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Interpolation Theorems for the Arboricity and the Vertex Arboricity of Graphs

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Abstract The arboricity $\boldsymbol{a}(G)$ (respectively, vertex arboricity $\boldsymbol{va}(G)$) of a graph G is the minimum number of subsets into which the edge set $E(G)$ (respectively, the vertex set $V(G)$) can be partitioned so that each subset induces a forest. In this paper, we study interpolation theorems for the arboricity and the vertex arboricity of graphs with size m and order n. We show that for $\rho \in \{\boldsymbol{a}, \boldsymbol{va}\}$, the values of $\rho(G)$ where $G \in \mathcal{G}(m, n)$ completely cover a line segment $[a, b]$ of positive integers such that $\mathcal{G}(m, n)$ is the class of all simple graphs with size m and order n. Then we say that ρ is an interpolation graph parameter over $\mathcal{G}(m, n)$. Thus for a graph parameter ρ, two variants a and b where

$$a = \min \{\rho(G) : G \in \mathcal{G}(m, n)\} \text{ and}$$
$$b = \max \{\rho(G) : G \in \mathcal{G}(m, n)\}$$

arise naturally. The extremal values a and b are obtained for all $\rho \in \{\boldsymbol{a}, \boldsymbol{va}\}$.

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1. Introduction

In this paper, our notation and terminology follow that of Bondy and Murty [1]. We only consider simple and finite graphs. Let $\mathcal{G}$ be the class of all simple graphs; a function $\pi : \mathcal{G} \to Z$ is called *a graph parameter* if $\pi(G) = \pi(H)$ whenever $G, H \in \mathcal{G}$ such that $G = H$. If there exist integers x and y such that $\{\pi(G) : G \in \mathcal{J}\} = [x, y] = \{k \in Z : x \leq k \leq y\}$, then a graph parameter π is called an *interpolation graph parameter over* $\mathcal{J} \subseteq \mathcal{G}$.

The question "*If a graph G possesses a spanning tree having m leaves and another having n leaves, where $m < n$, does G possess a spanning tree having k leaves for every k between m and n ?*" was raised by G.Chartrand during a conference on graph theory in 1980 [2]. In [3–6], this question was answered and it led to a host of lots of papers

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studying the interpolation properties of graph parameters with respect to the set of all spanning trees of a given graph. The interpolating character of many graph parameters over other classes of graphs was studied in number of papers.

In 1995, Zhou [7] investigated the interpolation property for a family of spanning subgraphs. In [8–10], Punnim proved that chromatic, clique, independent, covering, edge-covering, matching, forest, decycling and domination numbers are interpolation graph parameters over the class of d-regular graphs and d-regular connected graphs and determined the extremal values of these parameters.

2. Interpolation Theorems

For positive integers n and m where $0 \leq m \leq \binom{n}{2}$, let $\mathcal{G}(m,n)$ be the sets of all simple graphs of size m and order n. Let $G \in \mathcal{G}(m,n)$ with $e \in E(G)$ and $f \notin E(G)$. Define a *jumping transformation* $t = t(e,f)$ on G which produces the graph $G^{t(e,f)} = G - e + f$, simply written G^t. Let $\mathcal{T}(m,n)$ be a relation on $\mathcal{G}(m,n)$ defined by $(G,H) \in \mathcal{T}(m,n)$ if $G \ncong H$ and H can be obtained from G by a jumping transformation. Since $\mathcal{T}(m,n)$ is symmetric, it follows that the $\mathcal{T}(m,n)$-graph is simple. N. Punnim [11] proved that the $\mathcal{T}(m,n)$-graph is connected as the following results.

Theorem 2.1. [11] *Let $G, H \in \mathcal{G}(m,n)$. Then $G = H$ or there is a finite sequence of jumping transformations $t(e_1,f_1), t(e_2,f_2), ..., t(e_r,f_r)$ for some integer $1 \leq r \leq \binom{n}{2}$ such that $H = G^{t(e_1,f_1)t(e_2,f_2)...t(e_r,f_r)}$.*

Corollary 2.2. [11] *The $\mathcal{T}(m,n)$-graph is connected.*

As a result of Corollary 2.2, the following theorem is obtained.

Theorem 2.3. [11] *Let t be a jumping transformation on $G \in \mathcal{G}(m,n)$ and π is a graph parameter. If $|\pi(G) - \pi(G^t)| \leq 1$, then π is an interpolation graph parameter over $\mathcal{G}(m,n)$.*

The interpolation theorems for the arboricity and vertex arboricity of graphs are proved in the following theorems by using the useful fact about the arboricity that all the subgraphs of any graph cannot have arboricity larger than the graph itself.

Theorem 2.4. *Let t be a jumping transformation on a graph $G \in \mathcal{G}(m,n)$. Then $\boldsymbol{a}(G^t) \leq \boldsymbol{a}(G) + 1$.*

Proof. Let $ab \in E(G)$, $cd \notin E(G)$, and $t = t(ab,cd)$ be a jumping transformation on $G \in \mathcal{G}(m,n)$. Suppose that $\boldsymbol{a}(G) = l$. Since subgraphs of any graph cannot have arboricity larger than the graph itself, if we remove the edge ab from G, then $\boldsymbol{a}(G - ab) \leq l$. This implies that $E(G - ab)$ can be partitioned into k subsets $E_1, E_2, ..., E_k$ each of which induces a forest where $k \leq l$. If we add the edge cd into $G - ab$, then we shall consider $G - ab + cd$ (simply written G^t) in the following two cases.

Case 1. The subgraph of $G - ab$ induced by E_i contains a path from c to d for all $1 \leq i \leq k$. Then there must be a subgraph of $G - ab + cd$ induced by E_i for some i which contains a cycle. Hence $E(G - ab + cd)$ must be partitioned into at least $k+1$ subsets so that each subset induces a forest. That is $\boldsymbol{a}(G - ab + cd) \leq k + 1 \leq \boldsymbol{a}(G) + 1$.

Case 2. There exists E_i for some $1 \leq i \leq k$ where the subgraph of $G - ab$ induced by E_i contains no paths from c to d. If we add the edge cd to E_i, then a subgraph of $G - ab + cd$ induced by $E_i \cup \{cd\}$ is a forest. Thus $E(G - ab + cd)$ can be partitioned into k subsets each of which induces a forest. Consequently, $\boldsymbol{a}(G - ab + cd) \leq k \leq \boldsymbol{a}(G) + 1$. ■

Theorem 2.5. *Let t be a jumping transformation on a graph $G \in \mathcal{G}(m,n)$. Then $\boldsymbol{va}(G^t) \leq \boldsymbol{va}(G)+1$.*

Proof. Let G be a graph with $\boldsymbol{va}(G)=k$ and $t=t(ab,cd)$ be a jumping transformation. Then $V(G)$ can be partitioned into k subsets $V_1, V_2, ..., V_k$ each of which induces a forest in G. If we remove the edge ab from G, then it is clear that $V(G-ab)=V(G)$ and the subgraph of $G-ab$ induced by V_i is a forest for all $1 \leq i \leq k$. By adding the edge cd into $G-ab$, we shall consider $G-ab+cd$ or G^t in the following two cases.

Case 1. There exists V_p for some $1 \leq p \leq k$ such that the subgraph of $G-ab$ induced by V_p contains a path from c to d. Then $G^t[V_p]$ contains a cycle. Thus $V(G^t)$ must be partitioned into $k+1$ subsets each of which induces a forest in G^t. That is $V(G^t)=V(G)\cup V_{k+1}$ where $V_{k+1}=\{d\}$. Hence $\boldsymbol{va}(G^t) \leq k+1 = \boldsymbol{va}(G)+1$.

Case 2. The subgraph of $G-ab$ induced by V_p contains no paths from c to d for all $1 \leq p \leq k$. Hence there is no subgraph of G^t induced by V_p for all $1 \leq p \leq k$ containing a cycle. That is $V(G^t)=V(G-ab)$. Thus $\boldsymbol{va}(G^t) \leq k \leq \boldsymbol{va}(G)+1$. ■

As a consequence of Theorems 2.4 and 2.5, we have Corollary 2.6.

Corollary 2.6. *Let $\rho \in \{\boldsymbol{a}, \boldsymbol{va}\}$ and t be a jumping transformation on $G \in \mathcal{G}(m,n)$. Then $|\rho(G)-\rho(G^t)| \leq 1$.*

By combining the previous results and Theorem 2.3, it follows that the arboricity and vertex arboricity are an interpolation graph parameter over $\mathcal{G}(m,n)$. Consequently, the following corollary is obtained.

Corollary 2.7. *Let $\rho \in \{\boldsymbol{a}, \boldsymbol{va}\}$. There exist integers $a = min\{\rho(G) : G \in \mathcal{G}(m,n)\}$ and $b = max\{\rho(G) : G \in \mathcal{G}(m,n)\}$ such that there is $G \in \mathcal{G}(m,n)$ with $\rho(G)=c$ if and only if c is an integer satisfying $a \leq c \leq b$.*

3. Bounds on the Arboricities and Vertex Arboricities

S. st.J. A. Nash-Williams has published a well-known theorem for arboricity as stated in the following theorem.

Theorem 3.1. [12, 13] *A graph G has arboricity ℓ if and only if every non-trivial subgraph H has at most $\ell(|V(H)|-1)$ edges.*

Theorem 3.2. [13] *For every nonempty graph G, $\boldsymbol{a}(G) = max\left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$, where the maximum is taken over all non trivial induced subgraph H of G.*

Since a graph G is a subgraph of K_n for all $G \in \mathcal{G}(m,n)$, $\boldsymbol{a}(G) \leq \boldsymbol{a}(K_n)$. The following lemmas show the lower bound of the number of vertices and edges of a graph G with the prescribed arboricity.

Lemma 3.3. *Let $G \in \mathcal{G}(m,n)$ and $\boldsymbol{a}(G)=a$. Then $m \geq 2(a-1)^2+1$.*

Proof. Let $G \in \mathcal{G}(m,n)$ where $\boldsymbol{a}(G)=a$. By Theorem 3.2, there exists a subgraph H_0 of G where $\left\lceil \frac{|E(H_0)|}{|V(H_0)|-1} \right\rceil = \boldsymbol{a}(G)$. Since $\frac{(|V(H_0)|-1)(a-1)}{|V(H_0)|-1} = a-1$, $\left\lceil \frac{(|V(H_0)|-1)(a-1)+1}{|V(H_0)|-1} \right\rceil = a$. Hence if $\left\lceil \frac{|E(H_0)|}{|V(H_0)|-1} \right\rceil = \boldsymbol{a}(G) = a$, then $|E(H_0)| \geq (|V(H_0)|-1)(a-1)+1$.

Suppose that $|V(H_0)| \leq 2a-2$. Let $|V(H_0)| = 2a-k$ where $k \geq 2$. Since $\boldsymbol{a}(G)=a$, $|E(H_0)| \geq (|V(H_0)|-1)(a-1)+1 = (2a-k-1)(a-1)+1 = \frac{(2a-2)(2a-k-1)}{2}+1$. Since $k \geq 2$,

$|E(H_0)| \geq \frac{(2a-2)(2a-k-1)}{2} + 1 \geq \frac{(2a-k)(2a-k-1)}{2} + 1 = \binom{2a-k}{2} + 1$. It is impossible because $|V(H_0)| = 2a-k$. Thus $|V(H_0)| \geq 2a-1$. That is $|E(H_0)| \geq (2a-1-1)(a-1)+1 = 2(a-1)^2+1$. Since $E(G) \geq E(H_0)$, $m \geq 2(a-1)^2+1$. ∎

Theorem 3.4. *Let $G \in \mathcal{G}(m,n)$. Then* $\left\lceil \frac{m}{n-1} \right\rceil \leq \boldsymbol{a}(G) \leq \left\lfloor \frac{2+\sqrt{2m-2}}{2} \right\rfloor$.

Proof. As the result of Theorem 3.1, we observe that if a graph $G \in \mathcal{G}(m,n)$ has arboricity a, then $m \leq a(n-1)$. This implies that $\left\lceil \frac{m}{n-1} \right\rceil \leq \boldsymbol{a}(G)$. By Lemma 3.3, if $\boldsymbol{a}(G) = a$, then $m \geq 2(a-1)^2+1$. By using the quadratic formula, we have $a \leq \left\lfloor \frac{2+\sqrt{2m-2}}{2} \right\rfloor$. That is $\left\lceil \frac{m}{n-1} \right\rceil \leq \boldsymbol{a}(G) \leq \left\lfloor \frac{2+\sqrt{2m-2}}{2} \right\rfloor$. ∎

For complete graphs K_{2p} and K_{2p-1}, Beineke [14] proved that $V(K_{2p})$ can be partitioned into p subsets each of which induces a spanning path and $V(K_{2p-1})$ can be partitioned into p subsets such that $p-1$ subsets each of which induces a spanning path and the other subset induces a star with $2p-1$ vertices. This implies that $\boldsymbol{a}(K_{2p}) = \boldsymbol{a}(K_{2p-1}) = p$. These facts are useful to use in the proof of Theorems 3.5 and 3.6.

Let π be a graph parameter. Define $min(\pi; m,n) = \min\{\pi(G) : G \in \mathcal{G}(m,n)\}$ and $max(\pi; m,n) = \max\{\pi(G) : G \in \mathcal{G}(m,n)\}$.

Theorem 3.5. $min(\boldsymbol{a}; m,n) = \left\lceil \frac{m}{n-1} \right\rceil$.

Proof. Let $G \in \mathcal{G}(m,n)$. By Theorem 3.4, $\left\lceil \frac{m}{n-1} \right\rceil \leq \boldsymbol{a}(G)$. It is easy to see that if $m \leq n-1$, then $min(\boldsymbol{a}; m,n) = 1$. A graph of order n and size $m \leq n-1$ where the arboricity is equal to 1 is $P_{m+1} + \{v_1, v_2, ..., v_{n-m-1}\}$. Next we consider only when $m \geq n$ in the following three cases.

Case 1. $n = 2p$ and $m \leq \binom{2p}{2} = p(2p-1)$ for a positive integer p. Put $m = s(2p-1)+t$ where s and t are integers satisfying $0 \leq t \leq 2p-2$ and $1 \leq s \leq p$.

If $t \neq 0$, then $\boldsymbol{a}(G) \geq \left\lceil \frac{s(2p-1)+t}{2p-1} \right\rceil = s+1$. We now construct a graph G of order $n = 2p$, size $m = s(2p-1)+t$, and $\boldsymbol{a}(G) = s+1$ as follows. Since $\boldsymbol{a}(K_{2p}) = p$, $E(K_{2p})$ can be partitioned into p subsets, each of which induces a forest. Let $\{E_1, E_2, ..., E_p\}$ be a partition of $E(K_{2p})$ such that each $E_i; 1 \leq i \leq p$ induces a forest and $|E_i| = 2p-1$. Let $H_0 = K_{2p} - E_{s+2} \cup E_{s+3} \cup ... \cup E_p$. Then $|V(H_0)| = 2p$ and $|E(H_0)| = (s+1)(2p-1)$. If we delete $2p-1-t$ edges from E_{s+1}, then $|E_{s+1} - W_0| = t$ where W_0 is a set of those $2p-1-t$ edges. Therefore, the graph G can be obtained by deleting W_0 from H_0. In other words, $G = H_0 - W_0$ where $n = 2p$ and $m = |E(H_0 - W_0)| = s(2p-1)+t$. Moreover, $E(G)$ can be partitioned into $s+1$ subsets, each of which induces a forest. Let $\{E_1^*, E_2^*, ..., E_{s+1}^*\}$ be a partition of $E(G)$ where $E_i^* = E_i$ for all $1 \leq i \leq s$ and $E_{s+1}^* = E_{s+1} - W_0$. Thus $\boldsymbol{a}(G) = \boldsymbol{a}(H_0 - W_0) \leq s+1$. Since $\boldsymbol{a}(G) \geq s+1$, $\boldsymbol{a}(G) = s+1$.

If $t = 0$, then $\boldsymbol{a}(G) \geq \left\lceil \frac{s(2p-1)}{2p-1} \right\rceil = s$. We can now construct a graph G where $\boldsymbol{a}(G) = s$ by deleting $W_1 = E_{s+1} \cup E_{s+2} \cup ... \cup E_p$ from K_{2p}. It is clear that $K_{2p} - W_1$ is a graph of order $2p$ and size $s(2p-1)$. Observe that $E(G) = E(K_{2p} - W_1)$ can be partitioned into s subsets, each of which induces a forest. Let $\{E_1, E_2, ..., E_s\}$ be a partition of $E(G)$. Hence $\boldsymbol{a}(G) = \boldsymbol{a}(K_{2p} - W_1) \leq s$. Since $\boldsymbol{a}(G) \geq s$, $\boldsymbol{a}(G) = s$.

Case 2. $n = 2p-1$ and $m \leq \binom{2p-1}{2} - (p-1) = (p-1)(2p-2)$ for a positive integer p. Put $m = s_0(2p-2)+t_0$ where s_0 and t_0 are integers satisfying $0 \leq t_0 \leq 2p-3$ and $1 \leq s_0 \leq p-1$.

If $t_0 \neq 0$, then $\boldsymbol{a}(G) \geq \left\lceil \frac{s_0(2p-2)+t_0}{2p-2} \right\rceil = s_0+1$. We now construct a graph G of order $n = 2p-1$, size $m = s_0(2p-2)+t_0$, and $\boldsymbol{a}(G) = s_0+1$ as follows. Since $\boldsymbol{a}(K_{2p-1}) = p$, $E(K_{2p-1})$ can be partitioned into p subsets, each of which induces a forest. Let $\{D_1, D_2, ..., D_p\}$ be a partition of $E(K_{2p-1})$ such that each $D_i; 1 \leq i \leq p$ induces a forest and $|D_i| = 2p-2$ for all $1 \leq i \leq p-1$ and $|D_p| = p-1$. Let $H_1 = K_{2p-1} - D_{s_0+2} \cup D_{s_0+3} \cup ... \cup D_p$. Then $E(H_1) = D_1 \cup D_2 \cup ... \cup D_{s_0+1}$. Moreover, $|V(H_1)| = 2p-1$ and $|E(H_1)| = (s_0+1)(2p-2)$. If we delete $2p-2-t_0$ edges from D_{s_0+1}, then $|D_{s_0+1} - W_2| = t_0$ where W_2 is a set of those $2p-2-t_0$ edges. The graph G can be obtained by deleting W_2 from H_1. In other words, $G = H_1 - W_2$ where $n = 2p-1$ and $m = |E(H_1 - W_2)| = s_0(2p-2)+t_0$. Moreover, $E(G)$ can be partitioned into s_0+1 subsets, each of which induces a forest. Let $\{D_1^*, D_2^*, ..., D_{s_0+1}^*\}$ be a partition of $E(G)$ where $D_i^* = D_i$ for all $1 \leq i \leq s_0$ and $D_{s_0+1}^* = D_{s_0+1} - W_2$. Thus $\boldsymbol{a}(G) = \boldsymbol{a}(H_1 - W_2) \leq s_0+1$. Since $\boldsymbol{a}(G) \geq s_0+1$, $\boldsymbol{a}(G) = s_0+1$.

If $t_0 = 0$, then $\boldsymbol{a}(G) \geq \left\lceil \frac{s_0(2p-2)}{2p-2} \right\rceil = s_0$. We can construct a graph G where $\boldsymbol{a}(G) = s_0$ by deleting $W_3 = D_{s_0+1} \cup D_{s_0+2} \cup ... \cup D_p$ from K_{2p-1}. It is clear that $G = K_{2p-1} - W_3$ is a graph of order $2p-1$ and size $s_0(2p-2)$. Observe that $E(G) = E(K_{2p} - W_1)$ can be partitioned into s_0 subsets, each of which induces a forest. Let $\{D_1, D_2, ..., D_{s_0}\}$ be a partition of $E(G)$. Hence $\boldsymbol{a}(G) = \boldsymbol{a}(K_{2p-1} - W_3) \leq s_0$. Since $\boldsymbol{a}(G) \geq s_0$, $\boldsymbol{a}(G) = s_0$.

Case 3. $n = 2p-1$ and $(p-1)(2p-2) < m \leq \binom{2p-1}{2} = (p-1)(2p-1)$ for a positive integer p. Put $m = (p-1)(2p-2)+t_1$ where t_1 is an integer satisfying $1 \leq t_1 \leq p-1$. Then $\boldsymbol{a}(G) \geq \left\lceil \frac{(p-1)(2p-2)+t_1}{2p-2} \right\rceil = p$. We now construct a graph G where $n = 2p-1$, $m = (p-1)(2p-2)+t_1$, and $\boldsymbol{a}(G) = p$ as follows. Consider the partition $\{D_1, D_2, ..., D_p\}$ of $E(K_{2p-1})$ in *Case* 2. If we delete $p-1-t_1$ edges from D_p, then $|D_p - W_4| = t_1$ where W_4 is the set of those $p-1-t_1$ edges. Therefore, the graph G can be obtained by deleting W_4 from K_{2p-1}. In other words, $G = K_{2p-1} - W_4$ where $n = 2p-1$ and $m = |E(K_{2p-1} - W_4)| = (p-1)(2p-2)+t_1$. Moreover, $E(G)$ can be partitioned into p subsets, each of which induces a forest. Let $\{D_1', D_2', ..., D_p'\}$ be a partitioned of $E(G)$ where $D_i' = D_i$ for all $1 \leq i \leq p-1$ and $D_p' = D_p - W_4$. Thus $\boldsymbol{a}(G) = \boldsymbol{a}(K_{2p-1} - W_4) \leq p$. Since $\boldsymbol{a}(G) \geq p$, $\boldsymbol{a}(G) = p$. If $t_1 = p-1$, then a graph of order $n = 2p-1$, size $m = (p-1)(2p-2)+p-1$, and arboricity p is K_{2p-1}. That is $G \cong K_{2p-1}$. ■

Theorem 3.6. $max(\boldsymbol{a}; m, n) = \left\lfloor \frac{2+\sqrt{2m-2}}{2} \right\rfloor$.

Proof. Let $G \in \mathcal{G}(m,n)$. By Theorem 3.9, $\boldsymbol{a}(G) \leq \left\lfloor \frac{2+\sqrt{2m-2}}{2} \right\rfloor$. If $\left\lfloor \frac{2+\sqrt{2m-2}}{2} \right\rfloor = p$, then $\frac{2+\sqrt{2m-2}}{2} < p+1$. Combine this to Lemma 3.8, it is clear that if $2(p-1)^2+1 \leq m \leq 2p^2$, then $max(\boldsymbol{a}; m, n) \leq p$. We now construct a graph in $\mathcal{G}(m,n)$ with arboricity p where $n \geq 2p-1$ and $(2p-2)(p-1)+1 = 2(p-1)^2+1 \leq m \leq 2p^2$.

Case 1. $(2p-2)(p-1)+1 \leq m \leq (2p-2)(p-1)+(p-1)$. Let $m = (2p-2)(p-1)+t_0$ where $1 \leq t_0 \leq p-1$. We can construct a graph H_0 of order $n = 2p-1$ and size $m = (2p-2)(p-1)+t_0$ with $\boldsymbol{a}(H_0) = p$ from K_{2p-1}. Since $\boldsymbol{a}(K_{2p-1}) = p$, $E(K_{2p-1})$ can be partitioned into p subsets $E_1, E_2, ..., E_p$, each of which induces a forest, where $|E_i| = 2p-2$ for all $1 \leq i \leq p-1$ and $|E_p| = p-1$. Let $W_0 \subseteq E_p$ be the set of $p-1-t_0$ edges. Then $K_{2p-1} - W_0$ is a graph of order $2p-1$ and size $(2p-2)(p-1)+t_0$. Moreover, $E(K_{2p-1} - W_0)$ can be partitioned into p subsets, each of which induced a forest. Let

$\{E_1^*, E_2^*, ..., E_p^*\}$ be a partition of $E(K_{2p-1} - W_0)$ where $E_i^* = E_i$ for all $1 \leq i \leq p-1$ and $E_p^* = E_p - W_0$. Thus $\boldsymbol{a}(K_{2p-1} - W_0) \leq p$. Since $\boldsymbol{a}(K_{2p-1} - W_0) \geq \left\lceil \frac{(2p-2)(p-1)+t_0}{2p-2} \right\rceil = p$, $\boldsymbol{a}(K_{2p-1} - W_0) = p$. Thus $H_0 = K_{2p-1} - W_0$. Clearly, $H_0 = K_{2p-1}$ if $t_0 = p-1$. We can construct a graph H_0 of order $n > 2p-1$ and size $m = (2p-2)(p-1) + t_0$ with $\boldsymbol{a}(H_0) = p$ by adding $n - (2p-1)$ vertices into $K_{2p-1} - W_0$.

Case 2. $(2p-2)(p-1)+p \leq m \leq (2p-2)(p-1)+(3p-2)$. Equivalently, $(2p-1)(p-1)+1 \leq m \leq (2p-1)(p-1)+(2p-1)$. Let $m = (2p-1)(p-1)+t_1$ where $1 \leq t_1 \leq 2p-1$. We can construct a graph H_1 of order $n = 2p$ and size $m = (2p-1)(p-1)+t_1$ with $\boldsymbol{a}(H_1) = p$ from K_{2p}. Since $\boldsymbol{a}(K_{2p}) = p$, $E(K_{2p})$ can be partitioned into p subsets $D_1, D_2, ..., D_p$, each of which induces a forest, where $|D_j| = 2p-1$ for all $1 \leq j \leq p$. Let $W_1 \subseteq D_p$ be the set of $2p-1-t_1$ edges. Then $K_{2p} - W_1$ is a graph of order $2p$ and size $(2p-1)(p-1)+t_1$. Moreover, $E(K_{2p} - W_1)$ can be partitioned into p subsets, each of which induced a forest. Let $\{D_1^*, D_2^*, ..., D_p^*\}$ be a partition of $E(K_{2p} - W_1)$ where $D_i^* = D_i$ for all $1 \leq i \leq p-1$ and $D_p^* = D_p - W_1$. Thus $\boldsymbol{a}(K_{2p} - W_1) \leq p$. Since $\boldsymbol{a}(K_{2p} - W_1) \geq \left\lceil \frac{(2p-1)(p-1)+t_1}{2p-1} \right\rceil = p$, $\boldsymbol{a}(K_{2p} - W_1) = p$. Thus $H_1 = K_{2p} - W_1$. Clearly $H_1 = K_{2p}$ if $t_1 = 2p-1$ and $n = 2p$. We can construct a graph H_1 of order $n > 2p$ and size $m = (2p-1)(p-1) + t_1$ with $\boldsymbol{a}(H_1) = p$ by adding $n - 2p$ vertices into $K_{2p} - W_1$.

Case 3. $(2p-2)(p-1) + (3p-1) \leq m \leq 2p^2$. Equivalently, $p(2p-1) + 1 \leq m \leq p(2p-1) + p$. Let $m = p(2p-1) + t_2$ where $1 \leq t_2 \leq p$. We can construct a graph H_2 of order n and size $m = p(2p-1) + t_2$ with $\boldsymbol{a}(H_2) = p$ from K_{2p}. Since $\boldsymbol{a}(K_{2p}) = p$ and $|E(K_{2p})| = p(2p-1)$, $E(K_{2p})$ can be partitioned into p subsets $D_1, D_2, ..., D_p$, each of which induces a forest, where $|D_j| = 2p-1$ for all $1 \leq j \leq p$. If we add $n - 2p$ vertices $u_1, u_2, ..., u_{n-2p}$ to K_{2p} and join u_1 to each vertex $w_k \in V(K_{2p})$, $1 \leq k \leq t_2$, then we obtain the desired graph H_2. It is clear that $E(H_2)$ can be partitioned into p subsets $B_1, B_2, ..., B_p$, each of which induced a forest, where $B_j = D_j + w_j u_1$ for $1 \leq j \leq t_2$ and $B_k = D_k$ for $t_2 + 1 \leq k \leq p$. Thus $\boldsymbol{a}(H_2) \leq p$. Since $\boldsymbol{a}(H_2) \geq \boldsymbol{a}(K_{2p}) = p$, $\boldsymbol{a}(H_2) = p$. ∎

In consequence of Theorems 3.5 and 3.6, the bounds in Theorem 3.4 are sharp. Next, we determine the range for the vertex arboricity of $G \in \mathcal{G}(m,n)$. In [3], N. Achuthan, N.R. Achuthan and L. Caccetta verified that $\boldsymbol{va}(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ and determined the range for the size of a graph G of order n with the prescribed vertex arboricity as the following lemma.

Lemma 3.7. [3] *Let $G \in \mathcal{G}(m,n)$ and $\boldsymbol{va}(G) = p$. Then $m \geq \binom{2p-1}{2}$. Furthermore, if $m = \binom{2p-1}{2}$, then $G \cong K_{2p-1} \cup \overline{K}_{n-2p+1}$.*

As a result of Lemma 3.7, observe that any graph $G \in \mathcal{G}(m,n)$ with $\boldsymbol{va}(G) = p$ has at least $\binom{2p-1}{2}$ edges. Thus $\max(\boldsymbol{va}; m, n) \leq p$ if $\binom{2p-1}{2} \leq m < \binom{2(p+1)-1}{2}$. In the following theorem, we determine $\max(\boldsymbol{va}; m, n)$.

Theorem 3.8. *Let m, n, and p be positive integers and $\binom{2p-1}{2} \leq m < \binom{2p+1}{2}$. Then $max(\mathbf{va}; m, n) = p$.*

Proof. By above observation, if $\binom{2p-1}{2} \leq m < \binom{2p+1}{2}$, then $\max(\boldsymbol{va}; m, n) \leq p$. We now construct a graph $G \in \mathcal{G}(m,n)$ such that $\boldsymbol{va}(G) = p$. By Lemma 3.7, if $m = \binom{2p-1}{2}$, then $G \cong K_{2p-1} \cup \overline{K}_{n-2p+1}$. Clearly $\boldsymbol{va}(G) = p$. If $\binom{2p-1}{2} < m \leq \binom{2p}{2}$, then we construct a graph G from $K_{2p-1} \cup \overline{K}_{n-2p+1}$ by joining each v_i for $1 \leq i \leq m - \binom{2p-1}{2}$ to v_0 where $v_i \in V(K_{2p-1})$ and $v_0 \in V(\overline{K}_{n-2p+1})$. Since G is obtained from $K_{2p-1} \cup \overline{K}_{n-2p+1}$ by adding some edges, $\boldsymbol{va}(G) \geq \boldsymbol{va}(K_{2p-1} \cup \overline{K}_{n-2p+1}) = p$. Let $V(G) = V_1 \cup V_2 \cup ... \cup V_p$ such that $V_1 \cup V_2 \cup ... \cup V_{p-1} = V(K_{2p-1} - \{v_{2p-1}\})$ where $|V_k| = 2$ for all $1 \leq k \leq p-1$ and $V_p = \{v_{2p-1}\} \cup V(\overline{K}_{n-2p+1}))$. Hence $V(G)$ can be partitioned into p subsets, each of which induces a forest. Therefore, $\boldsymbol{va}(G) = p$. In the case $\binom{2p}{2} < m < \binom{2p+1}{2}$, we can construct G from the complete graph $K_{2p} \cup \overline{K}_{n-2p}$ by joining each u_j for $1 \leq j \leq m - \binom{2p}{2}$ to u_0 where $u_j \in V(K_{2p})$ and $u_0 \in \overline{K}_{n-2p}$. Since G is obtained from $K_{2p} \cup \overline{K}_{n-2p}$ by adding some edges, $\boldsymbol{va}(G) \geq \boldsymbol{va}(K_{2p} \cup \overline{K}_{n-2p}) = p$. Let $V(G) = U_1 \cup U_2 \cup ... \cup U_p$ such that $U_1 \cup U_2 \cup ... \cup U_{p-1} = V(K_{2p-1} - \{u_{2p-1}, u_{2p}\})$ where $|U_k| = 2$ for all $1 \leq k \leq p-1$ and $V_p = \{u_{2p-1}, u_{2p}\} \cup V(\overline{K}_{n-2p}))$. Therefore, $V(G)$ can be partitioned into p subsets, each of which induces a forest. Then $\boldsymbol{va}(G) = p$. From the 2 cases, $max(\boldsymbol{va}; m, n) = p$. ∎

To determine a formula for $\min(\boldsymbol{va}; m, n)$, we shall apply Turán's theorem. Turán provided the complete r-partite graph of order n whose partite sets differ in size by at most 1, usually called the *Turán graph* and denoted by $T_{n,r}$.

Theorem 3.9. (Turán's Theorem) *Among the graph of order n with no $(r+1)-$clique, $T_{n,r}$ has the maximum number of edges.*

In order to make an easy application of the Turán graph in our work, we would like to state the following facts.

1. If $n = rq + t$, $0 \leq t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.
2. Let $G \in \mathcal{G}(m, n)$. If $\omega(G) \leq r$, then $m \leq |E(T_{n,r})|$.
3. $|E(T_{n,r})| = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.
4. Let $t_{n,r} = |E(T_{n,r})|$. Then for a fixed n, by using elementary arithmetic, we have $t_{n,r-1} \leq t_{n,r}$ for all $2 \leq r \leq n$. In fact $t_{n,r} - t_{n,r-1} \geq \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

In [3], N. Achuthan, N.R. Achuthan and L. Caccetta defined the graph $Q_{n,p}$ as follows. Let n and p be given integers. Put $l = \left\lfloor \frac{n}{p} \right\rfloor$ and $l' = n - pl$. Define $Q_{n,p} \cong \bigvee_{i=1}^{p} T_i$ where T_i is a tree of order $l+1$ if $i \leq l'$ or of order l, if $i \geq l'$ and $\vee$ is the join operation of graphs. They determined an upper bound of the size of a graph G with the prescribed vertex arboricity as the following Lemma.

Lemma 3.10. [3] *Let $G \in \mathcal{G}(m, n)$ and $va(G) = p$. Then $m \leq \binom{n}{2} - l'(l-1) - p\binom{l-1}{2}$ where the equality holds if and only if $G \cong Q_{n,p}$.*

Clearly, $V(T_{n,p})$ can be partitioned into p subsets $V_1, V_2, ..., V_p$ each of which induces an empty graph, whose size differ by at most 1. Let $\nu_s = |V_s|$ and E_s be the set of $\nu_s - 1$ edges. By Lemma 3.10, it means that $V(Q_{n,p})$ can be partitioned into p subsets, each of which induces a tree and the cardinality of those subsets differ by at most 1. Observe that $Q_{n,p}$ is the graph obtained from $T_{n,p}$ by adding $E_1, E_2, ..., E_p$ into $T_{n,p}$ such that $T_{n,p}[V_s] + E_s$ is a tree for all $1 \leq s \leq p$. We illustrate graphs $Q_{7,3}$ and $Q_{9,4}$ in Figure 1.

It is clear that $|E(Q_{n,p})| = t_{n,p} + (\nu_1 - 1) + (\nu_2 - 1) + ... + (\nu_p - 1) = t_{n,p} + (\nu_1 + \nu_2 + ... + \nu_p) - p = t_{n,p} + n - p$ where $t_{n,p} = |E(T_{n,p})|$. Moreover, By Lemma 3.10, we have $\boldsymbol{va}(Q_{n,p}) = p$.

We can see that if $G \in \mathcal{G}(m, n)$ and $\boldsymbol{va}(G) = p$, then G has size as most $t_{n,p} + n - p$. This implies that if $t_{n,p-1} + n - (p-1) < m \leq t_{n,p} + n - p$, then $min(\boldsymbol{va}; m, n) \geq p$. We determine $min(\boldsymbol{va}; m, n)$ as the following theorem.

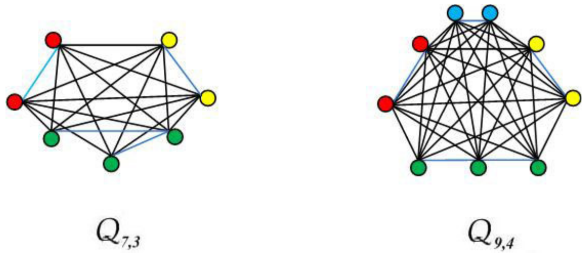

FIGURE 1. The graphs $Q_{7,3}$ and $Q_{9,4}$.

Theorem 3.11. *Let m, n, and $p \geq 2$ be positive integers and $t_{n,p-1} + n - (p-1) < m \leq t_{n,p} + n - p$. Then $min(\mathbf{va}; m, n) = p$.*

Proof. By the characteristic of the Turán graph and Lemma 3.10, we find that if $t_{n,p-1} + n - (p-1) < m \leq t_{n,p} + n - p$, then $min(\boldsymbol{va}; m, n) \geq p$. Let G be the graph of order n obtained by removing $t_{n,p} + n - p - m$ edges from $Q_{n,p}$. Since $G \in \mathcal{G}(m, n)$ is a subgraph of $Q_{n,p}$, $\boldsymbol{va}(G) \leq \boldsymbol{va}(Q_{n,p}) = p$. Because $\boldsymbol{va}(G) \geq \min(\boldsymbol{va}; m, n) \geq p$, $\boldsymbol{va}(G) = p$. Consequently, $min(\boldsymbol{va}; m, n) = p$. ■

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References

[1] J A. Bondy, U.S.R. Murty, Graph Theory with Applications (1st ed.), American Elsevier, Publishing, New York, 1976.

[2] G. Chartrand, H.V. Kronk, The point-arboricity of planar graph, J. London Math. Soc. 44 (1969) 612–616.

[3] C.A. Barefoot, Interpolation theorem for the number of pendent vertices of connected spanning subgraphs of equal size, Discrete Math. 49 (1984) 109–112.

[4] F. Harary, R. Mokken, M. Plantholt, Interpolation theorem for diameters of spanning trees, IEEE Transactions on Circuits and Systems 30 (1983) 429–432.

[5] E. Sampathkumar, L.P. Latha, The global set-domination number of a graph, Indian J. Pure Appl. Math. 10 (1994) 1053–1057.

[6] S. Schuster, Interpolation theorem for the number of end-vertices of spanning trees, Journal of Graph Theory 7 (1983) 203–208.

[7] S. Zhou, Interpolation theorems for a family of spanning subgraphs, Czechoslovak Mathematical Journal 48 (1998) 45–53.

[8] N. Punnim, Degree sequences and chromatic numbers of graphs, Graphs and Combinatorics 18 (2002) 597–603.

[9] N. Punnim, Interpolation theorems on graph parameters, Southeast Asian Bulletin of Mathematics 28 (2004) 533–538.

[10] N. Punnim, On maximum induced forest in graphs, SEAMS Bull. Math. 27 (2003) 667–673.

[11] N. Punnim, Interpolation theorems in jump graphs, Australasian Journal of Combinatorics 39 (2007) 103–114.

[12] C.St.J.A. Nash-Williams, Edge-disjoin spanning trees of finite graph, J. London Math. Soc. 36 (1961) 445–450.

[13] C.St.J.A. Nash-Williams Decomposition of finite graphs into forests, J. London Math. Soc. 39 (1961) 12.

[14] L.W. Beineke, Decomposition of complete graphs into forests, Magyar Tud. Akad. Mat. Kutato Int. Kozl. 9 (1964) 589–594.