_1h_{k-1}(x_1)$.

The sum of all elements in the set B_2 is $x_2h_{k-1}(x_1,x_2)$.

The sum of all elements in the set B_3 is $x_3h_{k-1}(x_1,x_2,x_3)$.

$\vdots$

The sum of all elements in the set B_n is $x_n h_{k-1}(x_1, x_2, x_3, \ldots, x_n)$.
Therefore,

$$h_k(x_1, x_2, \ldots, x_n) = x_n h_{k-1}(x_1, x_2, \ldots, x_n) + x_{n-1} h_{k-1}(x_1, x_2, \ldots, x_{n-1}) + \cdots + x_1 h_{k-1}(x_1).$$

This complete the proof. ■

In 2011, Cornelius [2] proved that any complete homogeneous symmetric polynomial can be expressed as the sum of rational functions.

Theorem 2.2. [2]. *The complete homogeneous symmetric polynomial of degree k in n variables, $x_1, x_2, x_3, \ldots, x_n$,*

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

can be expressed as the sum of rational functions:

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{l=1}^{n} \frac{x_l^{n+k-1}}{\prod_{m=1, m \ne l}^{n} (x_l - x_m)}.$$

3. Main Results

In this section, we derive a general expression for the entries of $n \times n$ upper triangular matrix A_n^m of the real upper triangular matrix A_n, where m is any positive integer.

Theorem 3.1. *Let $n \ge 2$ be a positive integer. For real numbers $a_1, a_2, \ldots, a_n$, define an $n \times n$ upper triangular matrix A_n,*

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ 0 & a_2 & \cdots & a_2 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix}.$$

Then, for each positive integer m,

$$A_n^m = \begin{bmatrix} b_{11}^{(m)} & b_{12}^{(m)} & \cdots & b_{1\ n-1}^{(m)} & b_{1n}^{(m)} \\ 0 & b_{22}^{(m)} & \cdots & b_{2\ n-1}^{(m)} & b_{2n}^{(m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1\ n-1}^{(m)} & b_{n-1\ n}^{(m)} \\ 0 & 0 & \cdots & 0 & b_{nn}^{(m)} \end{bmatrix},$$

where $b_{ij}^{(m)} = a_i h_{m-1}(a_i, a_{i+1}, \ldots, a_j)$ for all $i, j = 1, 2, \ldots, n$ and $j \ge i$.

Proof. For any positive integer m, let $P(m)$ be the statement that

$$A_n^m = \begin{bmatrix} b_{11}^{(m)} & b_{12}^{(m)} & \cdots & b_{1\ n-1}^{(m)} & b_{1n}^{(m)} \\ 0 & b_{22}^{(m)} & \cdots & b_{2\ n-1}^{(m)} & b_{2n}^{(m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1\ n-1}^{(m)} & b_{n-1\ n}^{(m)} \\ 0 & 0 & \cdots & 0 & b_{nn}^{(m)} \end{bmatrix},$$

where $b_{ij}^{(m)} = a_i h_{m-1}(a_i, a_{i+1}, \dots, a_j)$ for all $i, j = 1, 2, \dots, n$ and $j \geq i$.
For the base step, $m = 1$, we have $h_0(x_1, x_2, \dots, x_n) = 1$, and $a_i = a_i h_0(x_1, \dots, x_n) = b_{ij}^{(1)}$ for all $i, j = 1, 2, \dots, n$ and $j \geq i$.
Hence $P(1)$ holds.

Let k be a positive integer and suppose that $P(k)$ holds. Then

$$
\begin{aligned}
A_n^{k+1} &= A_n^k A_n^1 \\
&= \begin{bmatrix} b_{11}^{(k)} & b_{12}^{(k)} & \cdots & b_{1\,n-1}^{(k)} & b_{1n}^{(k)} \\ 0 & b_{22}^{(k)} & \cdots & b_{2\,n-1}^{(k)} & b_{2n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1\,n-1}^{(k)} & b_{n-1\,n}^{(k)} \\ 0 & 0 & \cdots & 0 & b_{nn}^{(k)} \end{bmatrix} \begin{bmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ 0 & a_2 & \cdots & a_2 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix} \\
&= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1\,n-1} & c_{1n} \\ 0 & c_{22} & \cdots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{n-1\,n-1} & c_{n-1\,n} \\ 0 & 0 & \cdots & 0 & c_{nn} \end{bmatrix}
\end{aligned}
$$

By using Lemma 2.1, then for $i = 1, 2, \dots, n$ and $j \geqslant i$,

$$
\begin{aligned}
c_{ij} &= a_i b_{ii}^{(k)} + a_{i+1} b_{i\,i+1}^{(k)} + \cdots + a_j b_{ij}^{(k)} \\
&= a_i(a_i h_{k-1}(a_i)) + a_{i+1}(a_i h_{k-1}(a_i, a_{i+1})) + \cdots + a_j(a_i h_{k-1}(a_i, a_{i+1}, \dots, a_j)) \\
&= a_i(a_i h_{k-1}(a_i) + a_{i+1} h_{k-1}(a_i, a_{i+1}) + \cdots + a_j h_{k-1}(a_i, a_{i+1}, \dots, a_j)) \\
&= a_i h_k(a_i, a_{i+1}, \dots, a_j) = b_{ij}^{(k+1)}.
\end{aligned}
$$

Therefore, $P(k+1)$ holds, and the proof is complete. ■

The proof of the following corollary immediately follows from Theorem 3.1 and Theorem 2.2.

Corollary 3.2. *Let $n \geq 2$ be a positive integer and let A_n be the upper triangular matrix as defined in Theorem 3.1 where $a_1, a_2, \dots, a_n$ are all distinct real numbers.*
Then, for each positive integer m,

$$
A_n^m = \begin{bmatrix} b_{11}^{(m)} & b_{12}^{(m)} & \cdots & b_{1\,n-1}^{(m)} & b_{1n}^{(m)} \\ 0 & b_{22}^{(m)} & \cdots & b_{2\,n-1}^{(m)} & b_{2n}^{(m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1\,n-1}^{(m)} & b_{n-1\,n}^{(m)} \\ 0 & 0 & \cdots & 0 & b_{nn}^{(m)} \end{bmatrix},
$$

where

$$
b_{ij}^{(m)} = a_i \sum_{l=i}^{j} \frac{a_l^{j-i+m-1}}{\prod_{k=i,k\neq l}^{j}(a_l - a_k)},
$$

for all $i, j = 1, 2, \dots, n$ and $j \geqslant i$.

Now Theorem 3.1 can be applied to a particular lower triangular matrix,

$$B_n = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 \\ a_1 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} & 0 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix},$$

by using $(B_n^m)^T = \left(B_n^T\right)^m$ and $\left(B_n^T\right)^T = B_n$.

Next, the following theorem shows a simple formula for trace of the m^{th} power of the real upper triangular matrix A_n.

Theorem 3.3. *For positive integer $n \geq 2$ and $n \times n$ upper triangular matrix A_n, defined as*

$$A_n = \begin{bmatrix} a_1 & a_1 & \cdots & a_1 & a_1 \\ 0 & a_2 & \cdots & a_2 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{bmatrix},$$

where $a_1, a_2, \ldots, a_n$ are real numbers. Then, for each positive integer m,

$$Tr(A_n^m) = \sum_{i=1}^{n} a_i^m.$$

Proof. It immediately follows from Theorem 3.1 and the definition of trace. ∎

Example 3.4. For positive integer $n \geqslant 2$ and $n \times n$ upper triangular matrix A_n, defined as

$$A_n = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 2 & \cdots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & n-1 & n-1 \\ 0 & 0 & \cdots & 0 & n \end{bmatrix},$$

Then, by using Theorem 3.3, for any positive integer m,

$$Tr\left(A_n^m\right) = \sum_{k=1}^{n} k^m.$$

Note that the sums for $m = 1, 2, \ldots, 6$ are

$$\sum_{k=1}^{n} k = \frac{1}{2}\left(n^2+n\right),$$
$$\sum_{k=1}^{n} k^2 = \frac{1}{6}\left(2n^3+3n^2+n\right),$$
$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\left(n^4+2n^3+n^2\right),$$
$$\sum_{k=1}^{n} k^4 = \frac{1}{30}\left(6n^5+15n^4+10n^3-n\right),$$
$$\sum_{k=1}^{n} k^5 = \frac{1}{12}\left(2n^6+6n^5+5n^4-n^2\right),$$
$$\sum_{k=1}^{n} k^6 = \frac{1}{42}\left(6n^7+21n^6+21n^5-7n^3+n\right).$$

Note that the sums $\sum_{k=1}^{n} k^m$ can be directly computed by the Faulhaber's formula

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k\, n^{m-k+1},$$

where $B_0, B_1, \ldots,$ are the Bernoulli numbers, as defined by

$$B_0 = 1,\ B_k = -\tfrac{1}{k+1}\sum_{i=0}^{k-1}\binom{k+1}{i} B_i \quad \text{for all } k \in \{1,2,\ldots\}.$$

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